DIFFERENT CULTURES OF THE YOUNGEST STUDENTS ABOUT SPACE (AND INFINITY) (*) [1]

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Abstract: *A blank sheet of paper is a simple and powerful artifact for each pupil from the beginning of school. It can be used in order to test 5 - 7 years old pupils' concept of Space. Children's protocols show different personal cultures.*

Sommario: *Il foglio bianco è un artefatto potente usato da ciascun bambino già dal suo primo approccio alla scuola. In questo lavoro è stato usato per saggiare il concetto intuitivo di Spazio di bambini tra i 5 e i 7 anni. I protocolli manifestano la presenza di culture personali diverse.*

Introduction

This is an experimental research by which I investigated how pupils 5 – 7 years old manage space (and infinity). In their answers pupils showed their beliefs about this (these) important mathematical and cultural object(s). I am not sure that the pupils' intellectual elaboration on this theme has yet transformed their beliefs into a concept of space. In my opinion we can glance at their culture about space. The term culture as a lot of meanings, cf. Kroeber and Kluckhohn (1952). I assume culture in the individual-related meaning (παιδεια) involving cultural aspects such as knowledge and beliefs, Tylor (1871), that are meant as the display of the abstract ideas of the mind, cf. Lévi-Strauss (1962).

The paper explores the youngest students' culture of space; I mean space to be related to a deep mind structure that can be revealed only indirectly by the means of the manifestations of beliefs cf. Leach (1978). I carried out the research using simple activities performed in three primary school classes (one first grade and two second grade) and in three kindergarten classes (5 years old pupils), during the years 2001 and 2002. These classes are situated in *Casalmaggiore* (C) and *Rivarolo del Re* (R), two small towns in an agricultural and small industrial area of Northern Italy [2]. The pupils involved (some of them at their first school day) were 28 kindergarten children (11 KR, 5 + 12 KC), 20 first grade (PR) and 36 second grade (PC) of primary school (84 in total, 13 of them not Italians).

(*) The contribution is executed in the sphere of Local Research Unity in Mathematics Education of University of Parma, Italy.

[1] I wish to thank Bill Barton for his kind revision of my "English" language.

[2] I wish to thank the teachers Francesca Isidori, (KC) Rosanna Soldi (KR), Maria Rosa Ragazzini (PC), and Laura Vezzoni (PR) for their precious co-operation. The acronyms KR, KC, PR and PC are used to denote kindergarten or primary school of Rivarolo del Re or Casalmaggiore, respectively.

Theoretical framework.

The notion of space is strictly related with geometry, but we can consider this topic as only one of the possible ways to conceive and to study space: mathematics use the noun “space” in association with several different adjectives: Euclidean, Cartesian, metric, topologic, functional, etc., each of them possibly conveying different intuitions. Moreover Manara (1988) points out that Euclid, in his work, never uses the word “space”.

Speranza (1994) explains several epistemological positions about space, for example the dichotomies: independence / non-independence, anisotropy / isotropy.

Independent space means that we first consider space and then we put the objects (figures) in it. In Plato’s *Timeo* we can find that space is a species that offers a seat to things; according to this interpretation, space itself is infinite, or better unbounded. On the contrary Aristotle maintains a non-independent (or relative) of space: his interpretation of space as the site of a *body* implies finiteness in the sense of limitation, or better, in topological language, it implies the finite feature of compactness (cf. the so-called Heine-Borel theorem on compactness in Euclidean spaces). Therefore space and infinity have a dialectic relationship.

The other relevant dichotomy is anisotropy / isotropy, i.e. whether space has privileged directions or not. Speranza (1994) states that the space of everyday experience is anisotropic (in horizontal and vertical directions); the same happens for Aristotle’s Cosmology Space (from the Earth to the fixed stars). The space of Euclidean geometry is isotropic and therefore the introduction of Euclidean geometry into physics, made by Krebs, Galileo and Newton, was a true revolution. Piaget and Inhelder (1956) stated that the anisotropic view of the Space is an achievement; Rozek and Urbanska (1999) investigated children’s protocols on these aspects. They underlined the importance of multilateral understanding of a squared rectangle as a necessary basic knowledge for handling tables and for the learning of commutative laws of the arithmetic operations.

In order to investigate whether these conceptions are present in pupils, in this research I used two instruments: the first one is a blank sheet of paper, the other one is represented by the *seriazioni* [3]. For the use of the second one some aspects of this paper are common with Pezzi (2002). *Seriazione* is an Italian word currently used in statistics for frequency distribution, but in the curriculum it is used with two different meanings:

1) individuation of the formation law of a sequence of which only a few of the first terms are given;

[3] In 1985 the Italian Education Board introduced the subject Logic in the national curriculum for primary schools (1st - 5th grades); the *seriazioni* are a topic of Logic to be taught in the first two grades. From this point of view my contribution should be considered in the ambit of Logic education. Examples of *seriazioni* can also be found in many items used to test mathematical-logical skills, for example in the determination of IQ.

2) construction of a sequence when the formation law is given.

Therefore the *seriazioni* present, at the same time, direct and inverse problems, cf. Marchini (2002). I can partially translate *seriazione* as “pattern sequence”. Sequences are mathematical entities that can be treated both with the notion of potential infinity, Brouwer (1975), and with the notion of actual infinity, Cantor (1932). This double nature is emphasized in Gilbert and Rouche (2001). Monaghan (2001) identifies the two aspects as a process and an object, as different sides of the same coin, Sfard (1991). It might be questioned whether (bounded) sequences would reveal infinity, but as Gilbert and Rouche (2001) point out, sequences in everyday context can hint about the “concrete” presence of infinity. From another point of view, the *seriazione* should be an application of a Basic Metaphor of Infinity, Lakoff and Nuñez (2000), since the sequence construction is a task that should imply «a process that evolved in an ordered string of stages that never ends, but at the same time some sort of conceivable completion of this process is suggested», Mamona-Downs (2001). The presence of infinity in my research is supported by the implicit suggestion of Fischbein et al. (1979) about the work of Piaget and Inhelder (1956).

Falk et al. (1986) investigated 5 - 12 years old pupils’ conceptions of infinity by using a game played with natural numbers; my research is focused on younger pupil’s conception of space (and of infinity in connection with space), by employing drawings (in particular, the *seriazioni*) instead of the subdivisions used by Piaget and Inhelder (1956).

Research hypotheses and problems.

Coherently with other researches developed by our Research Unity, I assume that:

Hypothesis 1. Beliefs of space and infinity are present in (some) pupils attending the last year of kindergarten and the first two years of primary school [4]. At these ages, the influence of school training on these subjects is, in my opinion, negligable, therefore pupils show their own personal culture, elaborated autonomously, out of school, and these cultures appear deeply different.

Hypothesis 2. A rough classification of these beliefs splits them in two main approaches, the Platonic and Aristotelian ones, that I have previously discussed. This conjecture needs specific tools in order to be proved, since the pupils’ knowledge, verbal means and representative skills are relatively scarce.

In order to prove or disprove these hypotheses I faced some problems:

Problem 1. It is obvious that the question: «What is your belief in space?» cannot be addressed to 5 - 7

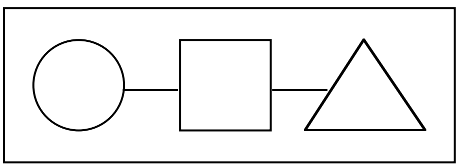


Figure 1

[4] This paper is a part of a long term longitudinal research of our Local Research Unity on the learning and teaching difficulties of limits, functions, and infinity.

years old pupils. The mediation tool I used to detect the "sense of space" in young people is a blank sheet of paper, combined with a drawing activity. Drawings must have suitable features: inducing pupils to fill the sheet by repeating the module, and somehow denoting whether the pupils conceive space as independent or not.

Such requirements can be satisfied by the *seriazioni* [5]: the sheet can be filled by repeating a ternary module similar to figure 1 [6]; the small lines [7] connecting the figures can be used in order to reveal (or simply to suggest) a continuation even out of the sheet, involving then the notion of infinity, cf. Pezzi (2002).

Problem 2. The "irruption" of infinity in this research on space complicates the inquiry. Monaghan (2001) points out that each investigation on pupils' concept of infinity has potential pitfalls, especially if the research is conduced by talking to children. For this reason, my instructions avoided the explicit or implicit use of infinity, such as *three dots*, and as expressions *etc*., *and so on*. From another point of view, the *seriazioni* offer pupils a dynamic context, since instructions ask for hand drawing on the sheet, performed in time.

Problem 3. Another question is whether *seriazioni* are suitable instruments for the youngest students. Therefore the research has to be supported by experimental evidence as to whether the *seriazioni* are well mastered tools for the children involved in the research.

Martina (KC)

Problem 4. This problem has another facet: are the children capable of following such instructions?

The first explorative protocols were obtained by the teacher Isidori (KC); in the year 2001 she gave five of her 5 years old pupils the instructions to draw two figures alternating two colours. Three pupils succeeded (e.g. Martina (KC)), other two showed difficulties with the repetition of the module (even if freely chosen by the children themselves).

Problem 5. How should the module of the *seriazione* (pattern) be chosen? Repetition of a binary module can be obtained using a local control of the difference: the sequence can be carried on, only by keeping in mind the last drawn element, changing (in these examples) the colour. For this reason, I decided that binary modules were not suitable for the research, and suggested that teachers try ternary modules. The right repetition of a ternary module implies control of enhanced

[5] A similar approach based on "pattern sequence" is presented in a newsletter for parents of the Wisconsin University (http://www.wcer.wisc.edu/MIMS/Parent_Newsletters/The_Shape_of_Space/newletters8.html). In this case the target is the primary school children's knowledge about shape (kindly suggested me by Daniel C. Orey).

[6] In a kindergarten classroom (KR), the adopted module was simpler: three rings with different colours, cf. Noemi's protocol.

[7] The idea of the small lines connecting the figures in the module is due to the teacher Ragazzini (PC).

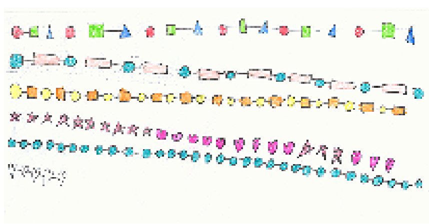
complexity, using a form of back and forth: back in determining which is the form and/or the colour needed (inverse problem), forth in drawing the element required (direct problem) [8]. Moreover the incomplete reproduction of the ternary module, because of the presence of the sheet boundaries, can imply the need for denoting the continuation of the *seriazione* out of the sheet.

Problem 6. The blank sheet itself could present another problem: this artifact hints an anisotropic interpretation of space: the sides can be used as vertical and horizontal directions. Therefore in an investigation of the pupils' beliefs on this aspect, the sheet of paper together with the intrinsic linearity of the ternary module should vitiate the problem results. Some of the protocols reported in Rozek and Urbanska (1999) contrast with this hypothesis, but the task they proposed was different.

Because of the small numbers of protocols (79), the results do not have a statistical meaning, but this contribution can only be considered as a case study.

Methodology.

The experiment used paper and pencils; the time devoted to this activity was variable, depending on the ability of each pupil. The teachers drew modules on the sheets (in kindergarten and in first grade of primary school) or gave explicit instructions for the drawing of the module (in second grade of primary school). Then they asked the pupils to continue or to go on by repeating the module. The key word of this research was just the verb "to continue" in drawing a repetition of the given module. When pupils reached the right edge of the paper, their only instruction was "to continue", without further specifications. The verb "to continue" could include an implicit use of infinity, as Monaghan (2001) remarks about Nuñez (1994), but in my research the context is concrete and not hypothetical as in Nuñez; therefore I do not think that this instruction suggests the notion of independent Space and of Infinity. Some protocols (e.g. Rebecca (PC)) should confirm my hypothesis. She interpreted "to continue" as the instruction of creating some other *seriazioni.*

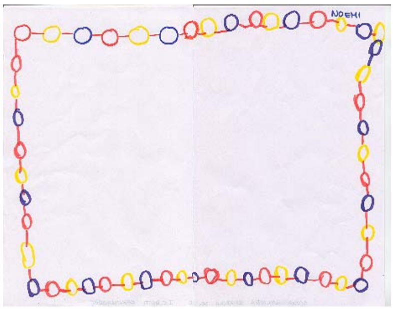


Rebecca (PC)

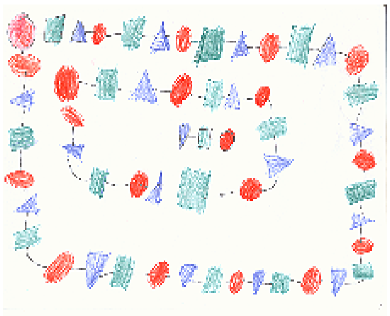
We obtained 23 protocols (11 KR and 12 KC) made on the first school day (with two children younger than 5 years old) and other 56 protocols, among which were 20 pupils from the first grade (PR) and 36 pupils from the second grade (PC).

[8] These aspects are not present in a binary or unary module.

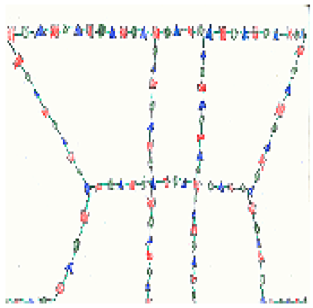
In order to assess the reliability of the *seriazioni* as a consistent tool for the 5 - 7 years old children involved, I recorded for each protocol the number of separate sequences, the number of elements of the module used, the length of each sequence, i.e. the number [9] of repetitions of the initial module, and the number of errors (or oddities) in the repetitions of the module. I separated the total length of ternary and binary sequences. I considered the sums of the lengths as level of the pupil's awareness in following instructions and using *seriazioni* in producing the task.



Noemi (KR)



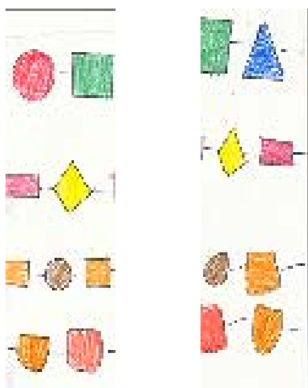
Arnaldo (PC)



Cristina (PC)

More complex, intriguing, and open for discussion was detecting the intuition of space. I assessed protocols presenting terminated sequences (e.g. Rebecca (PC)) or frames (e.g. Noemi (KR), Arnaldo (PC), Cristina (PC)) as the ones revealing a non-independent space belief. Conversely I interpreted the protocols showing the pupil's intent to continue outside the sheet as an independent Space conception. Pezzi (2002) reports some oral interviews in which the pupil bears witness of need for infinity. The two kindergarten teachers wrote all the oral interventions of pupils, and some of them actually supported the interpretation of an independent conception of space.

Enrico (PR)



Elena (PC)

Particulars of both top sheet left and right borders

In my opinion the presence of small lines on one or two sides of the sequence, in the vicinity of the sheet boundaries, reveals an independent conception of space; the

[9] The integer part of the number indicates the complete reproduction of the module, the decimals the incomplete reproduction. Either when two sequences were reproduced in different parts of the sheet without connections (small lines) or with different modules (change of number, forms or colours of the elements of the module), I held them as separate sequences. If the sequence was built by one element only, I just recorded that the module was unary.

same is true for the partial representation of elements of the module, interrupted by the boundary. I also consider that the presence of infinity in the drawings is coupled with the conception of the independent space.

The anisotropic conception is signalled by either vertical or horizontal disposition of the drawings.

Results of the experiment.

I summarise below the results of the experiment in which 79 pupils were involved; the first five KC obtained protocols are not included. Kindergarten protocols are generally not easy to interpret.

Andrea (KC)

Result 1. Some pupils appeared able to perform the task; other ones showed hindrance in reproducing forms and/or colours. The protocols of Noemi (KR) and Andrea (KC) (4:6 years old) presented two opposite examples.

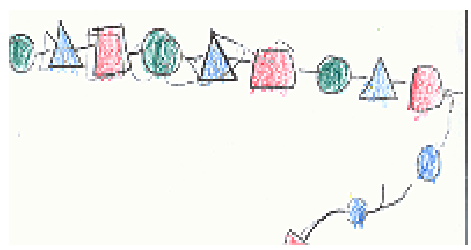
School	n. pupils	n. seriazioni	average ternary length	average binary length
Kindergarten R	11	15	6,66	0,37
Kindergarten C	12	35	7,73	0,17
Primary 1st deg R	20	153	24,96	0
Primary 2nd deg C	36	133	15,47	7,83

The calculus of ternary length on KC, KR and PR protocols is naturally affected by the interventions of teachers who drew the module on the sheets. Another typical phenomenon of children at this age is the lack of consistency in writing and drawing direction, reported by Isidori, and evident in some protocols.

The numbers of correct *seriazioni* and of pupils performing at least one "correct" execution of the ternary module are shown in the following scheme.

School	n. pupils	n. correct *seriazioni*	n. pupils correctly performing at least one ternary *seriazione*
Kindergarten R	11	7	5
Kindergarten C	12	26	11
Primary 1st deg R	20	107	19
Primary 2nd deg C	36	99	36

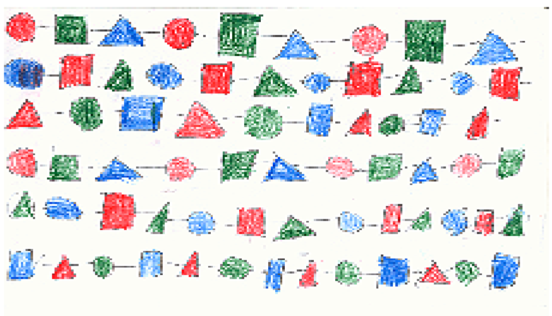
Result 2. These data show that the ability to correctly reproduce or create a ternary module is already present in 5 - 7 years old pupils and this skill improves with the school training. There could also be an innate component, as some 5 years old pupils, for example Noemi (KR), appear mature already, and other 7 years old ones, for example Mirko (PC), are not yet mature.



Mirko (PC) particular of sheet top right corner

Foreign pupils obtained shorter average ternary length than that of the whole class; for example in 2nd grade (PC), seven foreign pupils got 10,99 instead of 15,47, but two of them, Otman (PC) and Arnaldo (PC) obtained better results than their class average.

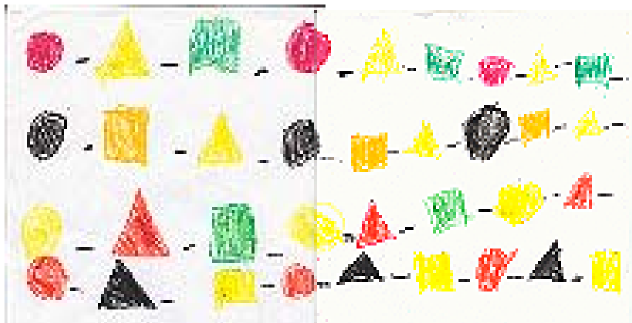
Result 3. Protocols show the presence of the two main beliefs about space (Plato's and Aristotle's ones). Following the previous protocols interpretation, I isolated those ones suggesting an independent space belief [10] from the non-independent ones, but this criterion was not so sharp, as there were "mixed" answers. The practice of writing should justify the pre-eminence of the continuation on the right side of protocols revealing an independent conception of space.



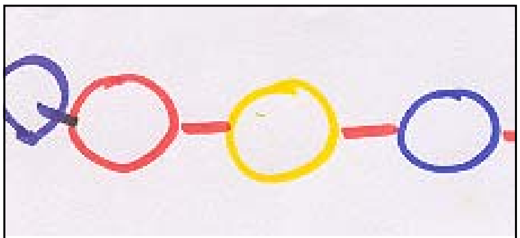
Otman (PC)

School	n. pupils	n. mixed	n. non-independent pure	n. independent pure
Kindergarten R	11	0	7	4 (3**r**,1**b**)
Kindergarten C	12	7 (7**r**)	3	2 (2**r**)
Primary 1st deg R	20	2 (1**r**,1**b**)	16	2 (2**b**)
Primary 2nd deg C	36	7 (5**r**,1l,1**b**)	21	8 (2**r**,1l,5**b**)

Kindergarten teachers reported the pupils' statements and some of them seemed important for the research. For example I interpreted Lisa (KR): *It is impossible to go on*; Andrea (KC): *There is no room enough for the drawings*; Mejo (KC): *The sheet ends, what do I do?*; interventions as the revelation of their non-independent space beliefs; their protocols show terminating sequences, confirming this interpretation. On the contrary, the statements of Laura (KC): *There are no **spaces** on the sheet!*; Luca (KC): *The sheet ends **but** I stop myself*; Silvia (KR): *I continue below, otherwise I would paint the desk*; Daniele (KR): *I attached another sheet as I wanted to make it long*, revealed their independent Space belief. Some protocols present the "continuation" on the right side, other ones on the left side; only one from the primary school continues on the bottom. Some



Laura (KC)

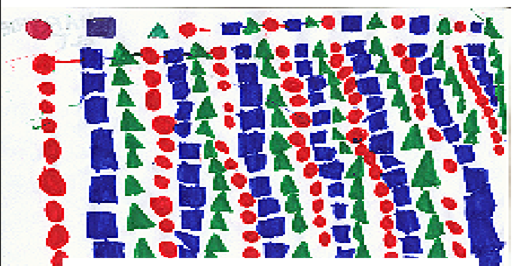


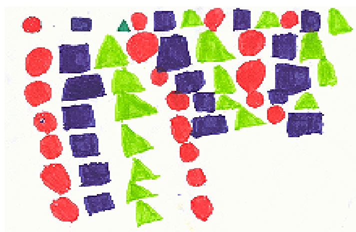
Silvia (KR)-particular of the top sheet left corner

[10] I specify by "**r**", "**l**" and "**b**" the continuation of the *seriazioni* either on the right, on the left or on both sides.

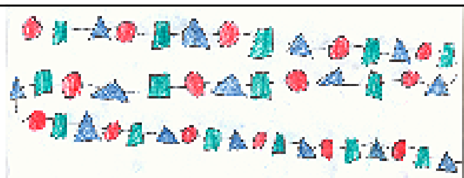
pupils interpreted the task of continuing by going back to the first element and adding a new ring to it on the left (e.g. Silvia (KR)): this seems rather interesting. In some primary school protocols the continuation on the left of the first module is performed by adjoining small lines when the drawings already reach the right edge of the sheet (e.g. Enrico (PR)).

Result 4. The influence of the writing strongly supports the anisotropic interpretation of the Space. Another interesting interpretation of the module was given by several pupils of 1st grade primary school, e.g. Cristian (PR) and Harpreet (PR): they reproduced the module elements vertically, without control of the horizontal lines. The protocol of Harpreet (PR) is an example of a sort of non-Euclidean model: the module is repeated by varying both sizes of its elements and distances among the elements, as closer the drawing goes to the right edge of the sheet. I assess the 20 protocols with much more branched drawings (e.g. Cristina (PC)), as expressions of an isotropic space belief, or better an incompletely anisotropic interpretation of space, according to Piaget and Inhelder (1956) assertion.


Harpreet (PR)


Cristian (PR)

Nine pupils, seven of them in PC, show a strange use of a “writing style” that recalls the ancient Greek and Latin inscriptions, e.g. Ilyass (PC).

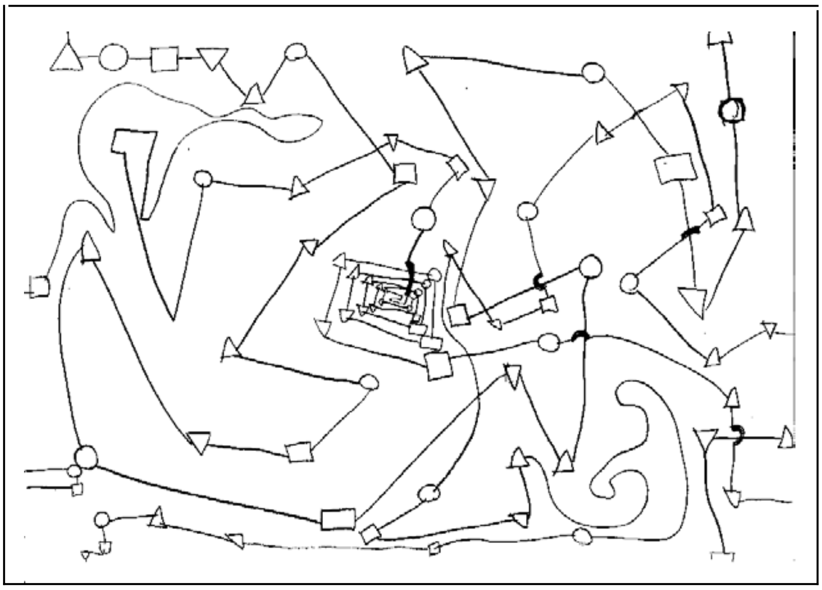

Ilyass (PC)

Conclusion.

The experiment shows that in the same classroom there can be distinct conceptions and cultures about space and infinity. The teacher can detect them by suitable tools (like the one I proposed) in order to be aware of these aspects and to avoid the evolution of these beliefs into epistemological obstacles in the following school years.

The blank sheet of paper combined with *seriazioni* may be a weak or scarcely sensitive tool, but it is very hard to build an inquiry instrument for detecting these conceptions in children so young. The teacher can improve the tool by observing and listening to the pupils while they execute the task, paying particular attention when the drawing is close to the boundaries. This delicate moment can reveal the conception of space of their pupils. From the protocols I observed the manifestations of different cultures, about space and infinity.

I presented and discussed some protocols with prospective teachers. One of them (A. Speroni) presents the same question to first year middles schools students. In these students' protocols I found the manifestations of the same beliefs about space, with one exception, Matteo's protocol in which Space appears unbonded by the four sheet edges and infinity appears in the center of the sheet:



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