

PRE-SERVICE ELEMENTARY TEACHERS AND THE FUNDAMENTAL AMBIGUITY OF DIAGRAMS IN GEOMETRY PROBLEM-SOLVING

Bernard Parzysz

IUFM Orléans-Tours & Equipe DIDIREM (université Paris-7)

Abstract: In elementary geometry problem-solving, in most cases two different points of view on diagrams are in interaction : as physical objects and as images of theoretical objects. An ongoing research is studying if, and how, pre-service elementary teachers are conscious of this double status. It appears that most of them do not have a clear view of it, which leads to a number of confusions and may prevent them helping their pupils in an effective way in their learning of geometry.

Three years ago, we started a research at the IUFM¹ Orléans-Tours about the initial training of preservice elementary teachers in geometry, after having observed that, like many secondary students, they had difficulties in that area of mathematics, the crucial point seeming to be how they consider the diagrams used in geometry problem-solving [Parzysz & Jore 2002].

1- THEORETICAL FRAMEWORK AND RESEARCH HYPOTHESES

Many authors have written on students' cognitive evolution in elementary geometry, beginning with P. M. & D. van Hiele [van Hiele & van Hiele-Geldof 1958; van Hiele 1986]. More recently, Fischbein developed the idea of *figural concept* [Fischbein 1993] and Lakoff & Nunez that of *embedded cognition* [Lakoff & Nunez 2000]; Monaghan discussed the way in which students perceive various quadrilaterals, and particularly the connection between the perception and description of a given geometrical figure [Monaghan 2000]. The study of such a connection appears indeed to be fundamental for researchers, because it is symptomatic of how a given individual deals with geometry, as I hope to show in what follows with students wishing to become elementary teachers. I shall begin by giving indications² on our specific theoretical framework.

Following other researchers [Houdement & Kuzniak 1998], our basic postulate is the –mostly unconscious- co-existence of two paradigms within these students³, as well as within secondary students :

- *spatio-graphic* geometry (G1), in which the objects in play are physical (models, diagrams, computer images...) and the proofs are of a perceptive nature (eyesight, comparison, measure...)

¹ Institut Universitaire de Formation des Maîtres (Academic Institute for Teachers' Training).

² More details can be found in [Parzysz 2001].

³ Who have already spent three years at university, but mostly in non-scientific curricula.

- *pre-axiomatic* geometry (G2), in which the objects in play are theoretical (their existence proceeds from axioms and definitions) and the proofs are theoretical as well (moving in fact from perceptive to hypothetico-deductive as the students' geometrical knowledge grows up)⁴.

Indeed, an “expert” in elementary geometry uses a diagram alternately in two ways when solving a geometry problem, and his/her cognitive path is made of a series of moves between G1 and G2. Namely, he or she :

- uses the wording (G2) to draw a diagram
- considering this diagram as a physical object (G1), uses it to make conjectures by mere eyesight, or by using instruments (ruler, compasses, etc.) to make a better guess;
- uses the data of the wordings to prove this guess (G2)
- (if necessary) comes back to the diagram to select useful elements for the proof (G1 _ G2)
- (eventually) uses the diagram to check some results (G2 _ G1)
- and so on.

This process shows that the diagram is a crucial element in the process of solving a geometry problem : it is considered both as a physical object on which physical operations can be made and as an image of a theoretical object on which thought operations only can be performed. This fundamental duality can be related to van Hiele's visual and descriptive levels: “*on the lower level, the visual level, shapes are recognised by seeing: ‘This is a square because I see that it is one’.* *On the higher level a shape is recognised by its properties: ‘This is an isosceles triangle because it has three sides and two of them are equal’.* *This level I have called the descriptive level.*” [van Hiele 2002, p. 30]

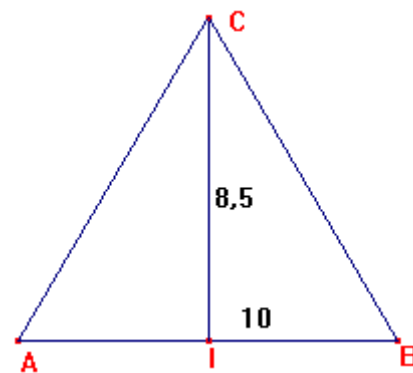
Such an ambiguous status can possibly trouble the students who begin to study geometry, and there is a risk that they may get mixed up.

Now, if we consider how geometry is taught, we can observe that, at primary school, students construct diagrams, using specific instruments, and they learn geometrical properties through the study of these diagrams : for instance, they learn that a rectangle triangle is a triangle which has a right angle and later on they can assert that a triangle is rectangle from the mere use of their straightedge, such a statement being accepted by the teacher (G1). But some years later it will no longer be accepted, and they will be asked to produce another kind of proof(G2) which, however, will also be based –as seen above- on a study of the diagram. The reason is that a major aim of the teaching of geometry through elementary and secondary schools is to help students to move from a G1 to a G2 point of view, since geometry appears to be –at least in France- a favourite place to put hypothetico-deductive proofs into play.

⁴ The term “pre-axiomatic” means that this paradigm appears in fact as a simplified, vulgarized version of a complete axiomatic theory (G3), such as those which have been built by Hilbert.

Daily practice led us to guess that the situation could be the same for pre-service elementary teachers as for middle school students, since most of them did not seem to take the metaphorical dimension of geometry diagrams into account when solving problems. We thought that such a situation was not suitable for elementary teachers, since they are supposed to accompany their pupils' first steps on their way from G1 towards G2. Indeed, if the teacher does not have a clear view of the process of problem solving in elementary geometry, one can imagine that it will be difficult for him/her to help the students.

Let us imagine for instance that a pupil finds experimentally that, starting from a 10 cm long segment $[AB]$ and drawing a 8,5 cm long perpendicular line $[IC]$ from its middle I , he/she gets an equilateral triangle. Indeed, this triangle looks fine.



Let us imagine now that, in the same class, another student has used the same process, but with 15 cm and 13 cm respectively. His triangle looks equilateral as well. However both triangles cannot be equilateral, since the ratio $8,5/10$ is different from $13/15$, so who is right and who is wrong? This should puzzle the teacher, and make him/her feel uneasy if he/she cannot answer this question. It is only by shifting from G1 to G2 that he/she will be able to overcome this unpleasant feeling ... and find that both constructions are only approximations: within G2, triangle ABC is indeed isosceles, but not equilateral, since (from Pythagoras' theorem) in the first case AC is longer than AB, while in the second case it is shorter.

As described above, when an 'expert' solves an elementary geometry problem he or she has, in most cases, to shift several times from G2 to G1 and *vice versa*. What enables him/her to perform the task accurately is that he/she always knows if he/she is working within G1 or within G2 and, according to this, he/she uses diagrams in an adequate way. We think it most urgent to help pre-service teachers to become somehow 'experts' in that domain. This does not mean that we want them to be able to solve difficult problems within G2 -even if that could be very useful to them-, but to know at any moment 'where they stand', and especially to distinguish between diagram as a physical object and diagram as an image of a geometrical concept.

2- EXPERIMENTAL SITUATION

Even before thinking of implementing changes in the students' syllabus, we needed to ascertain our initial intuition about the relation of preservice elementary teachers to geometry diagrams. For that purpose, we imagined an experimental problem-solving situation and put it into play in five groups of pre-service elementary teachers, each

group being composed of 20 to 25 students. This situation was part of their geometry course.

Each student was given a sheet of paper with the wording of the task, such as :

Draw a straight line, d . Let O be a point of this line.

Draw a circle C_1 with O as its center and 2 as its radius length. This circle intersects d in two points A and B .

Draw a circle C_2 with O as its center and 3,5 as its radius length.

Draw a circle C_3 with A as its center and 4 as its radius length. This circle intersects C_2 in two points C and D .

What means can you put into play to know whether the line (CD) is the perpendicular line of $[AB]$ or not?

The students worked at first in teams of 4, each student in a given team having a sheet with the same wording, but with different numerical data : for two of them, the radii r_1, r_2, r_3 of C_1, C_2, C_3 derived from Pythagorean triplets or PT (*i.e.* $r_1^2 + r_2^2 = r_3^2$), and for the other two, from pseudo-Pythagorean triplets or PPT (*i.e.* $r_1^2 + r_2^2 = r_3^2 \pm 1$) :

version	values of radii	corresponding triplet	nature of triplet
A	1 1 1,5	(2, 2, 3)	PPT
B	2,5 6 6,5	(5, 12, 13)	PT
C	2 4 4,5	(4, 8, 9)	PPT
D	4 7,5 8,5	(8, 15, 17)	PT

(*N.B. : the example of wording given above refers to the PPT (4, 7, 8), which was not used in the experiment.*)

Of course the students were not aware of this feature, which was designed in order to cause them to question perceptive evidence, since in all cases the figure looked the same:

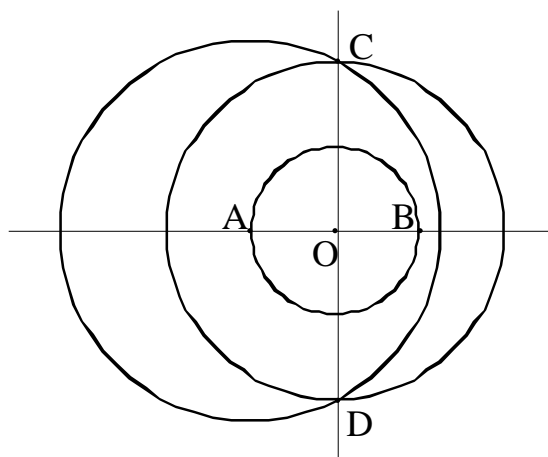


Diagram corresponding to PPT (4, 7, 8)

Comments :

- 1) The unit of length is not given, in order to enable the students to change the size of the diagram.
- 2) (Within G2, for an 'expert') According to the symmetry of the construction, (CD) is in each case perpendicular to (AB). The only question is then to know whether point O belongs to (CD) or not.
- 3) The wording makes no allusion to a precise paradigm (G1 or G2) : students are only asked to "put means into play".
- 4) Moreover, they were not asked to solve the problem in any way, but to think of means for that purpose.

The students were asked to work on their own and give their answer on their sheet. Then they were asked to produce a single answer for the group, which would be written down on a poster. Such a task is unusual for the students, since they were asked, not to give the answer, but to indicate means of reaching the answer. By doing so we wanted to see :

- what kinds of proofs they could think of (belonging to G1 ? to G2 ?)
- whether discussion led them to a conflict about what was a 'valid' proof.

After the posters were made, all of them were pinned up on a wall and a general discussion took place.

3- RESULTS AND DISCUSSION

We finally got 31 posters in all (5 to 7 by group). A thorough analysis of these productions cannot be made here, but we can nevertheless highlight some results.

(i) We could at first distinguish two main types of means proposed by the students:

- those belonging to G1 (e.g. drawing the perpendicular line of [AB] and observing its coincidence with (CD))

Example of the G1 category:

Definition of the perpendicular bisector:

It is a line which cuts a segment perpendicularly through its midpoint.

Thus, any point situated on the perpendicular bisector of a segment [AB] is the same distance from A and B.

Means:

* check that (CD) goes through the midpoint of [AB] :

- O midpoint of [AB]. Does (CD) goes through O?
- with compasses or graduated ruler : $CA = CB?$ or $DA = DB?$

* check that $(CD) \perp [AB]$:

- with set square

* check that (CD) is the perpendicular bisector of (AB) :

- construct the perpendicular bisector of $[AB]$, if it coincides with (CD) , then (CD) is the perpendicular bisector of $[AB]$

- construction of the rhombus $ACB'D$. $B = B'$?

[Although the vocabulary is taken from G2, all the means proposed belong clearly to G1. The use of 'Thus' (line 3) shows that the students are not 'experts' in G2.]

- those belonging to G2 (e.g. making use of Pythagoras' direct and reciprocal theorems).

Example of the G2 category:

The perpendicular bisector of the segment $[AB]$ cuts it perpendicularly through its midpoint, O

If the triangle AOC has a right angle in O, then the line (CD) is the perpendicular bisector of $[AB]$. One applies the reciprocal theorem of Pythagoras' theorem :

If $AC^2 = CO^2 + OA^2$, then the triangle AOC has a right angle in O.

Same thing with the triangle AOD.

If both triangles AOC and AOD have a right angle in O, then C, O, D stand on a straight line and (CD) is the perpendicular bisector of $[AB]$.

[This is a 'classical' proof within G2.]

(ii) Some posters proposed several means: if all of them were of the same kind, we classified the poster in the corresponding category; if some of them were G1 and some others were G2, we looked for signs indicating that a distinction was made between them; when this was the case we classified the poster as G2, and otherwise we classed it in a distinct category (G1-G2).

Example of the G1-G2 category:

Arithmetical method :

Perpendicular bisector: goes through the midpoint of $[AB]$ and perpendicular

* Supposing $CD \perp AB$, and O midpoint of $[CD]$.

Thus if $OB \perp OC$, from Pythagoras in triangle BOC right-angled in O we have :

$$BC^2 = OB^2 + OC^2$$

$$(4,5)^2 = 4^2 + 2^2$$

20,25 ? 20 * Thus triangle BOC is not right-angled in O and

* BO is not perpendicular to OC

Thus (CD) is not the perpendicular bisector of [AB]

Geometrical method :

* Perpendicular bisector : any point on the perpendicular bisector is equidistant to [AB]

Verification with compasses $AC = BC$

* From the figure by drawing the perpendicular bisector of [AB] going through O one can observe that (CD) and the perpendicular bisector are not identical.

[The so-called 'arithmetical' method is situated in G2 : it proves by reducing to the absurd, using the C version of the wording. The 'geometrical' method is clearly situated in G1. In fact, 'arithmetical' seems to refer to calculation, whereas 'geometrical' seems linked to the use of instruments. Both 'methods' appear to be on an equal footing.]

(iii) To end with, we met with a fourth category : a proof was given, which was clearly in G2 ; but, at some moment, an implicit slip towards G1 could be observed because a perceptive feature of the figure was taken as a datum, which it was not (e.g. O was said to be on (CD)). We named this category CKS (*contamination of 'knowing' by 'seeing'*, see [Parzys 1988]).

Example of the CKS category :

C_1 is the circle whose centre is O : all the points of the circle are the same distance from point O.

Thus: * $OA = OB$

* O midpoint of [AB]

Since A and B \in to (d), points A, O, B are on the same line.

- C and D are on circles C_2 et C_3 , whose respective centres are O et A.

Thus: * $OC = OD$, O midpoint of [CD]

* $AC = AD$.

- triangle ACD is isosceles in A because $AC = AD$. The half-line [Ad) starting from A cuts [CD] in O. Since O is the midpoint of [CD], [AO] is the perpendicular height and perpendicular bisector of [CD].

- We can deduce that [AB] and [CD] are perpendicular in O. Thus [CD] perpendicular bisector of [AB].

[This proof in four points shows indeed that these students can be considered as 'experts' in G2: vocabulary and symbolism are used correctly (including the half-line), the argument is developed and exposed in a clear way, the assertions are

justified, with the exception of just one (the fact that C, O, D are on the same line): unfortunately, CKS has pulled down the carefully built intellectual construction.]

Finally, the overall distribution was as follows :

<i>category</i>	G1	G2	G1-G2	CKS
<i>number</i>	8	10	6	7

Discussion : This table shows that an important number of students do not seem to make any difference between the two geometrical paradigms. We could learn more from the general debate which took place after the exposition of and comments on the posters : we could then see that many students did indeed put perceptive and theoretical proofs on the same level. More precisely, they used their G2 knowledge (definitions, theorems) in their argumentation, together with G1 results (measures). Moreover, when trying to produce a formal proof, they are not always aware of introducing perceptual observations.

What can be said on the whole is that the students show a consistent knowledge about G2, as can be inferred from the vocabulary, the definitions and theorems they use, but their behaviour is still far from being that of 'experts' in elementary geometry, as this term has been defined above.

4- CONCLUSION

I quite agree with Godino & Recio when they state that *"it is necessary to somehow articulate the different meanings of proof, at different teaching levels, thereby developing progressively among students the knowledge, discriminative capacity and rationality required to apply them in each case"* [Godino & Recio 1997, 319]. And, as van Hiele says: *"if a teacher wants something better than instrumental thinking he will have to take account of the difference between the two levels"* [van Hiele 2002, 30]. These are main reasons why elementary school teachers must be aware of the various paradigms within which their pupils will have to work (now and later on), and be able to know and distinguish between the various kinds of proofs which can be given to validate a given assertion. As our results show, this is not quite the case yet.

Our team is currently working on another problem-solving situation which will not make use of a 'paper + pencil' environment, but of a 'software' one (Cabri-geometry). There are indeed some major differences between the two environments, since the feedback is very different in each case ; that is the reason why we now want to study if -and how- the dynamic aspect of the software may enable our students to become aware of the G1 / G2 distinction, even if other problems arise ([Laborde 1995], [Love 1996]). We could then elaborate and develop a didactic engineering -including real life, paper-pencil and software environments- designed to help them to make their future teaching more effective.

REFERENCES

- FISCHBEIN, E. (1993): The theory of figural concept, in *Educational Studies in Mathematics* 24, 139-162
- GODINO J. D. & RECIO A. M. (1997): Meaning of proof in mathematics education. *Proceedings of the 21th International Conference for the Psychology of Education*, II: 313-320
- HOUEMENT C. & KUZNIAK A. (1998): Géométrie et paradigmes géométriques. *Petit x*, 51: 5-21
- LABORDE C. (1995): Designing tasks for learning geometry in a computer based environment. The case of Cabri-géomètre. *Technology in Mathematics teaching*: 39-52. L. Burton & B. Jaworski, eds. Chartwell-Bratt, London
- LAKOFF, G. & NUNEZ, R. (2000): *Where mathematics comes from: how the embodied mind brings mathematics into being*. Basic Books, New York
- LOVE E. (1996): Letting go: an approach to geometry problem solving. *Proceedings of the 20th International Conference for the Psychology of Education*, III: 281-288
- MONAGHAN, F. (2000): What difference does it make ? Children's views of the differences between some quadrilaterals, in *Educational Studies in Mathematics* 42 : 179-196
- PARZYSZ, B. (1988): 'Knowing' vs 'seeing'. Problems of the plane representation of space geometry figures, in *Educational Studies in Mathematics* 19: 79-92
- PARZYSZ, B. (in press): Articulation entre perception et déduction dans une démarche géométrique en PE1, in *Actes du colloque COPIRELEM de Tours*
- PARZYSZ, B. & JORE, F. (2002): What is geometry for French pre-service elementary schoolteachers ?, in *Proceedings of the 26th PME Conference* I: 218
- van HIELE, P.M. (1986): *Structure and insight*. NY Academy Press
- van HIELE, P.M. (2002): Similarities and differences between the theory of learning and teaching of Richard Skemp and the van Hiele levels of thinking, in *Intelligence, learning and understanding in mathematics. A tribute to Richard Skemp* (D. Tall and M. Thomas, eds). Post Pressed, Flaxton (Australia)
- van HIELE, P.M. & van HIELE-GELDOF, D. (1958): A method of initiation in geometry at secondary schools, in *Report on methods of initiation into geometry*, 67-80. H. Freudenthal (ed.). J.B. Wolters, Groningen