THE NATURE OF MATHEMATICS AS VIEWED FROM COGNITIVE SCIENCE

Laurie D. Edwards

ST. Mary's College of California

ledwards@stmarys-ca.edu

Abstract: The theory of embodied cognition proposes that the source of many ideas, including mathematical ones, lies in the common physical experience of human beings. Through mechanisms such as conceptual mappings and metaphors we are able to construct new ideas and understandings that are more abstract, in the sense of being less tied to physical experience. The purpose of this paper is to review work on the theory of embodied mathematics and to describe an application of the theory to understanding how transformation geometry is learned.

My goal in this paper is to characterize one perspective on the nature of mathematics, drawing primarily from recent cognitive science and in particular the work of George Lakoff and Rafael Núñez (Lakoff & Núñez, 2000), and to offer an example of the application of this perspective in my own research. *A* theory of how human beings think and understand mathematically should be able to account for both the correct and consistent mathematical ideas of professional mathematicians, and for the variety of ideas that diverge from this body of shared knowledge. In addition, unless mathematics education wants to set mathematics aside as a special case, such a theory should draw on conceptual primitives and mechanisms that are not unique to mathematical thinking. This paper presents a perspective that addresses the nature of mathematics by drawing from work by a number of researchers in contemporary cognitive science, including Lakoff and Núñez (2002), Johnson (1987), Varela, Thompson and Rosch (1991), Dehaene (1997), Fauconnier and Turner (2002) and others.

The theory of embodied mathematics

The theory of embodied mathematics states in essence that:

> *(1) mathematics emerges through the interaction of the mind with the world, and*
>
> *(2) that each of these elements (the mind and the world) offers constraints on what kind of mathematics human beings are able to be constructed.*

A specific aspect of the theory describes a conceptual mechanism that has been used to understand thought and language in areas outside of mathematics, that of conceptual metaphor. As Lakoff and Núñez state, *(3) "A large number of the most basic, as well as the most sophisticated, mathematical ideas are metaphorical in nature" (Lakoff and Núñez, 2000, p. 364).*

What Lakoff and Núñez refer to as the theory of mind-based mathematics does not deny the objectivity of the material world; in fact, it accords to this world a constraining influence on the kind of mathematics that humans are able to construct. The world behaves consistently, fairly reliably, predictably - when you put a collection of two objects together with another collection of two objects, you are reliably going to end up with a collection of four objects, not one of five or three. Human infants behave as if they have some awareness of this kind of predictable behavior, though in their case, this awareness extends only to the very smallest of collections, namely, single objects combined with single objects (cf., Dehaene, 1997). Evolutionarily, it seems to have been important for humans to be able to subitize, to keep track of small collections of objects, and this provides part of what the mind brings to the picture. There is a small amount of innate or virtually innate arithmetic that humans share with a few other kinds of animals, for example, chimps and parrots (Dehaene, 1997). But this starting point is very restricted, and one of the questions addressed by the theory of embodied mathematics has to do with identifying the conceptual mechanisms and building blocks that allow humans to go from this starting point to develop the incredible richness of contemporary and historical mathematics.

Conceptual mechanisms and primitives

In addressing the nature of mathematical thought, Lakoff and Núñez offer a set of mechanisms and cognitive primitives drawn from existing work in cognitive linguistics and embodied cognition. These primitives and mechanisms include prototypes, image schemas, aspectual concepts, conceptual metaphor, conceptual blends, and metonymy (for details, see Lakoff, 1987 and Lakoff & Núñez, 2000). Conceptual metaphor is an important (but not the only) mechanism proposed to account for how mathematical understandings are connected to the world and to each other. Conceptual metaphor is an unconscious mapping between a well-understood source domain to a less-well-understood target domain, one which carries with it the inferential structure of the source domain, thus allowing an understanding of the target domain to be constructed. A non-mathematical example might be the way in which research is conceptualized as a process of physical construction: we speak of someone's work as having a firm foundation, or as extending previous work, even when there is clearly no actual process of physical building going on. Within mathematics, a simple example of conceptual metaphor would be the "arithmetic as object collection" metaphor, where our common, embodied experience of grouping or collecting objects serves as the source domain for constructing the arithmetic of natural numbers.

It should be clarified that not all conceptual mappings draw from direct physical experience, or have to do with the manipulation of objects. In fact, only the most basic level of mathematics is constructed from metaphors that link to physical experience. The metaphors utilized at this level are called "grounding metaphors." The majority of the concepts and processes of mathematics, according to the theory, are constructed through mappings between and among existing mathematical domains. An example of this kind of linking metaphor will be given later, when space is conceptualized in terms of sets of points. A third type of metaphor, not based on physical experience, is the redefinitional metaphor, defined as "metaphors that impose a technical understanding replacing ordinary concepts" (Lakoff & Núñez, 2000, p. 150).

Conceptual metaphor is only one of the building blocks proposed in the theory of embodied mathematics. Conceptual blends are another. Blends consist of mappings that draw from more than one source domain or element to allow the construction of a target domain which is isomorphic to neither of the sources, but which draws from the inferential structure of each. An example would be the "numbers as points on a line" blend, where drawing from previously-constructed understandings of both numbers and lines, new entities, "number-points" are created that have characteristics of both.

An embodied perspective on the learning of transformation geometry

The theory of embodied cognition can provide a framework for understanding how both children and adults learn initial concepts within the domain of transformation geometry. In a series of studies beginning in 1989, I worked with eleven-to-fifteen year old students using a computer environment I designed for exploring transformation geometry (see Figure 1). Later, with Rina Zazkis, I investigated adult undergraduates' learning of the same subject. In the first, most extensive study, I introduced three euclidean transformations, translation, rotation and reflection, to a whole class of sixth-grade (eleven- to twelve-year-old) students, utilizing sheets of overhead transparencies to illustrate them and to elicit the students' own description of these motions. Then, I worked with twelve of these students for a period of five weeks, videotaping their interactions with the transformation geometry microworld (Edwards, 1991, 1992). A similar study was carried out two years later with a group of ten high school students (fifteen-year-olds). In this study, the students were introduced to the transformations in pairs, rather than in a whole group, using drawings on a sheet of paper (Edwards, 1997). Finally, Rina Zazkis and I examined the responses of fourteen college undergraduates, first to a paper-and-pencil task requiring them to predict the outcome of several transformations of the plane, and then, through videotapes of their use of a modified version of the microworld (Edwards & Zazkis, 1993). Although there were methodological differences across these studies, and some differences in specific results, overall, the responses of all of the participants, whether middle-school, high-school or adult, were remarkably similar. In particular, the students all seemed to have had the same initial expectations of how the transformations would work, and they made the same kinds of errors. This raises the question: why is there such consistency in how students, of various ages, learn transformation geometry? My initial expectation was that older students and adults would be less subject to "misconceptions" about geometric transformations and better able to carry out independent investigations, due to the more advanced state of development of their mathematical thinking. Yet this was not the case. The question of why students of various ages respond in a similar way to a "new" mathematical topic can, I believe, be productively investigated utilizing the theory of embodied mathematics. To do so will require examining the domain of transformation geometry from the perspective of contemporary mathematicians, and from the perspective of a learner who is meeting the domain, in a formal sense, for the first time.

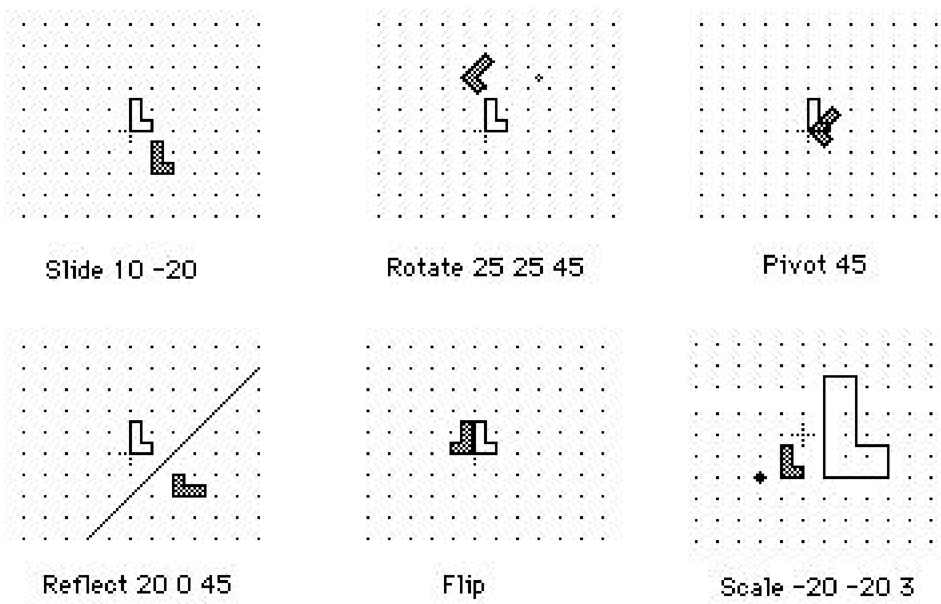

Figure 1: Transformations in the computer microworld

Transformation geometry from the perspective of contemporary mathematics

From the point of view of contemporary mathematicians, geometric transformations are mappings of the plane, and a "geometry" is "a study of the properties (expressed by postulate, definitions, and theorems) that are left invariant under a group of transformations" (Meserve, 1955, p. 21). This definition is in contrast to the notion of geometry current before the end of the 19^{th} century. Prior to Felix Klein's Erlangen program, geometry was seen as the investigation of points, lines, planes, and three-dimensional space, utilizing the tools of deductive logic and formal proof. The latter elements, logic and proof, remain central to all of mathematics, including transformation geometry. However, the Erlangen program, which changed the focus of geometry from specific objects with concrete, visualizable referents (points, lines, planes, etc.), to notions of invariance, group theory, and mappings, is an example of what Lakoff and Núñez call "the discretization program" which has permeated mathematics since the late 19th century. This evolution within the discipline established a new conception of space (and of the plane), one that differs substantially from how space is conceptualized outside of formal mathematics. Briefly, prior to the 19th century, and outside of formal mathematics today, space (and planes, and lines) are conceptualized as naturally continuous, and as the background settings within which objects are located. Space, planes, and lines exist independently from any objects that are located in or on them, for example, geometric figures. Points are located at particular places in space, or on a plane or line (Lakoff & Núñez, 2000, p. 260); however, points do not exist independently of the places where they are located.

The discretization program was based on the metaphorical reconceptualization of numbers as points, in which discrete numbers are associated with discrete points on a line (and vice versa). In conjunction with the matching of numbers with discrete symbols, this made possible the progressive arithmetization of mathematical activity involving space or the plane, as in analytic geometry or calculus. The discretization program is based on a central metaphor, as follows (ibid., p. 263):

A Space is a Set of Points

Source Domain A Set with Elements	*Target Domain* Naturally-Continuous Space with Point-Locations
A set	-> An n-dimensional space—for example, a line, a plane, a 3-dimensional space
Elements are members of the set	-> Points are locations in the space
Members exist independently of the sets they are members of	-> Point-locations are inherent to the space they are located in
Two set members are distinct if they are different entities	-> Two point-locations are distinct if they are different locations
Relations among members of the set	-> Properties of the space

Under this metaphor, the objects of traditional geometry are radically reconceptualized. As stated in Lakoff and Núñez, "There is nothing inherently spatial about a 'space.' What are called 'points' are just elements of the set of any sort...Like any members of sets, points exist independently of any sets they are in. Spaces, planes and lines—being sets—do not exist independently of the points that constitute them...A geometrical figure, like a circle or a triangle, is a subset of the points in a space, with certain relations among the points" (ibid., p. 263).

The transformation geometry microworld was built based on this "modern" understanding of geometry. The transformations are intended to be understood as mappings of the whole plane. The geometric figures that appear in on the computer screen are simply specific subsets of points of the plane that serve to illustrate, graphically, the effects of a given transformation. When the middle school students were first shown the transformations, they were asked to describe what was the same and what was different before and after each transformation. Thus, the notion of invariance was introduced explicitly into the context of the activity. A task used near the end of the study with the middle and high-school students required them to find the symmetries of a set of geometric figures and sort them into groups having the same symmetries. In general, the goal was to involve the students in working with geometric transformations the way that a mathematician would, as mappings of the plane, rather than as simple motions of geometric figures.

Transformation geometry from the learner's perspective

Although the microworld may have been designed with a contemporary mathematical perspective on geometry in mind, those who use the software as a learning environment may interpret it in a quite different way. That is, the software tool is a different instrument to the designer and to the learner, who have different expectations about how it will work and different interpretations of what they see on the computer screen (Lagrange, Artigue, Laborde & Trouche, 2001). The learners who worked with the transformation geometry microworld, whether young people or adults, brought to it not the mathematician's commitment to the power of formalism, but instead, a set of intuitions honed through activity and experience in the physical world. In other words, their understanding of geometry was an embodied one. This is supported by examining the nature of their interpretations and "errors" in the microworld.

The first design of the microworld included only the most general version of three euclidean transformations, translation, rotation and reflection. To use the REFLECT command, for example, you need to specify the mirror line "over which" the plane is reflected; to use ROTATE, you need to enter the location of a center point and an angle amount for the rotation. When the microworld was pilot-tested, however, many of the students had difficulty with these two commands, and so simpler, "local" versions were implemented, versions that worked only with the block letter "L" shape that was the default figure shown on the screen. Specifically, a FLIP command was created that always reflected across the long "back" of the "L," and a PIVOT command was created that always turned around the bottom left corner of the L-shape. These specialized commands were introduced as easier versions of

REFLECT and ROTATE, and they made it possible for the students to begin working with the microworld. However, the learning activities created in the microworld required that the students come to understand the more general commands.

In learning about the general transformations, a number of students exhibited what I, at that time, referred to as a conceptual "bug" regarding rotation. Instead of seeing rotation as mapping all the points of the plane around any center point on the plane, these students expected that a ROTATE command would slide the shape to the given center point, and then pivot it around it (see Figure 2). In other words, they had a hard time seeing rotation as occurring "at a distance" from the object. This "misconception" cropped up among approximately a third of the students in the first study with middle-schoolers, and among a smaller number of high school and adult students. In contrast to other mistakes made when using the microworld, the students did not find it easy to correct their own understanding by further exploration, and I generally needed to step in and explain how the ROTATE command actually worked.

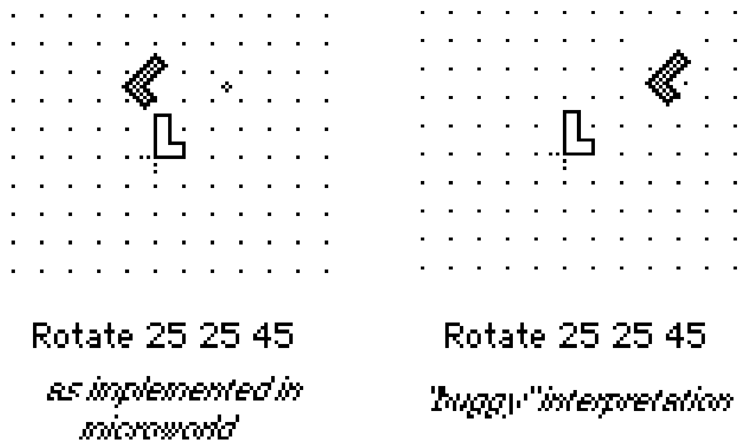

Figure 2: The rotate "bug"

I first thought about the rotate "bug" simply as a misconception, something idiosyncratic occurring with a few specific students that could be overcome through better instruction or through redesigning the microworld. Later, I thought that the source of the "misconception" was social. That is, the conventions that mathematicians had developed for this concept, derived from the values of coherence and consistency, were not obvious to the students, but that once these conventions were explained, the students would have no problem with rotation.

However, after finding that the "rotation bug" showed up in sixth graders, high school students, and adults, and that it was not easy to teach the "correct" version of the transformation, it seemed that something more fundamental was going on. These results from the studies of transformation geometry could be accounted for within an embodied perspective.

Transformations: motions or mappings?

Data on how students interacted with the computer microworld indicate that the conceptualization of transformation geometry of contemporary mathematicians and that held by learners are quite different. The students' responses to the computer microworld grow out of an embodied understanding of space (and the plane) as naturally continuous. When they look at the graphical window of the microworld, they see a continuous (but normally "invisible") plane that provides a background, *on* which geometric figures are located. The commands (the geometric transformations) are used to *move* the geometric figures around on top of this plane. By contrast, for the mathematician, the plane consists of an infinite number of discrete points, and transformations are simply mappings of those points that preserve some properties and change others (euclidean transformations preserve size and shape; affine transformations preserve area, and so on). From the point of view of the discipline of mathematics as it has evolved since the discretization program, there are no motions involved in transformations of the plane. These two conceptualizations are contrasted in the table below:

Learner's Transformations	**Mathematician's Transformations**
The plane is an empty, invisible background Geometric figures sit *on* the plane Transformations are physical motions of geometric figures on top of the plane The "objects" of geometry are points, lines, circles, triangles, etc.	The plane is a set of points Geometric figures are subsets of points of the plane Transformations are mappings of all the points of the plane The "objects" of geometry are groups of Transformations.

In fact, in introducing the computer microworld, I would emphasize that transformations were mappings of the whole plane, and that the "L" shape shown on the screen was simply a subset of those points, shown as dark dots in order to be able to easily see the effects of a given transformation, Yet this explanation did not prevent students from consistently talking about transformations as movements, nor from interpreting the general mapping ROTATE as a sequence of simpler, local motions, with rotation always occurring around a point that's a part of the object.

The learner's general understanding of transformations, as well as the rotate "bug" in particular, both arise from our embodied experience. As human infants, our first physical actions of turning always occur around a center point located within our own bodies. Our experience of the space we move through is as continuous. When we interact with objects on a table (a physical analogue to the mathematician's plane), they sit *on* the table, they are not embedded "subsets" of it. Other early experiences with rotation, including the turning of wheels, phonograph records, and the like, are also of this "local" nature; the rotation occurs around a center within the object itself. Even when there is a "turning" around a center point at a distance, for instance, on a swing, there is always some kind of physical connection between the object and the center of the rotation. Thus, it is difficult for students to understand a general mapping of the plane around an arbitrary center point, detached from the geometric figure that constitutes the object of their attention.

This tension between the mathematician's understanding of transformations as mappings, implemented in the microworld, and the embodied, natural understanding of motion that the learners brought to the experience is the source of their "misconceptions." These "misconceptions" are in actuality, conceptions that are adaptive and functional outside of the context of formal mathematics.

Conclusion

The purpose of this paper was to briefly review the central elements of the theory of embodied cognition as applied to understanding mathematics, and to examine a specific case in which embodiment can clarify a persistent student "misconception." The idea that mathematics is constructed from conceptually simpler building blocks seems to me to be relatively uncontroversial within the mathematics education research community. The controversy, or perhaps, better, the scholarly enterprise, comes in attempting to build a coherent theory about how this occurs, to discover the nature of the mechanisms and entities involved. The theory of embodied mathematics, as demonstrated in the case of transformation geometry, can provide a coherent and plausible account of how mathematics is built from the conceptual raw material available to us as human beings with bodies, acting in the physical world.

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