

THE NEED OF TEACHING THE LIMITS AND THE POSSIBILITIES OF THE REPRESENTATION SYSTEMS THAT OFFER THE SURROUNDING SUPPORT FOR COMPREHENDING THE CONCEPT OF FRACTION

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Abstract: It is known that rational numbers and fractions are subject to many different interpretations. Research results⁽¹⁾ lead us to affirm that a full understanding of all interpretations takes place at the formal operations stage. At this level, these operations are mental, and mental operations need surrounding support⁽²⁾. External representation systems offer this surrounding support. And since they are objects of our natural world, therefore, obey some specific structural constraints that establish their limits and their possibilities.

In this paper we will argue that, if we want our students to have the conceptual meaning to support the written symbols and the symbol manipulation rules for fractions, then, we have to teach them the utilization, limits and possibilities for each one of these representation systems along with their connection with the written symbols and the symbol manipulation rules.

Fractions “personalities”

When fractions and rational numbers are looked at from a pedagogical point of view, they take on numerous “personalities”⁽¹⁾ Kieren(1980)⁽⁷⁾ identifies 5 ideas as basic, namely

Part whole In this case some area, object, or dimension is split up in b parts and we take the a parts of them.

Quotient Here, a objects share equivalently in b groups. a could be smaller, equal, or greater than b .

Distance Here, the $1/b$ fraction is used repeatedly in order to determine an a/b distance.

Ratio This personality states a relation between two different objects.

Operator This personality shows the transformation of an object into another homogeneous to the original one.

Comprehending the fractions “personalities”

“...Kieren(1988) presents a theoretical model of rational number knowledge represented by four concentric rings. The inner ring consists of the basic knowledge that one acquires as a result of living in a particular environment... Moving outward, the next ring suggests a level of intuitive knowledge... The third ring represents technical symbolic language that involves the use of standard language, symbols and algorithms. The outer ring represents axiomatic knowledge of the system. An important observation about this model of rational number knowledge is that it is thought to be dynamic, organic, and interactive; that is, a mature rational number knower must be able to engage in the whole range of thought and action and interrelate thought and action at one level with thought and action at other levels...”⁽¹⁾ Thus, according to Kieren’s theory⁽⁸⁾, full comprehension of the fractions personalities seems to take place on the highest stage of Piaget’s theory, that is, on the formal operations stage.


Rational operations and visualization

At the above level mental, operations dominate. According to Piaget, these operations “...are actions for they are operating on objects before operating on symbols”⁽²⁾ Therefore, mental operations are created from the handling of objects, and they are preserved and developed as long as this handling continues. The importance of this seems to be due to the fact that the student will possibly develop operations systems, which are autonomous. During the whole developing process, and even until its maturity, the mental operations must have a surrounding support.

The surrounding support that mental operations need occurs with the help of information visualization where, “...Information visualization is the static or dynamic presentation of information in an external representation such that the information can be processed by efficient human visual mechanisms.... The information in a visualization resides in external representations...”⁽⁴⁾

From the above we conclude that the role of the surrounding support is dependent on the full comprehension of the concept of fractions. Thus the limits and the possibilities of the surrounding support must be clarified to both the students and the teachers, in order neither to underestimate the importance of the visualization, nor to consider it equivalent to the concept of fractions.

Research on school textbooks

One of the major representation systems used by school textbooks for students of ages 6-12 in most countries is system A. System A consists of all external representations of the form .

Research results show that system A:

1. Originally, it is used on German and Greek school textbooks for students of ages over 5 for the visualization of natural numbers as well as their operations where its utilization is absolutely consistent, and it does not lead to misconceptions.
2. It is also used on Greek school textbooks for students of ages 7-12, in the presentation of fractions. The percentages on the following table show how system A is distributed on the fractions personalities

Part-Whole	Measure	Quoti	Operator	Ratio
14	0	7	1	2

3. It is not specified how it can visualize:
 - a. fractions greater than one
 - b. operations with fractions

From the above we conclude that Greek students may have the risk facing visualization operations between fractions in the same way that they face visualization operations between natural numbers.

4. It is not used on German school textbooks for students of ages 11-12, in the presentation of fractions.

Students conceptions concerning representation system A

We found that:

- Greek students learn about fractions at the ages 7,8,9,10,11,12 and are familiar with the system A.
- German students learn about fractions at the ages 11,12 only and are not familiar with the system A.

Based on the above we realized the importance of carrying out a research in order to verify whether, with the use of representation system A, Greek and German students: a) can recognize Fractions < 1 , b) can construct Fractions > 1 , c) accept that the system

Age	11	12	13	14	15
Greece	33	76	65	44	17
Germany	22	25	17	20	28

A visualizes subtraction

The table shows how many students from each country responded to our questionnaire.

For the question

For each of the fractions that are on the right of the picture do the following. If the fraction shows the coloured part of the picture check "CORRECT". If the fraction does not show the coloured part check "WRONG" Otherwise check "I DON'T KNOW"

	$\frac{3}{4}$			$\frac{4}{3}$			$\frac{3}{7}$		
	CORRECT	WRONG	I DON'T KNOW	CORRECT	WRONG	I DON'T KNOW	CORRECT	WRONG	I DON'T KNOW
	WHY?			WHY?			WHY?		

The percentages were the following

AGE		wwc	??c	ccw	cc?	?cc	c??	w?c	cww	???	w??	cwc	c??	ccc	www	?wc	wcc	?cw	cw?	wcw
11	GERMANY	64	14						9	9					4					
	GREECE	70	6	9				3	3			6								3
12	GERMANY	52	12	4	8		4	4	8	4	4									
	GREECE	82	11	1		1				3		1	1							
13	GERMANY	88	12																	
	GREECE	82	8	3	2			2		1		1				1				
14	GERMANY	65		5				5	10	5				10						
	GREECE	82	7		2					3					2		2	2		
15	GERMANY	93												3	3					
	GREECE	94	6																	

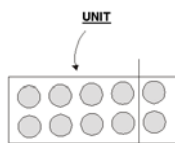
w=wrong, c=correct, ?=don't know. e.g "wc? Means 1st answer is wrong, 2nd is correct, 3rd don't know

The first column gives us the percentage of those students that the picture is

AGE		CATEGORIES					Explanation of categories
		a	b	c	d	e	
11	GERMANY	5	55	9	18	13	a=they say "not possible"
	GREECE	18	42	22	15	3	
12	GERMANY	4	52	12	24	8	b=no action
	GREECE	4	59	18	16	3	
13	GERMANY	6	65	18	11		c=
	GREECE	11	42	23	22	2	
14	GERMANY		85	5	10		d=
	GREECE	9	59	16	14	2	
15	GERMANY	18	39	8	21	14	e=other constructions
	GREECE		59	29	6	6	

not representing $\frac{3}{4}, \frac{4}{3}$ but represents $\frac{3}{7}$. Here we see how most students correspond representations of the A system to fractions < 1. The table on the left shows how students responded to the question, "Is it possible to color the picture so that the new picture may show the fraction $\frac{10}{8}$?" If

we take as a unit 10 balls then the construction is impossible. If we take as a unit 8



balls then is the correct construction. A small percentage (d) colored all

balls without justifying their answers. A small percentage (a) says with certainty that “it is not possible”. An important percentage (b) is not responding. Thus, it seems like this question is not accepted by the students either because it isn’t understandable or because it makes them feel uncomfortable. This may be due to the fact that students are not used to discussing about the limits and possibilities of the representation systems. Finally, categories c, d, e show that it is a considerable percentage (∪30%) that considers this construction is possible. Students may not realize how the A system can represent fractions greater than one. For the question:

For each picture check "CORRECT", "WRONG", "DON'T KNOW" if it corresponds to the relation on the left

$\frac{3}{4} - \frac{2}{3}$									
	CORRECT	WRONG	DON'T KNOW	CORRECT	WRONG	DON'T KNOW	CORRECT	WRONG	DON'T KNOW
	WHY?			WHY?			WHY?		

The percentages were the following:

AGE	11		12		13		14		15	
	GERM	GRE	GERM	GRE	GERM	GRE	GERM	GRE	GERM	GRE
THE PICTURE CORRESPONDS TO THE RELATION	27	52	20	36	35	37	20	34	10	53
THE PICTURE DOES NOT CORRESPONDS TO THE RELATION	73	37	44	44	65	49	70	57	86	23
I DON'T KNOW		11	36	20		14	10	9	4	24

The 1st column of the above table shows that a considerable percentage of German students and a small percentage of Greek students gave correct answers.

If we isolate the answers concerning whether the picture corresponds to $\frac{3}{4} - \frac{2}{3}$ then we will have the following table.

The above table shows us that Greek students have become familiar with the system A even though they do not understand in depth its limits and possibilities. On the other hand, German students—in a larger percentage than Greeks—gave correct answers although their school textbooks did not include such system. The answers that Greek students gave do not depend on their age (e.g.37-44-49-57-23). On the contrary, German students’ answers depend on their age(73-44-65-70-86). Some of the students who gave correct answers justified their answers (e.g $\frac{3}{4} - \frac{2}{3} = \frac{1}{12} \neq 1$). From

the table on the right, which gives their percentage, we see that Germans are more critical than Greeks in their judgments concerning operations on system A.

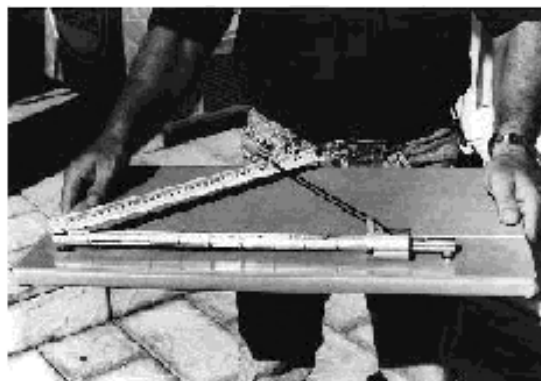
AGE	11	12	13	14	15
Germany	63	55	27	86	75
Greece	28	12	16	24	27

Conclusions

External representations are objects of our natural world, since at least we see them, and therefore, they obey certain structural constraints which rule their limits and possibilities. The variance in inferential potential of these representations is largely attributable to the different ways in which these structural constraints match with the constraints on targets of these representations⁽⁵⁾. Currently, to teach concepts of rational numbers, traditional representation systems are utilized; some of them are self-inconsistent⁽⁹⁾, some of them are over-specific⁽¹⁰⁾ and some constitute mathematical models of the field of rational numbers⁽⁶⁾. For each of those systems the utilization, limits, and possibilities must be taught, along with their connection with the written symbols and symbol manipulation rules. This way students will have conceptual meaning to support written symbols and symbol manipulation rules.

References

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9. A non consistent representation system is the musical representation system.

This system is capable of producing contradictory situations. For example, the picture on the left shows us that on this system $\frac{3}{4} \neq \frac{6}{8}$

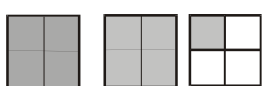
Die Summe der Notenwerte gibt einen Hinweis auf den Takt. Es ist $\frac{3}{4} = \frac{6}{8}$, aber in der Musik wird zwischen einem $\frac{3}{4}$ -Takt und einem $\frac{6}{8}$ -Takt unterschieden. Ein Unterschied liegt in der Anzahl der Grundschläge. Taktarten mit einer geraden Anzahl von Grundschlägen nennt man **gerade Taktarten**, solche mit einer ungeraden Anzahl von Grundschlägen **ungerade Taktarten**. Der $\frac{3}{4}$ -Takt ist ein ungerader Takt, der $\frac{6}{8}$ -Takt ist ein gerader Takt.

Beispiel für einen $\frac{3}{4}$ -Takt:

> gibt den Grundschlag an.

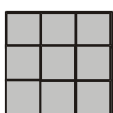
Beispiel für einen $\frac{6}{8}$ -Takt:

10. Any representation system, which uses two or three-dimensional figures, is over-specific since it is capable of producing multiple representations of a problem's solution. For example on the left hand figure there are some of the possible representations of the problem's



(The side of each rectangle = x)

"Find $\frac{3}{4}$ of the rectangle of side x" solution.



(The side of the rectangle = $\frac{3}{2}$)



(Base = 3 & altitude = $\frac{3}{2}$)