FUNCTIONS OF NATURAL LANGUAGE IN THE RESOLUTION OF A PLANE GEOMETRY PROBLEM

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Abstract: The study reported in this paper concerns the development of a research work presented at CERME 2. Our aim is to analyse a classic situation of problem solving in the domain of plane geometry: what is the role of natural language in the evolution of problem solving? The analysis is focused on the dialectic relationships between graphical domain and theoretical domain, which are the main domains involved in geometrical problem solving. In this paper we try to answer following question: how and under which conditions does natural language act as a mediator between graphical domain and theoretical domain, in order to promote the evolution of problem solving?

Introduction

The early results of our research showed that natural language can play an important role in geometrical problem solving. In particular, we have analysed the influence of natural language on the relationship between the operational handling of drawing[1] and the theoretical reference to which it is related. In a given geometric theory, we define "theoretical reference" theorems, definitions and relations of that theory, which are related to the figure by the student who is solving the problem. Since solving a plane geometry problem involves reciprocal relationships between drawing and theory, we note a two-way relationship, mediated by the natural language, between the handling of the drawing and the choosing of a particular theoretical reference: choosing a particular theoretical reference leads to the operational handling of the drawing and vice versa, the operational handling of drawing can suggest choosing of a particular theoretical reference.

The aim of this paper is present different functions that natural language fulfils in these relationships between drawing and theory. In particular, we will describe how the functions of natural language allow the construction and management of deductive thinking in a situation of geometrical problem solving and the conditions under which this is accomplished.

1. The research hypothesis and research questions

[1] We consider the "operational handling" of a drawing in sense of Duval's theory (Duval, 1994): operational handling of drawing ("*appréhension opératoire*") involves an immediate perception of the drawing and its different variations ("mereologiques", optical or of position). A variation is called: "*mereologique*" when it divides the drawing into parts; *optical* if it is an enlargement or a reduction of the drawing *positional* when the figure background changes position

In our research, natural language plays two different roles: it is a tool for the researcher, because it can reveal students' cognitive process and, at the same time, it is a tool for the students when they use it to develop problem solving. Taking care of this idea we have formulated the following general hypothesis:

a) Natural language is a revealer of the students' cognitive process (in this sense it is a tool for the researcher)

b) Natural language is a tool for the construction and the management of thinking in geometrical problem solving (in this sense it is a tool for the students who are solving the problem)

Hypothesis a) and b) bring in particular to the following questions, related to an experimental situation of "dyadic" problem solving (i.e. cooperative problem solving by a pair of students):

a') What kind of linguistic tools allow us to point out how the verbal exchanges among students support the evolution of their problem solving?

b') In which way verbalization promotes the passage from the simple remarks about the drawing to the construction of a deductive thinking?

In order to test the above hypotheses and to answer the related questions, we need both a theoretical framework and a research methodology to perform the investigation.

2. The theoretical framework

Since natural language plays a double role, we refer to Duval and Bronckart's theories, in order to support the idea of natural language as a tool for research, and we refer to Vygotsky and Bachtin's theories, in order to support the idea of natural language as a tool for the students who are solving the problem.

Duval defines two modes of progression of discourse: accumulation and substitution. Taking into account these Duval's definitions we consider:

- *Accumulation,* the progression of discourse made by juxtaposition of independent propositions concerning geometrical information. Duval assigns a value to propositions of discourse. In the accumulation mode, the value of the propositions is exclusively depending to their content; for example, the content appears as possible, probable, certain….

- *Substitution,* the progression of discourse made by means of non-changeable order of propositions: for instance, the conclusion of a deductive step becomes the premise of the next step. In the substitution mode, the value of the propositions is a logical value: true, false, indeterminate value.

Bronckart defines the concept of "linguistic unities". They are words, connectors as " if…so", verbal tens; their occurrences in a discourse allow to identify a particular kind of discourse. Taking care of this idea, we have identified some main linguistic unities characterizing the discourse product during a geometrical problem solving. Then, we have characterized their particular use. Finally, we have associated their different uses to the "accumulation" and "substitution" modes of progression of discourse. In this way, the occurrences of the linguistic unities and their uses allow us to identify the progression of discourse (from accumulation mode to substitution mode)

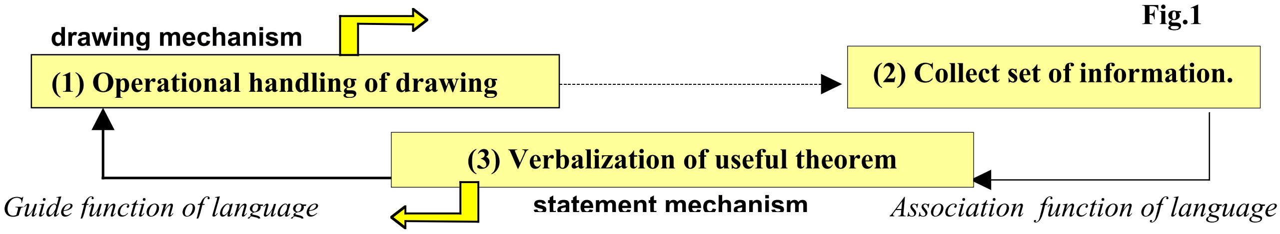
We consider Vygotsky and Bachtin's definition of "function of language" in order to support the idea of language as a tool for the construction and management of thinking. They consider language as a tool that can help to think. In particular, human thinking develops in interpersonal interactions (that need language as a communication tool). In others words, the language is essential element for the human activity of thinking. Taking care of these two theories we have developed our study about functions of natural language as problem solving tools.

3. Research methodology

Basically, we tried to elaborate a methodology that should be suitable for taking into account the fact that language is a tool for the researcher and, at the same time, it is a tool for students. For this reason, we elaborated models for analysing protocols, based on the use of language as a revealer (see modes of progression of discourse and linguistic unities), which allowed us to point out the role of language as a problem-solving tool for the students. The models are based on the assumption that problem solving is mainly expressed through two registers: the linguistic and the figural. Therefore, we distinguish between two kinds of problem solving: one moving from the drawing (after checking the text of problem), and the other moving from the question posed in the text of problem or from the sub-questions obtained by transforming that question. We named these two kinds of problem solving "drawing mechanism" and "statement mechanism" respectively. The criteria adopted in order to elaborate these models have been mainly two: to put the relations between graphical and theoretical domains into evidence and, at the same time, to analyse the role of natural language in these relations. Mechanisms are cyclic models; they can be performed several times in order to describe the whole solving process.

Principal phases of these mechanisms are the following.

Fig.1



(1) The operational handling of drawing (from which the "drawing mechanism" starts) involves an immediate perception of drawing and its different manipulations in order to isolate same figural unities or same sub-configurations of the drawing. In this phase, the figural unities isolated on the drawing are described in spatio-graphical terms by means of deictic words, used instead of names (for example: "This one…that one…"). The aim of the operational handling of drawing is to construct a "work environment" by means of a list of information **(2)**[2]. In this phase (2), the figural unities are described in geometrical terms because they are recognized as mathematical objects (for example in Fig.2, the segments "AE" and "OD" are named and identified as *diagonals* of the quadrilateral OADE). Moreover, the figural unities can be composed in sub-configurations described in geometrical terms. The propositions haven't a status of premises or of conclusions because they are simply juxtaposed. They have a semantic value. For this reason we recognize an "accumulation mode" of progression of discourse. This list of information allows the students associating -and then verbalizing **(3)** - the useful theorem in order to solve the problem (from this verbalization the "statement mechanism" starts). Moreover, in the passage [(2)→(3)] we consider the action of the *association function of language* that will be described in the following. The verbalization of the theorem (phase 3) allows to associate the status of premises and of conclusion to the propositions of the statement of the theorem and it allows to associate to the propositions of the list the status of potential premises. This fact allows the students to compare the two sets of premises (those of the theorem and potential premises) because the propositions have the same status, and it allows to identify the premises that have to be proved. The students came back to the operational handling of drawing **(1)** searching for the only geometrical relations inherent in the premises that have to be proved. For this reason we consider that now, the operational handling of drawing is oriented towards the verbalization of theorem. The value of the propositions-premises is not determinate until they are not proved. In this phase it can appear the "substitution mode" of discourse because the propositions can be linked in the deductive way. In the passage [(3)→(1)] we consider the action of the *guide function of language.* This function is carried out by the verbalization of the theorem, which allows isolating the premises to be proved. The guide function of natural language will be described in the following.

In this paper, we will concentrate mainly on describing "drawing mechanism" and showing how we used it in order to analyse the students' protocols[3].

4. Experimental situation and the analysis of a students' protocol

In order to develop a study about the functions of natural language I considered a situation of communication between two students who were solving a plane geometrical problem. Communication is necessary to address to an interlocutor, but, at the same time, just because we address ourselves to an interlocutor, language can

[2] How do students get the information (theoretical references, geometrical relations, properties, etc) for making their "list of information"? Our research shows that the information can be collected from the drawing, through operational handling or through the perception of it, and they can also be collected through a local inferences. These inferences are named "local" because they are not linked in a deductive way: the conclusion of a deductive step is simply a new information which can be added to the information list and it is not the premise for the next step.

[3] We consider the students' protocols as the students' written texts and the transcriptions of students' oral texts

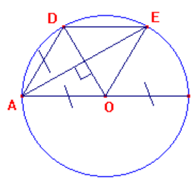
support a conceptual evolution. By analysing the student's oral and written texts using the "drawing mechanism" presented above, we try to describe how the natural language acts in order to promote the resolution process of plane geometrical problem.

4.1 The experimental situation

Given a circle C; its centre O; its diameter AB; D is a point on this circle, so that AD = AO.

The perpendicular to DO through A meets the circle C again in point E.

Prove that OADE is a rhombus



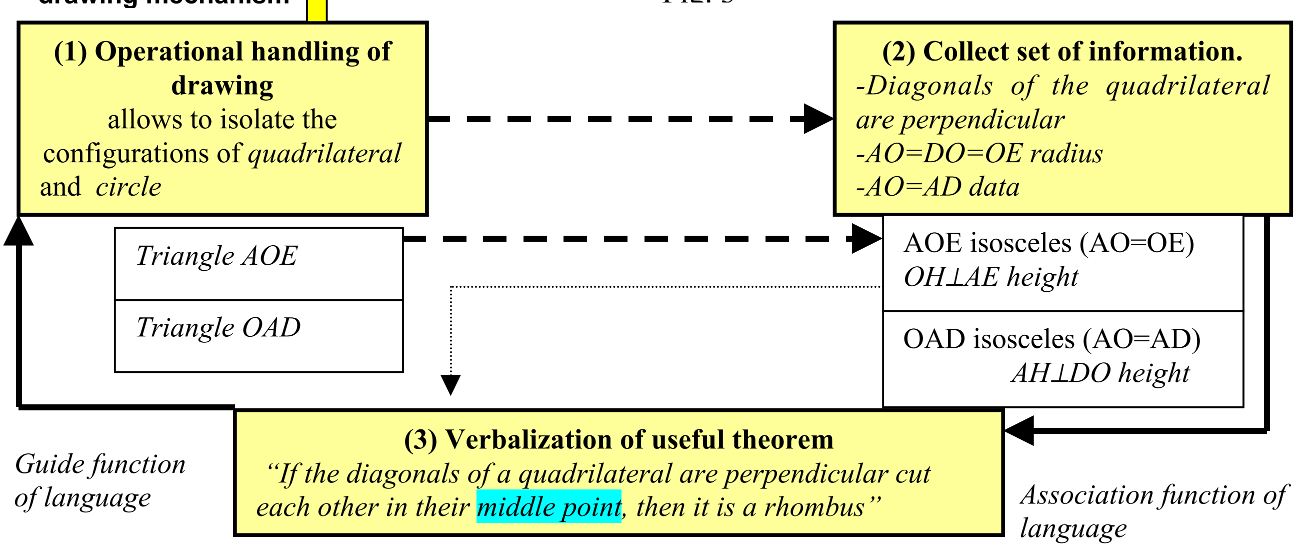
We performed an experiment involving some pairs of Italian Scientific Upper School students and some pairs of French Grade X students. Here is the task.

During the experiment different versions of this task were proposed: for instance, without drawing or with different data. We are presenting now only the above version.

4.2 Drawing Mechanism: model to analyse student's output

As pointed out above, this mechanism allows us to describe problem solving process that moves from drawing. Here is the functioning of mechanism that we have identified in a pair of students' protocol.

Fig. 3



(1) The operational handling of drawing allows the students isolating the configuration of the quadrilateral OADE and the figural unities of AE and OD, from which the students obtain the information "perpendicular diagonals" (2), and the configuration of circle, from which the students obtain the information "AO=OD=OE because radius of the same circle" (2).

The set of information allows the students to **associate** and then verbalize the useful theorem *"If the diagonals of a quadrilateral are perpendicular and cut each other in* *their middle point, then it is a rhombus"* (3). Verbalization of the theorem will associate the status of premises to the propositions "the diagonals are perpendicular" and "the diagonals cut each other in their middle point", and the status of conclusion to the proposition "the quadrilateral is a rhombus". Just because the propositions of the theorem take over a status, the propositions belonging to the information list take over a status too: that of potential premises. Thus, the students can compare the set of information, now potential premises, with the set of premises of the theorem. In this way the students identify the premises, which have not been proved yet, in our case, "the diagonal cut each other in their middle point". Since the premise to be proved is identified, the students came back on the drawing (1) in order to identify the geometrical relations inherent in this premise. We consider that the verbalization of theorem acts as a **guide** for this new operational handling of drawing. The students try to isolate some sub-configurations inside the drawing having the diagonals as figural unities in order to prove the relation "diagonals cut each other in their middle point". These configurations are the triangles AOE and OAD and their heights OH and AH.

5. Some research results: functions of natural language.

Among the results of our research there are various functions of natural language involved in problem solving: guide function, association function, control function, planning function…We will present now only the guide function and the association function of language, trying to answer the following questions:

- How the guide function and the association function of language act?
- Which conditions are necessary in order that these functions of language can act?

5.1 Association function of natural language

How the association function of natural language acts?

By means of an example, we will show how, starting from an information list (2), the association function of language allows the students to evoke and then verbalize the statement of the useful theorem (3). The results of our research show that, in general, this is possible by the joined action of a particular word and a particular configuration. Thus, linguistic and graphical representations of a concept are associated to certain theoretical references. We consider "concept" in the Vergnaud's (1990)[4] sense. In his theory, the concept is a set of three components: reference situations- that give sense to the concept; geometrical properties; linguistic representations or graphical representations[5].

[4] « [Le concept est] un triplet de trois ensembles :
C = (S, I, $\mathfrak{I}$)
S : l'ensemble des situations qui donnent du sens au concept (la référence)
I : l'ensemble des invariants sur lesquels repose l'opérationalité des schèmes (le signifié)
$\mathfrak{I}$: l'ensemble des formes langagières et non langagières qui permettent de représenter symboliquement le concept, ses propriétés, les situation et les procédures de traitement (le signifiant) » (Vergnaud, 1990, p.145)

[5] For more details see § 1.1 of our Cerme2 paper "Verbalization as a mediator between figural and theoretical aspects"

Fig. 3

(2) Information list.

[1].E: all of these triangles *(AHO, AHD, DHE, OHE)* are equal among themselves because AD is equal to AO, then AO is equal to OE because they are radii and ED is equal to the radii because we have demonstrated it…. and then?

…

[2].A: and…the **diagonals** are perpendiculars

[3].E: then, the **segments AH and HE are equal** and… **DH and HO are equal** too

(3) Verbalization of the theorem

[4].E: so hence, … there is a theorem about the diagonals of the parallelogram which cuts them in their middle point. Is it so?

[5].A: Yes, "if the diagonals of a quadrilateral cut each other in their middle point, then it is a parallelogram"

Association function of language

In this example, the association function of language acts by means of the **configuration** "crossing of the diagonals" [3], and by means of the **word** "diagonals" [2]. This word and this configuration play a key role for the subject who refers to a concept belonging to his/her knowledge system: they remind the student of a particular graphical representation and of a specific geometrical property that are related to the concept of "parallelogram". Thus, the joined action of this configuration and this word allows the students to evoke the useful theorem. We named these particular words and configurations "key words" and "key configurations"

Which conditions are necessary because the association function of language acts? The joined action of a "key word" and a "key configuration" is one of the principal conditions in order that the association function of language acts. Moreover, the analysis of the protocols showed that an information list is necessary to support the evocation of the theorem. However, in some cases, some pertinent theorems were not evoked, although the students had collected an appropriate list of information. We observed that, in this case, students choose to use the theorem that has lower "cost" of application. This "cost" depends on the students' knowledge, on their customs and, obviously, on the task. For example, the theorem "A quadrilateral having four equal sides is a rhombus" is easy to prove by means of the "isometric triangles" (belonging to the Italian scholastic program) but it is not easy to prove by means of the "geometrical transformations" (belonging to the French scholastic program). For this reason, the theorem is more used in Italy than in France. We observed that the same words or configurations can evoke different useful theorems. For example, the key word "median", joined to the configuration of perpendicular diagonals, allows the French students to evoke more frequently the theorem concerning the "Addition of vectors"; at the contrary, it leads the Italian students to evoke the theorem "A quadrilateral having the perpendicular diagonals cutting each other in their middle points is a rhombus". This theorem belongs to the scholastic programs of the middle class in France and in Italy too; instead, the theorem "Addition of vectors" belongs only to the French program. For this reason we consider the students' knowledge a fundamental condition for the association function of language.

5.2 Guide function of natural language

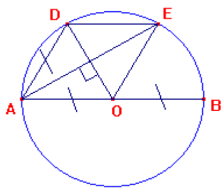
How the guide function of natural language acts?

By means of an example we will show how the verbalization of the useful theorem (3) (in particular the theorem whose conclusion is the answer of the problem) leads the operational handling of the drawing (1) in order to identify the only geometrical relations asked in the premise to be proved.

(1) Operational handling of drawing

[3].C: wait! AO is equal to AD… and *what if we could prove that* triangle DAO is isosceles? Because, you know, **it is important with reference to this one *DO*.**
[4].G: yes, because it is the height and it is also the median ….Yeees!!! it is the median!!!
[5].C: and this means that it is an isosceles triangle because the height is equal to the median… AE is perpendicular to OD and AH is the height in the triangle ADO (*H intersection of the two diagonals*).
[6].G: then AE cuts OD in the middle.

Fig.4



[2].C: wait! Perpendicular it's OK. yes, but what we have to say is that this one is the middle point …it is the middle point of this one and of that one (*DO and AE*)

(3) Verbalization of the theorem

[1].C: yes but, for a rhombus we have to say that… the diagonals of the quadrilateral are perpendicular and that they cut each other in their middle point, is a rhombus

Guide function of language

We observe that in the intervention [2], C. identifies the only premise of theorem that has to be proved ("diagonals cut each other in their middle point") by comparing the premises of the theorem with the data of the problem. The students came back to the operational handling of drawing in order to identify the geometrical relation between diagonals asked in the premise. Thus, the student C. isolates the sub-configuration of triangle OAD [3] in order to prove that AE cuts OD in its middle point [6].

Which conditions are necessary because the guide function of language acts?

The results of our research show that verbalization of the theorem is a necessary but not sufficient condition in order that the guide function of language acts. Others conditions are necessary, for instance, an information list supporting the verbalization of the theorem but, above all, the re-verbalization of the premises of the theorem during the resolution process. These re-verbalizations allow to identify the premises that have to be already proved during the resolution process.

Moreover, we observed that the guide function of language acts in different ways depending both the difficulty of the task and the particular pair of students. We observed two kinds of situations between the students: in the first one, each student can verbalize the theorem and this verbalization acts as a guide for the schoolmate; in the second one, the verbalization of the theorem, made by one of the students, can lead the schoolmate. In this case there are two possible situations: either the student who verbalizes the theorem uses it to lead the schoolmate; or the student who has not verbalized theorem uses the schoolmate's verbalization in order to lead himself in the problem solving.

6. Conclusion

Our research provides some results about both aspects of natural language, that' to say both the language as a revealer, and the language as a tool for the construction and the management of students' deductive thinking. In the first case, we have focused on useful models to analyse the students' cognitive process considering the progression of discourse (from accumulation mode to the substitution mode) and the use of linguistic unities. In the second case, we have characterized different functions of natural language as problem solving tool for the students. Moreover, we think that our research can provide some didactical implications too about the necessary conditions in order that the functions of language act. As a matter of fact, we showed that the functions of natural language involved in geometrical problem solving are promoted by particular conditions.

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