SOME EXAMPLES OF THE RELATIONSHIP BETWEEN THE USE OF IMAGES AND METAPHORS AND THE PRODUCTION OF MEMORY IN THE TEACHING AND LEARNING OF MATHEMATICS

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Abstract: *In this paper, we give some examples of using metaphors and images in teaching and learning logarithmic and exponential functions at the end of high school level. A possible explanation for their use is based on using memory to construct a milieu to teach or learn mathematical concepts. This use of memory appears to initiate an ostensive didactical contract or to create a private milieu to understand concepts.*

I. Introduction

This paper is based on the theoretical framework of the didactics of mathematics. At present, this framework is built on thirty years of work, research and results based on theses, on journal publications (Recherches en Didactique des Mathématiques, three publications every year since 1980), regular attendance at ARDM seminars, regular work in Summer Schools (the 12th will take place in August 2003), and biannual meetings of 80 French and 40 foreign didacticians. In this paper, we use very few theoretical didactical terms, because they are not known yet in English language, in contrast to other countries using Romance languages. To satisfy the curious who would like to know more about this theoretical framework, it is possible to read, as a beginning, Brousseau 1998 or Chevallard 1985-1991.

For other reasons of language, we don't give a definition of "memory :" even in scientific publications, it is not so easy to understand what kind of meaning is used, because many scientific fields have taken memory as one of their research objects. To study memory, we only follow a recollection's phenomenology, as it has been proposed by Ricœur (2000), based on its relation with time. We also do not give more definitions of didactical categories of memory that we have built; to know them is not necessary in this paper (to know more *cf* Matheron 2000 & 2001).

In the article, we don't make any distinction between images and metaphors, considered as instruments. Both are considered in the same way: a process of communication, with others or with oneself, which consists in a modification of meaning realised by an analogic substitution or representation.

We consider some examples as representative of this kind of category. They come from audio- or videotaped lesson observations, during maths courses at the high-school level in the French educational system. In all cases, pupils are engaged in the highest level of scientific studies, at the end of high school, in which mathematics is one of the three most important subjects they study. Their school year finishes with an exam: the scientific A-levels called “baccalauréat scientifique”. The examples are extracted from the teaching of exponential and logarithmic functions.

II. Two examples during the teaching process

In the first example, the teacher shows how to solve two logarithmic equations: $(\ln x)^2+\ln x=2$ and $(\ln x^2)+\ln x=2$. To explain how, the teacher uses images referring to the bijective property of the logarithmic function. She says: “You have to write logarithm of *truc* (thingamajig) equals the logarithm of *machin* (whatchyamacallit)” and write: “ln of a square equals ln of a circle”, drawing on the blackboard a small square and circle in the equation.

The second example is about solving $\ln x^4+\ln x^2=0$. A few pupils propose wrong solutions, so the teacher explains the reason why: “What is the problem with what you have done? It’s the same thing when some one is working with square roots, and two years ago (in the level named *seconde* in France), you got an explanation about it. You wrote $\ln x^2$. It exists when x does not equal 0. You, you wrote $\ln x^2=2\ln x$; which is not always right. When is it not right ?” A pupil answers that x has to be different from 0.

The teacher continues : “If x is -3, it is wrong. It’s only when x is a strictly positive number. It is exactly the same as when you write [the teacher writes on the blackboard]

$$\sqrt{a^2} = a$$

It’s right only if “a” is a positive number. If a=-3, that is wrong! You have encountered these phenomena before. You are only allowed to write it if x is positive. All right? And that is the explanation of your mistake.…”

Continuing teaching, the teacher comes back using this metaphor to solve

$\ln(x^2-1)=\ln 2$, after having solved $\ln(x+1)+\ln(x-1)=\ln 2$: “The difference between the first and the second equation is their definition sets; they are no longer the same. This function also exists on… You see, it’s exactly like $\sqrt{(-2).(-3)}$. $x+1$ is negative and $x-1$ is negative, so every logarithm does not exist, even if the logarithm of the product of both exists.”

III. Analysis of the two examples: ostensive contract and public memory

It is quite easy to consider together the first two examples: in each case, the mathematical concept needed to understand the mathematical technique is missing in the mathematical knowledge coming from the didactic transposition. In the first example, the missing knowledge is an immediate consequence of the definition of bijectivity to solve the equation. In the second one, the missing concept is the concept of homomorphism from G-group to G'-group (the concept of group is also missing) ; and therefore the work demanded to get any x and any y in G, in order to write $f(xy)=f(x)f(y)$.

Brousseau (1996) named and described the technique used by the teacher during the two examples; it is included in an ostensive contract. Brousseau gives the following definition of the ostensive contract : "The teacher "shows" an object, or a property, the pupil agrees to "see" it as the representative of a class whose elements have to be recognized in other circumstances. […] It is implied that the object is the generic element of a class the pupil must imagine using play with some often-implicit variables. Therefore, the base of this contract is an empirical and realistic hypothesis whose two parts are in agreement. The teacher can communicate an element of knowledge by doing a way with action situations where it appears, by doing a way with its formulation and corresponding knowledge organisation.[…] Even if the ostensive contract is founded on a "wrong" epistemology, it is yet much used by teachers because it operates very well in most cases where a mathematical definition would be too difficult or useless."

In the first example, from the symbolism used by the teacher (square and circle) to the vocabulary which implies that *truc* and *machin* refer to a class of algebraic formulation, the teacher wants to show the most important steps of the technique for solving the equation. Using this ostensive didactic technique, the teacher discriminates and institutionalises these different steps.

From the institutional point of view, after the teacher has shown the right technique, knowledge for the classroom pupils must be homogeneous, and practices must be standardized. Doing that, the teacher carries out, in directing the community of the classroom, a work of past reconstruction which is a kind of memory work. This memory work is done by the teacher in the open, and so it does not guarantee that institutional and pupils' private relationships to knowledge will coincide. For pupils it requires, at least, personal work of past reconstruction, that is, work using their own memory for mathematical practices. So, using the ostensive contract, only the fiction of practices common to teacher and pupils, to the same class of mathematical objects is guaranteed. Even so, the natural function of institutionalisation phases is to indicate which practices must be learned and, consequently, which ones can or must be forgotten. In the case of the example, the institutionalisation phase is based, under the teacher's responsibility, on the creation of public memory; the didactic technique for this creation is based on using an ostensive didactical contract where images and metaphors take place.

The second example is about solving $\ln x^4+\ln x^2=0$, $\ln(x^2-1)=\ln 2$ and $\ln(x+1)+\ln(x-1)=\ln 2$; to be understood, the teacher uses the metaphor of

$\sqrt{(-2).(-3)}$ which exists, even if neither $\sqrt{-2}$ nor $\sqrt{-3}$ exist. In this case, what is offered by the metaphor is built on the analogy between two practices coming from the same mathematical concept, homomorphism, which is missing. Showing it, the teacher asks the pupils to remember the well-known practice used with square roots three years before; pupils are once more engaged in an ostensive contract. As we saw for the first example, the teacher builds a public memory for the institution in which he figures along with the pupils, but two differences appear.

The first concerns the means for the construction of memory. Contrary to the first example, the teacher asks pupils for recollections of a similar practice; he asks pupils to remember what we have called their practical memory (Matheron 2000 & 2001). Now they have to adapt to a new kind of situation. The second difference concerns the function of the constructed public memory. Before tackling this point, we must first give some didactical definitions.

Brousseau (1988) defines the *milieu*, "as the set of external conditions within which a human being behaves and grows". He specifies : "It plays an important role in the determination of the knowledge that the subject - its antagonist - must develop in order to control a situation for action." An important part of the work of devising didactical engineering in Didactical Situations Theory is to find a fundamental situation for teaching a mathematical concept, which will be the point of departure to create an antagonistic system for pupils. Bound by the didactical contract, pupils know they have to behave in a given situation by acting on it. Acting creates retro-actions and from this dialectical process pupils' knowledge is born. So teaching, in this theory, needs this antagonistic system which is named the milieu. Chevallard (1989), in an institutional problematic, explains that in ordinary teaching, during the temporal process, "institutional relationships to [some kinds] of [mathematical] objects are going […] to stop their own evolution, to become robust against external disturbances, to become "natural" and transparent for the institutional actors. […] These kinds of sub-systems of objects are going to assume, for the actors of the institution, the function of milieu. The latter appears to possess an objectivity that is not under the institution's control or intentions : we can say that this milieu is an "a-institutional" one. The actor's "play" with these objects will appear to him as a single player game, a game against "Nature", that only depends on intrinsic properties of Nature and on his own choices (and not coming from any agreement with Nature)".

Coming back to the second example, we can now tackle the second function of the public memory constructed by using the metaphor of the square root in this logarithmic equation. In modern teaching, which excludes the use of an *ex cathedra* exhibition of knowledge, new knowledge introduced in the didactical system by the teacher must provoke the pupils' personal activity allowing for its appropriation. So, the teacher has to guarantee the existence of this institutional milieu by suitably selecting knowledge and practices of knowledge that have been learned before. The teacher also has to show that the pupils can easily get this knowledge. So, to construct the institutional milieu, the teacher can use the ostensive contract: he can recall, showing by language (oral, written, gestural) and using images and metaphors, mathematical objects taught before, or revive them through practice. In this example, this revival is obtained using metaphor and by realising the dialectic between memory and institutional milieu. The pupils' entrance into the ostensive contract assures the possibility of forgetting, because the teacher's speech and the direction he gives to the study, evacuate personal relationships into their private fields, to conform to the new official relationship.

So, memory appears as a cognitive process implemented to construct the milieu, and conversely memory is (re)constructed from relationships to new and original object organizations taken for the milieu's construction. In this point, we find again the central thesis of the sociology of memory, expressed by Halbwachs (1925, 1994) and followed by Douglas (1987): collective memory is a continuing construction of the past to answer the needs of the present.

IV. Images and metaphors as techniques to create a milieu for action using memory

We are now able to understand a third example, where the teacher is using the metaphor of the logarithmic curve and the image of a turning straight line, to teach the asymptotic direction of the exponential curve. The logarithmic function has been studied, the teacher wants the pupils to find the asymptotic direction of the curve of the new exponential function. During interaction with pupils, the teacher uses the analogy with the curve of the logarithmic function. But, this analogy quickly reaches its own limits: as they know that the horizontal straight line gets a directory coefficient equal to zero, and consequently the logarithmic curve gets a horizontal asymptotic direction, the situation is not so easy when the limit of the ratio $f(x)/x$ is not a number! Once some pupils have found that the direction of the exponential curve will be the same as a vertical line, the teacher is obliged to explain:

Teacher : […] If there is a direction, what is the direction the curve rises into?

Pupil : Into a vertical asymptote

Teacher : Yes, it rises into a vertical direction. The logarithmic function rose into a horizontal direction, therefore $\ln x/x$ gets 0 as a limit. The exponential function rises with a vertical direction. So, what is the directory coefficient of a vertical straight line ? Il doesn't exist, but if you look at the line, if you make it turn and go up (do you see it, do you see the line ?) [The teacher gesticulates with her hands the imaginary movement of an imaginary straight line]… I consider the Cartesian system and a straight line, and I make it turn and rise; what happens for its directory coefficient ?

Pupils : It is increasing

Teacher : It is increasing towards +∞. We shall know that e^x/x goes to +∞ as the limit. It is dependent on this vertical direction. To prove that e^x/x goes to +∞ as the limit, we are going to compare functions, as we have done a short while ago with the comparison to x, we shall compare e^x not to x, but to x^2. So, you will have to work alone a little, because you must know how to do it, and we have just done it with e^x-x and it is not very different. So, try to compare e^x to x^2. Therefore, you study the sign of the difference…

A pupil : Oh, my god!

In this example, we can see the same didactical technique as in the two first examples, combining in a dialectic the pupils' memory, which is publicly spoken, and ostension to create an institutional milieu to teach the exponential curve. The last pupil's sentence seems to reveal that, at least for him, the problem's devolution is realised. It is the sign that he can identify a milieu in which he feels uncomfortable, maybe because he knows that the recollection with which it is built reminds him of old difficulties, or may be because the organization of recollection into the milieu appears problematic for him.

So, we can see a particularity of the kind of memory used in teaching and understanding mathematics. The chronological suite of mathematical and taught objects does not reproduce the objects exactly as they have been taught, but only the objects needed for our present practices can reappear. And the reason for their reappearance is not to be found in themselves, but only in the relations they bear to our questions, research, activities in solving present mathematical problems which require the construction of a milieu, and which are the consequences of the wish to teach, learn and study new mathematical objects.

A final example shows that using images and metaphors is not only a teaching practice, but that it is used in private mathematical practices. Four pupils, Alexandre, Aurélie, Sarah and Ludivine, are meeting with a didactical researcher, talking together about the last lesson, on the logarithmic function; at this time, the pupils have not been taught the exponential function, and the researcher certified that they didn't know any thing about it before. It is an audio recording. During the conversation, the researcher asks the pupils if they think the number e they met writing $\ln e=1$ has a future:

60. Alexandre : I don't know

61. Aurélie : Yes, sure !

62. Researcher : Yes, why ?

63. Alexandre : To solve the equation: when we have 3, we must translate in ln*e*...

64. Aurélie : hmm, hmm

65. Researcher : In fact, you use it in equations which…

66. Sarah : of course…

[…]

74. Researcher : Have you any idea about how the lesson can continue?

75. Aurélie : I think we go deeper into the study of *e*, because it's of great use

[…]

80. Aurélie : We use it when we do 1 equals,… in fact 2 equals $\ln e^2$, it's used. It allows to do, …, well, the opposite of ln

[…]

83. Researcher : What do you mean with "opposite of ln" ?

84. Aurélie : Well, if we come, from a number to find ln again, well I don't know… We have not done that but it seems to me very logical !…

[…]

96. Researcher : And what did you mean by "opposite", what did it mean ?

97. Aurélie : Well, it allows us to come back to ln

[…]

106. Aurélie : As "primitive" is the contrary of "derived", well, we find again… that *e* is the contrary of ln, well, something like that

We stop this transcription at this point; a deeper examination shows that, unlike the three others pupils, she is the only one who enters into a functional logic, while the others remain in an algebraic logic, with the number *e* (Matheron, 2000). She uses two metaphors to explain how to work with it : "the opposite", and "primitive" is the contrary of "derived". The concept shown by using them, the opposite f^{-1} of a bijective function *f*, is actually missing for her because it will be taught in the beginning of the next lesson: on precisely the exponential function. It is interesting to notice that, once again, the use of metaphors is needed to describe a mathematical practice, and that the objects of metaphors themselves are practices. Also in this example, using metaphors is directly connected to memory practice. We don't know if Aurélie used these metaphors before, on her own, to understand the functions of the number *e*, or if the use of metaphors has prompted by the researcher's questions. But, it really does not matter, because it always plays the same function: using metaphors is needed to create, with the help of memory, a milieu, in order to be understood by others (which is a kind of teaching) or to get retroactions, for personal use, to control one's own mathematical actions with a new object.

V. Conclusion

In all the examples, we have shown the same function for metaphors and images: how they help create a milieu. This does not mean that it is the only one. However, in these examples, using metaphors and images appears as a didactical technique to create, ostensively, a contract and a milieu. The need to resort to memory for the construction of the milieu explains the use of metaphors and images as easy. The use of metaphors and images simplifies access to memory for the construction of the milieu. The way is cleared to study the effectiveness of this technique, or the effectiveness of metaphors and images used specifically to teach a specific mathematical concept. It is from the point of view regarding the constructed milieu's efficiency for studying the different mathematical objects that this study can be conducted.

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