BRIDGING PERCEPTION AND THEORY: WHAT ROLE CAN METAPHORS AND IMAGERY PLAY?

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Abstract: *In this paper I present a study investigating ways in which secondary school students (9th grade) conceive mathematical objects (like functions and variables), in modelling activities. The focus is on how mathematical knowledge is structured in the pupils and what mental dynamics play on activating mathematical thinking. In particular, I analysed students' cognitive processes within the embodied cognition perspective. My aim is to observe how metaphorical thinking and imagery can foster the transition from perception to theory, and the construction and communication of mathematical thinking.*

INTRODUCTION

Recent Mathematics Education researches have given an emphasis on the intuitive and embodied nature of mathematical ideas and symbols. These researches have the purpose of studying the cognitive foundations (and conceptual structures) of Mathematics. They suggest connections with studies on the behaviour and structure of the brain, in particular in terms of the modalities by which the brain manages and elaborates perceptions. As such, their approach involves other fields, like biology, physiology and neuroscience, and provides reasons why mathematical knowledge seems to be deeply rooted in biological, neurological, cognitive mechanisms, and is prone to emotional, historical and cultural constrains, linked to daily experience (Berthoz, 1997; Dehaene, 2000; Lakoff & Núñez, 2000). Therefore, "the portrait of mathematics has a human face" (Lakoff & Núñez, 2000), which means that human beings reason in a certain way because they are made in a certain way. *Embodied cognition* supports such believes by denying the mind-body split and considering these not as two separate aspects but rather as depending on each other. Roughly speaking, the activity of the mind has its roots in the activity of the body.

In this perspective, the following are crucial matters:

- How is mathematical knowledge structured in the students?
- How do students activate mathematical thinking?

In order to tackle these issues, I consider some problems performed by students of the same grade (the 9th grade), but attending different schools. These activities are modelling tasks within the context of introducing algebraic symbolism, functions and graphs at this school level. Starting from the idea that metaphorical thinking and imagery seem to play an essential role in the processes of constructing and communicating knowledge, the analysis pays attention to the passage from perception to theory. Particularly, the paper focuses on how metaphorical thinking and mental images appear in students' (oral and written) language and gestures, and more generally on how they affect the construction of mathematical meanings.

THEORETICAL FRAMEWORK

In the framework of cognitive science and linguistics, Lakoff & Núñez, investigating on where mathematics comes from, use *embodiment* (which means "*to put the body into the mind*"; Lakoff & Núñez, 2000) to explain the human mind's manifestations closely related to mathematical activity. They assert that our mind is deeply embodied: human concepts, especially mathematical concepts, are structured by the brain and by the nature of the body. Furthermore, abstract notions are organised through metaphorical thinking: they are conceptualised in concrete terms through precise inferential structures and ways of reasoning based on the sensory-motor system.

The *conceptual metaphors* and *image schemas* are elements of metaphorical thinking. The conceptual metaphors are fundamental cognitive mechanisms which allow to understand abstract concepts in terms of concrete concepts, i.e. deep nets of conceptual mappings that sistematically organise the concepts and preserve the inferences of the net structure. They "*project the inferential structure of a source domain onto a target domain*" (Núñez, 2000); these domains are ontologically different, but inferentially equal. Lakoff & Núñez distinguish various kinds of conceptual metaphors: they talk about *grounding metaphors*, *linking metaphors* and *redefinitional metaphors*. The grounding metaphors are the most interesting for my research, because they "*ground our understandings of mathematical ideas in terms of everyday experience*" (Núñez, 2000). Their target domain is mathematical, while their source domain lies outside mathematics.

The *image schemas* are (universal) topological and dynamic structures, which characterise spatial inferences and relate language to visual and motor experience, to perception and motion (Johnson, 1987). Their inferential structure is preserved under metaphorical mappings, like grounding metaphors. The very important feature is that "*image schemas have a special cognitive function: they are both perceptual and conceptual in nature. As such, they provide a bridge between language and reasoning on the one hand and vision on the other*" (Núñez, 2000).

My interests focus on a particular image schema linked to motion. This schema is part of the category of Source-Path-Goal Schemas (that is the schemas characterised by a starting point, the source of the motion, a trajectory, which represents the path of the motion, and a target, the intended goal of the motion). Lakoff and Núñez (2000), using the same words introduced by Talmy (1988), name this schema *fictive motion*. When one conceives of the static and mobile aspects of a continuous curve, one activates the so-called *fictive motion metaphor*. The definition given by Núñez is: "*a line is the motion of a traveler tracing that line*" (Núñez et al., 1999). Therefore, fictive motion is a metaphorical manifestation of a line, thought in terms of motion. As such, it linguistically appears in everyday language, for example in sentences like "the path *crosses* the woods", "the red line of the underground *goes* to the center", and so on. Using the same cognitive mechanism, in mathematics both students and mathematicians speak of "functions *going up*", "graphs *reaching* a maximum at a certain value", "two lines *meeting* at a point", etc. Fictive motion concerns a *trajector* (a dynamic entity) and a *landscape* (a static entity), in which the trajector moves. The trajector's motion produces a static line; using a terminology introduced by Sfard (1991), the line is an object, obtained through a process, linked to the movement, like that of a hand (the entity moving according to Eulero) or a pencil. As a consequence, a crucial issue is how a static situation can be conceived in a dynamic way (as an object and a process at the same time, that is as a procept, if we refer to the work of Gray & Tall, 1994).

I would like to add to metaphorical thinking an important dynamic, which was developed by Simon (1996) in his search for a sense of knowledge. Simon argues that students, while "doing mathematics", do not only use inductions and deductions. A natural inclination towards a third kind of reasoning, named *transformational reasoning,* seems to appear. Transformational reasoning is based on "*the ability to consider, non a static state, but a dynamic process by which a new state or a continuum of states are generated*" (Simon, 1996). Simon describes it as something which involves, at its core, seeing, at a mental level, one or more transformations of a mathematical situation and the results of those transformations. The process is constituted of three key ingredients: a mental enactment, a physical enactment, and an envisioning of the final state. The mental enactment results in a series of operations performed on mental images. According to Piaget & Inhelder, these images can have a *reproductive* or *anticipatory* nature (in Gruber & Vonèche, 1977). In the first case, they refer to a previous perception; in the second one, they precede transformations not previously perceived. Transformational reasoning is supported by such kind of images. However, it can also be characterised by a further physical enactment which leads to looking for a model to explore the final results of a transformation.This step requires an anticipation in mental imagery.

There is another important factor, which justifies the choice of analysing students' language and gestures. Some ongoing studies bear the importance of gestures in teaching and learning at all levels, because they are considered a deep and leading feature of cognitive development (Radford, 2000; Roth, in print; Roth & Lawless, 2002). Using gestures, in fact, students can already communicate without yet having the right words and express new levels of understanding before expressing the new understandings through language. Furthermore, pupils can refer to previous sensory-motor actions carried out in the activities they were engaged in (see also the proposal by Arzarello & Paola, accepted in Group 9 of this CERME). On the other hand, in his study about the relationships between body motion and graphing, Nemirovsky gathers that "*when one participates in a conversation, one does not distinguish a gesture as belonging to the body or to the mind*" (Nemirovsky et al., 1998).

Setting my research within this framework, my intention is to discuss analogous cognitive behaviours in students who are trying to solve different tasks (in the context mentioned above).

THE ACTIVITIES

the introduction of algebra is a crucial point in the teaching and learning of mathematics at the very beginning of secondary school. On the one hand, algebraic symbolism brings with it some cognitive difficulties and obstacles for the students used to thinking and reasoning in arithmetical terms. On the other hand, it really involves complex key concepts, like those of variable and function, which constitute the curriculum at all levels, already in the elementary and middle grades, and not only in the field of calculus. These concepts play a fundamental role when students face modelling situations or compare graphs of functions; therefore modelling can be a meaningful educational choice to approach these concepts (and graphs) from a semantic point of view. Instead, the epistemological choice lies in conceiving a function as the possible model of a (mathematical or extra-mathematical) phenomenon.

In this paper I consider three different modelling activities, each presented to an Italian class of 9^{th} grade students:

- the ‘Two Squares on segment’ problem, solved by 18 students of a technically oriented secondary school;
- the ‘Biggest Area’ problem, solved by 25 students of a scientifically oriented secondary school;
- the ‘L’ problem, solved by 20 students of a technically oriented secondary school.

Methodology

From the methodological point of view, these activities have some points in common. They were all part of long-term teaching experiments carried out during the whole school year. Each activity lasted two hours, in which the students worked in small groups (generally two to three pupils) and collaborating with each other: pupils learn in a social context, interacting with each other and sharing their understandings.

The usual routine of the classes consisted of: briefly reading the task, together with the teacher, in order to tackle problems or doubts; writing (in the classroom or at home) group or individual results; a final classroom discussion on the groups’ solutions guided by the teacher aimed to share the discoveries and to institutionalise the knowledge. Each group handled the problem using a technological device or tool, namely Derive in the first and third case, and a symbolic-graphic calculator (TI92 Plus) in the second case.

It is important to highlight that at the moment of the experiments the students had not acquired or constructed any formalised algebraic understandings yet. Moreover, the role of the teacher (besides guiding the discussion) was to help pupils in overcoming blocks, lacks, or troubles with the instruments. When possible, more people were present in the classroom to observe and videotape a group and the discussion.

The 'Two Squares on segment' problem

> *A segment AC is 20 cm long. Point B belongs to segment AC. Construct the squares ABGF and BCDE. Consider the perimeter of the figure constituted by these two squares [ACDEGF]. If the position of B changes, how does the perimeter change?*

A possible configuration of the task appears in figure 1 on the right (pupils had other two configurations on paper). The picture on the left shows the table given to the students to complete.

k distance AB	PERIMETER (k) perimeter of the figure
0	
1	
2	
3	
k	

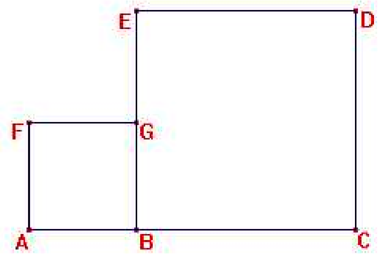

Figure 1

The function which models the situation is a piecewise linear function. It reaches a minimum value when k=10 and its graph is composed of two different segments which are symmetric with respect to this point.

This is a brief excerpt of a student, Dario. It is part of the written protocol (that he made himself at home after the classroom activity).

> *It has helped me a lot to write the data on the figure and the segment.* [he refers to figure2 drawn on paper]
>
> *Only this way, I found the correct function: P(k)=3k+[(20–k)·4–k]*

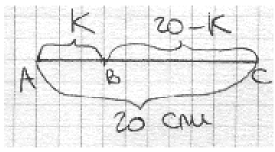

Figure 2

The function Dario found is not correct, because he only considers one part of the whole algebraic expression. However, the first important point lies in his use of the adverb 'only'. It emphasises that *only* when Dario introduces the symbol k on the segment AC, to indicate AB, he also understands that the remaining part, BC, can be denoted in terms of the same symbol (namely, as 20-k). This way, he is able to find an expression for the perimeter. The following observation of the teacher is crucial. When Dario thinks of the *static* point B *moving* on AC, he *moves his fingers* from left to right and vice versa some times, in order to show a series of shifts of B on the segment. The segment AC (the landscape) is thought of as generated by point B which, as a trajector of the segment, moves on it, tracing its length. Therefore, in

Dario's gestures, *fictive motion metaphor* appears and induces the use of the symbol *k*.

The teacher observes this cognitive behaviour, in similar shapes, in the whole class: most of the students do gestures to refer to the motion of B on the segment AC; supported by their hands, they think of the static point B enacting in this movement as the trajector which traces the length of AC. Activating fictive motion allows them to conceive *the static situation as if it possesses dynamic features*. This dynamic interpretation leads pupils to the need of finding a general rule, starting from what observed in a particular case (for example, if AB is 5 cm long then BC is 15 cm long or if AB is 8 cm long then BC is 12 cm long, and so on). As a consequence, there are both the introductions of *k* and (20-*k*) on the segment AC. However, the generalisation brings the students back to a static interpretation of the problem: it is represented by the expression of the perimeter in terms of *k* (Bazzini et al., in print).

The 'Biggest Area' problem

> *Among all rectangles with perimeter 16 cm, find that (or those) having maximum area.*

In this case, the teacher paid special attention to the reading of the text, analysing the following steps: finding the rectangle/s means knowing its/their base and height; it can be helpful to sketch examples satisfying the condition (perimeter of 16 cm); it is better to introduce the symbol *x* to indicate the base and express the height and the area in terms of the same letter *x*. There was no problem in accepting the possibility to use decimal numbers to measure the sides; on the contrary, a student proposed to draw on the blackboard the rectangle with base of 7.1 cm and height of 0.9 cm.

I will now consider a small excerpt of the initial classroom discussion.

Teacher: *Therefore how many are the rectangles having perimeter 16 cm?*

Filippo: *There are infinite rectangles.*

The first conjectures are aimed to find the rectangle solving the task. The square with side of 4 cm is identified as the limit case of all rectangles with perimeter 16 cm.

Alessio: *The more similar to a square, the bigger is the area.* [his hands are open and his fingers outline the shape of a rectangle; he moves his hands up, changing the dimensions of the rectangle until he gets a square; then he quickly puts them away and come back to the initial configuration]

Teacher: *What will be the height if the perimeter has to be 16?* [she draws on the blackboard a generic rectangle having base *x*]

Alessio's gestures are very important. Supported by the *movement* of his hands (back and forth), he imagines *subsequent transformations* of the initial rectangle and *consequent changes* in the value of its area (as pointed out by his words). Three consecutive steps can be identified. A mental enactment is embedded in the thinking of different rectangles having the same perimeter: this marks the first step. This precedes the physical enactment (the second step) which arises in the gestures and anticipates the envisioning of the final step (that consists of drawing a square). *Transformational reasoning* features the mental process. It supports the hypothesis that the square has the biggest area, even before finding the solution through the use of *x* and the related mathematical expressions (the use of the symbols is introduced by the teacher).

The 'L' problem

> *In a rectangle (whose sides are 4 and 3 cm long respectively), one of the longer sides is divided into two halves: the first one is raised and the second one is lowered simultaneously, of the same length. You obtain a polygon that resembles the letter L. Express the general perimeter of this polygon in terms of the length of raising/lowering.*

The task also enclosed other questions, as for example finding, if possible, a maximum or minimum for the raising/lowering and the perimeter (for further information, see: http://www.bdp.it/set/area1_esperienzescuole/cm131/5.htm).

During the classroom discussion, some doubts arise from the case of the maximum value for the shift (up and down). The teacher highlights the focus of the problem, asking: "*Is the final figure a rectangle with sides 2 and 6 cm long?*" or "*Is it necessary to add the (double?) segment, on which lowered half side is supported?*". These questions lead to the need of understanding (the teacher poses the attention on the following issues): "*Does it make sense to speak about the perimeter of a rectangle degenerated in a segment?*" and "*What is the perimeter of a figure?*". One of the students (Andrea) argues that: "*If an ant walks around the boundary of the figure, to come back to the starting point, it also should cover the segment twice*".

On the one side, the *walking* of the ant reveals the presence of the *fictive motion metaphor*. The ant represents the trajector which, moving along the figure (the landscape), covers its "perimeter". The rectangle (as a shape) can be thought of as generated, step by step, by the movement of the ant. This provides a dynamic interpretation of the static figure. On the other side, together with the fictive motion, Andrea shows the evidence of a *transformational way* to consider the relationships between the shifts and the L polygon. Andrea does not see the L as a static figure having particular dimensions. On the contrary he is able to think of the process that generates *subsequent L polygons*, while a half of the longer side (the base) goes down to the bottom and the other one goes up to the top. Therefore, Andrea knows that it is necessary to "go around" the L and to return back (to the starting point), in order to calculate the perimeter. This dynamic mental model allows Andrea to imagine the *consequent transformations* on the starting figure till the last configuration (corresponding to the maximum shift), as the limit case of the process. As a consequence, the ant has first to walk on the segment in one direction, and then to cover the same way in the opposite direction.

As a result, *transformational reasoning* and the *fictive motion metaphor* prompt the need for counting the segment twice in the calculation of the perimeter of the "extreme" L polygon.

CONCLUSIONS

The issues raised in the paper are preliminary reflections about similar cognitive behaviours of pupils dealing with different mathematical situations. As such, they are still open problems. To investigate more deeply the role of metaphorical thinking and mental imagery in the construction of mathematical knowledge, the following are crucial suggestions.

Can metaphors and imagery be mathematical thinking and acting tools? Can they enhance the teaching and learning of mathematics and allow students overcoming obstacles and difficulties?

From the point of view of Mathematics Education research, it is also important to reflect on how the role of technology can affect the creation and the use of metaphorical and imaginative thinking (see also the proposal by Robutti, accepted in Group 9 of this CERME).

To conclude, the fact that a mental anticipation (in imagery) comes before a physical enactment (through gestures) reminds us of the idea that perception precedes action (Berthoz, 1997). In this perspective, can the idea of Berthoz be a way of giving a neuro-biological explanation of the fact that metaphors and transformational reasoning are natural ways of reasoning?

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