TEACHERS' IMAGES OF THE 'EQUATION' CONCEPT

Iiris Attorps
University of Gävle, Sweden

Abstract: The purpose of this study is to analyse what kind of conceptions secondary teachers in mathematics have of the equation concept and what kind of experiences gave rise to their own concept learning. Ten secondary school teachers of mathematics participated in the study. Data was gathered by interviews and questionnaires. The 'phenomenographic' research method was applied in interpreting the results of the investigation. The results indicate that the teachers' conceptions of the equation concept differ from the formal definition of the equation concept. At school they spent most of their time developing procedural skills instead of mathematical understanding.

INTRODUCTION

Teaching of, and achievements in, mathematics have been criticised in several countries during the last decade. It is generally concluded that school mathematics focuses on developing algorithmic skills instead of mathematical understanding (Soro & Pehkonen 1998) and that teachers devote much less time and attention to developing conceptual knowledge than procedural knowledge (Porter 1989). Pupils learn several basic concepts in arithmetic and algebra superficially without understanding (Soro&Pehkonen 1998). Teaching of mathematics for understanding is an extremely complex process (Hiebert and Lefevre 1986). Teaching for understanding relies on teachers having mathematical and pedagogical knowledge and skills (Shulman, 1987). The studies of prospective elementary and secondary teachers' mathematics subject knowledge indicate that they lack a conceptual understanding of many topics in elementary mathematics (Ball 1990a, 1990b, Cooney 1999).

In this paper I discuss questions about mathematical knowledge, a theory of learning of mathematical concepts and a theory of concept definition and concept image. I will illustrate the theoretical framework with an empirical study. This investigation revealed mathematics teachers' experiences of learning the mathematical concept 'equation' and the kinds of conceptions they developed.

THEORETICAL FRAMEWORK

It is common to use the terms 'procedural knowledge' and 'conceptual knowledge' to denote a distinction between two forms of mathematical knowledge (Hiebert and Lefevre 1986). Procedural knowledge refers to computational skills and knowledge of procedures for identifying mathematical components, algorithms and definitions. Procedural knowledge of mathematics has two parts: (a) knowledge of the format and syntax of the symbol representation system and (b) knowledge of rules and algorithms useful in mathematical tasks. Conceptual knowledge refers to knowledge of the underlying structure of mathematics. It is characterised as knowledge rich in relationships and includes the understanding of mathematical concepts, definitions and fact knowledge. Both procedural and conceptual knowledge are considered as necessary aspects of mathematical understanding (Hiebert and Lefevre 1986).

Sfard (1991) has analysed the development of various mathematical concepts, definitions and representations from an historical and a psychological perspective. Sfard's analysis has shown that abstract notations such as a rational number, function etc. can be conceived in two fundamentally different ways: operationally as processes or structurally as objects. According to Sfard the process of concept learning includes three stages: 1) interiorization: a learner performs operations or processes on lower level mathematical objects, 2) condensation: a learner has an increased capability to alternate between different representations of a concept and 3) reification: a learner can conceive of the mathematical concept as a complete, "fully-fledged" object. At the stage of reification the new entity is detached from the process that produced it and the concept begins to receive its meaning as a member of certain category. The first two stages represent the operational aspect of mathematical notation and the last stage its structural aspect. Sfard suggests that the structural conception of a mathematical notation is static whereas the operational conception is dynamic and detailed. To understand the structural aspect of a mathematical concept is difficult for most people because a person must bridge the ontological gap between the operational and structural stage. Sfard also distinguishes between the words "concept" and "conception".

... the word "concept" (sometimes replaced by "notion") will be mentioned where a mathematical idea is concerned in its "official " form.. (Sfard 1991, p3)

Sfard emphasises thus that mathematical concepts have an official, formal, side. Conceptions are regarded as a part of human knowledge (Ponte 1994). Sfard considers conceptions to be the private side of mathematical concepts, which every human being has in his or her mind:

the whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective "universe of human knowing" – will be referred to as "conception"

(Sfard 1991, p3.)

This seems to draw on Tall and Vinner's (1981) theory of concept image and concept definition. They suggest that when we think of a concept something is evoked in our memory. Often these images do not necessarily relate to a concept definition even if the concept is well defined theoretically. The collection of memories is called the concept image. In Tall and Vinner's theory, concept image is the whole cognitive structure that is associated with the concept. Concept images are generated by previous experiences and conceptions and by memories of tasks in which the concept definitions have been tested in the teaching and learning of mathematics (Tall and Vinner 1981, Vinner 1991). Mathematical experiences in everyday life and experiences of learning mathematics have an important role in our mathematical thinking. They give us, however, a limited conception of the innermost nature of mathematics (Hatano 1996). Stacey's (1997b) investigation has shown that students' interpretations of algebraic symbolism are based on experiences that are not helpful for them. For example 5s can mean 5 seconds in everyday life, but in algebra it means 5 multiplied by s. Kuchemann (1981) also found that a very small percentage of the 13- to 15-year-old pupils were able to consider letters as generalised numbers and even fewer were able to interpret letters as variables. The majority of students either treated letters as concrete objects or ignored them.

Pupils encounter the equals sign early in their life. However, the limited view of the equals sign causes difficulties for students in understanding mathematical notations. Outside the mathematics classroom the equals sign is often used to mean "is" (for example MATH = DIFFICULT) or "gives" (for example HARD WORK = SUCCESS). Students associate the equals sign with questions such as "find the solution to the following equation: 5x - 2 = 10". The equals sign is also often taken as a left-to-right directional signal or to mean "do something". In algebra this meaning of the equals sign is not correct because there is often no "question" on one side of the equals sign and no "answer" on the other (Kieran, 1992, p393). Kieran (1992) points out that the traditional approach in algebra focuses on procedural issues, which allows the students to bypass the algebraic symbolism when solving equations. Wagner, Rachling, & Jensen (1984) also show that many algebra students have an operational interpretation of algebraic expressions, because they try to add " = 0" to any expressions they where asked to simplify.

In contrast to concepts in everyday life, mathematical concepts are well defined. Concept definitions are the body of words used to designate the concepts. For example the concept 'equation' is defined as:

a formula that asserts that two expression have the same value; it is either an identical equation (usually called an identity), which is true for any values of the variables, or conditional equation, which is only true for certain values of the variables (the roots of the equations). For example, $x^2 - y^2 = (x - y)(x + y)$ *is an identity, and* $x^2 - 1 = 3$ *is a conditional equation with roots* $x = \pm 2$

Borowski and Borwein 1989, p194 (se also Karush, 1989, p95)

Formal definitions like this could be part of a concept image, but they do not guarantee the understanding of the concept. When learners have formed their concept images or their subjective conceptions of mathematical concepts the definitions become unnecessary. Empirical studies also indicate that students interpret mathematical concepts operationally as processes even if the concepts were introduced structurally using definitions (Vinner and Dreyfus 1989, Sfard 1989). The majority of students do not use definitions when solving tasks because their everyday-life thought habits take over and they are unaware of the need to consult the formal definitions. In most cases referring to the concept image is successful. It is suggested that only non-routine problems, like the identification of examples and non-examples of a given concept, problem solving and mathematical proofs, can encourage students to use the formal concept definitions (Vinner 1991).

METHOD

Selection: Ten secondary school mathematics teachers participated in the current study. Five teachers were newly graduated (less than one year's experience) and five were experienced (between 10 and 32 years' experience)

Procedure: Data were gathered by interviews and questionnaires. The interviews were recorded in the schools where the teachers worked. The teachers told me first about their memories and experiences of concept learning from their secondary school up to their university level. After the interview they answered the question: *Which of the following statements do you consider to be an equation?* There were 18 expressions with examples (e.g. $e^{x+y}=1$) and non-examples (e.g. $x^2-5x-10$). They answered Yes (Y) or No (N) on a scale from 1(unsure) to 5 (sure). An answer "N5" thus means that the teacher is quite sure that the statement is not an equation. During a second interview the teachers could elaborate their thoughts on the 'equation' concept and explain their answers. Each interview lasted about two hours. The interview quotations have been marked in the following way: For example I1, p1 = interview 1, page 1; E1, I2, p2 = example number 1 in the questionnaire, interview 2, page 2.

Analysis method: Tapes were transcribed and interpreted into categories of conceptions by the phenomenographical research method. The aim of phenomenography is to describe how phenomena are experienced and understood. "Conception" is the central concept in phenomenography (Marton and Booth 1997). In my investigation I use Sfard's definition: the "conception" is a persons' subjective picture of the concept (Sfard, 1991, p3).

RESULTS

My teachers described their concept learning as: a mechanical drill, learning by heart, learning of a model and focus on routine problems. They felt that concept learning was a mechanical drill particularly at the secondary and the upper secondary school level. University teachers assumed that students already understood the equation concept and they continued to teach procedures. One of the teachers said:

Studies at the upper secondary school were similar to those at the primary school: examples only from the textbook… I had no time to enter deeply into my studies at the university … it is important to save my skin and to learn by heart certain typical examples and to pass an exam".(I1, p3)

The teachers also said that they had worked mostly with routine problems involving first, second and third degree equations, higher order equations, trigonometric equations, logarithmic functions and equations etc. Some teachers said that understanding came later, especially in connection with applications in physics.

When analysing the interviews I have identified a number of conceptions of the equation concept that are not identical with the concept definition. The following five categories were not regarded as equations: 1) identities, 2) non-algebraic equations, 3) equations that include one or more unknowns, 4) trivial equations, 5) functions. The final category includes those expressions regarded as equations: 6) inequalities and expressions.

In the following I present examples from each main category.

1) Conceptions of identities: $\cos^2\alpha + \sin^2\alpha = 1$

Subcategories for No-answers: a rule, a formula, a result, an identity, a unit circle, a Pythagorean identity. E.g.:

No (N3), *"It is a rule or a formula. I do not remember what you call it. It is a result of something"*. (E11, I2, p5)

No (N5), *"...because there is no unknown factor ".(*E11, I2, p5).

2) Conceptions of non-algebraic equations: $\int f(x)dx = x^2 + C$

Subcategories for No-answers: an integral, an integral and a derivative, a derivation, an area under the curve, a surface, a function, an interval. E.g.:

No (N3), *"I associate this to the integral. It is some kind of area. I don't remember what it is. I feel it is an area under the curve and it can't be an equation"*. (E9, I2, p4).

No (N5), *" No, it is not an equation. There are integrals and derivatives, I don't connect them with equations"*. (E9, I2, p4).

3) Conceptions of equations with one or more unknown factor: $2x + 5y = \sqrt{a}$

Subcategories for No-answers: a formula, impossible to solve it. E.g.:

No (N1), *"I am a bit unsure if I can solve it.... I feel there are three unknown variables. Maybe a formula, but you can't solve it because you don't know any values of variables"*. (E2, I2, p1).

No (N5), *" x and y are unknown. I need one more equation to solve this"*. (E2, I2, p1).

4) Conception of a trivial equation: $x = 2$

Subcategories for No-answers: "an answer" i.e. a solution, only an expression for what x is, a value. E.g.:

No (N5), *" I am sure about this, it is only an answer, you have got an answer"*. (E7, I2, p4).

No (N5), *"You see a value of the unknown factor already in the beginning. It is not an equation*". (E7, I2, p4).

5) Conception of a function: $f(x) = 2x + 1$

Subcategories for No-answers: a function, a straight line. E.g.:

No (N), *"It is a function. It is a straight line. You can draw it".* (E10, I2, p5).

No (N3), *"It is a function. Equal sign says that it is an equation, but I don't know mathematically, if you regard it as an equation".* (E10, I2, p5)

6) Conceptions of inequalities and expressions: $x + |x - 3| \geq |x - 1| + 2$

Subcategories for Yes-answers: an equation, an inequality. E.g.:

Yes (Y5)*"It is an equation... I can solve x here. I have a goal... I must have a goal and in the end I can solve x".* (E15, I2, p8).

Yes (Y3), *"Yes, it is an equation. It is an inequality. I wonder if it is an equation or not, but I think inequalities are treated in the same section of the textbook as equations, so it is some kind of equation*". (E15, I2, p8).

When I asked what does the concept "equation" mean for you, one of the teachers said:

"*When you ask me now whether this is an equation or not...I get a feeling that I want to solve it. I want to get an answer, i.e. a solution. I have learnt, if you get a right answer, it's good, you are capable. This I have learnt at school".* (I2, p10).

Another teacher said:

"*The left hand side is equal to the right hand side ...I have not before reflected on what the concept equation means ...7+ x = 9, something like this. You try to find an unknown factor, you solve an equation..."* (I2, p13).

DISCUSSION/CONCLUSIONS

Textbooks treat equations and inequalities in the same chapter and therefore it is perhaps natural that the teachers look upon 'equation' and 'inequality' as being the same concept. Identities like $\cos^2\alpha + \sin^2\alpha = 1$ are not regarded as equations. They are rules and it is not possible to solve them. Teachers say that the concept 'equation' means "*equality*". It is also worth noting that sometimes they cannot identify which of the statements can be regarded as equations, because they are unsure of mathematical symbols like $\sqrt{\ }$, $f(x)$, $\int$, $y(x)''$, $=$, $\leq$ and unsure of the meaning of the letters like α or *a*. They may also be unsure of solving procedures (Kuchemann 1981, Kieran 1992).

Some of my teachers don't regard statements like $2x + 5y = \sqrt{a}$ as equations because they think that it is not possible to solve them if they don't know the values of the other factors. The solution for them must be a fixed number. To many of my teachers statements like *x = 2* are "answers" or results but not equations, because they feel that *x* is already solved.

For these teachers an equation does not constitute a mathematical statement or a complete, "fully-fledged" object (Sfard, 1991). The process-object duality of the mathematical notation creates fundamental problems for teachers. They have considerable problems in leaving the process level and in entering the object level (Sfards 1991). Many of my interviewed teachers have an operational or procedural conception of the equation concept in their own concept image (Hiebert & Lefevre, 1986; Vinner 1991). Teachers' previous experiences from teaching and learning of mathematics indicate that they have spent most of the time at school developing procedural skills instead of mathematical understanding. Their experiences of learning have formed their concept images (Vinner and Dreyfus 1989). These teachers' conceptions of the concept 'equation' seem to be based on their experiences when they first learned the process of solving equations.

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