GEOMETRIC FIGURES FROM MIDDLE TO SECONDARY SCHOOL: MEDIATING THEORY AND PRACTICE

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Abstract. *This paper concerns a recent work carried out with a group of 6 expert teachers who changed their school level at the beginning of this year. The work is a continuation and an adaptation of an experience that has been going on for some years with several teachers from high schools. The theoretical aspects involve questions on the teaching of elementary geometry, van Hiele's theory of levels and teachers' conceptions on the role of definitions and their teaching. We conducted four discussion sessions on these arguments following a designed precise plot in order to encourage teachers to reflect on their own practice, on their curricular activity and on their teaching aims.*

Prologue, or the first "match" between theory and practice

Between 1998 and 2001 the authors used Van Hiele's theory as a means to facilitate secondary school teachers in discussing students' performances on agreed worksheets on geometry (Cannizzaro & Menghini, 2001). The main purpose was to enhance their didactical awareness in managing pupils' transition from the perception of an elementary figure to its *definition* and from definitions to inclusive definitions.

The work took off from a curriculum design formulated for the first year of high school (age 14) in a school-university working group. The origin of the project was the acknowledgement that pupils entering secondary school, even if they know geometric figures, are not familiar with their properties and are not always able to point out specific differences expressed in the definitions. The teachers needed to acknowledge the information that students had, to direct it towards a higher level of precision in language and towards a real comprehension of the role played by definitions. We designed a series of 6 worksheets, for a total of about 30 exercises, which highlight the properties of triangles and quadrilaterals. Requested constructions or geometric manipulations were alternated with observations of figures already drawn. The practical work with concrete artefacts was notably reduced with respect to that foreseen by van Hiele & Geldof (1958) for the lower scholastic levels, but it was not totally abandoned.

Our contribution was to give theoretical support to the teachers. We used van Hiele's theory to encourage them to reflect on their own practice. They recognised in van Hiele's theory a framework corresponding to their purpose, i.e. to move gradually from the sphere of visualisation and description, or recognition of properties, to the sphere of refinement of language and definitions.

They were particularly struck by the idea of the passage from the *symbol* to the *significant signal*. Van Hiele's *symbol* (1958, 1974) represents the level of *perception* (level 1), into which the pupils condense all properties of a geometrical figure with which they have had experience. The figures have the character of images, i.e. a symbolic character. Van Hiele's *signal* represents the level of *description* or *analysis* (level 2), when perceptions are translated into descriptions, but without specific linguistic properties. One property emerges that will be significant in the description of the figure (the *significant signal*). The third level is that of *definition* (or descriptive definition). One begins to observe the various relations from a logical point of view: implication, and therefore definition, takes on significance within the realm of geometrical relations. This, according to van Hiele, is the *essence of geometry*. Van Hiele's further level (level 4) is that of *Euclidean deduction*, when the pupils begin to understand what is meant by proof and, for example, to distinguish between a proposition and its inverse.

Without doubt, the fact that we rendered explicit van Hiele's levels 1, 2, 3 and 4 allowed teachers to better clarify their teaching aims, even if that specific work was concerned only with the *first three levels*, i.e. with the period *preceding* Euclidean proofs. In particular the passage "step by step" from exclusive to inclusive definitions, and the role of the latter in starting deduction, became clear to them. This theoretical support allowed them to modify and to use the worksheets in a specifically aimed manner.

Clements & Battista (1992) emphasize the difficulty of the passage from middle school to high school, holding that the major focus of standard elementary and middle school curricula is on recognising and naming geometric shapes, writing the proper symbolism for simple geometric concepts, developing skills with measurement and construction tools such as a compass and protractor, and using formulas in geometric measurement. On the one hand, "these curricula consist of a hodgepodge of unrelated concepts with no systematic progression to higher levels of thought - levels requisite for sophisticated concept development and substantive geometric problem solving" (ibid. p. 422); on the other hand, "at the secondary level the traditional emphasis has been on formal proof, despite the fact that students are unprepared to deal with it".

In effect the purpose of the work set for the beginning of high school was to favour the gradual passage from intuitive geometry to the rational geometry in the high school by covering levels 1, 2, and 3 again.

Nevertheless, as Clements and Battista too referred in 1992, most researchers agree that achieving level 2 and 3 thinking is an important goal of pre-secondary instruction.

The problem of present research

Van Hiele himself reiterates that the levels of thought do not refer to the biological evolution of the individual (van Hiele, 1986) but to the development of thought itself, this should mean that they can be activated with different modalities within the various scholastic levels (see also Burger & Shaughnessy, 1986).

In this work we tried to analyse the argument from a new point of view, and that is to say from the viewpoint of a middle school teacher, in order to highlight the diversity of approach, the diversity of objectives and the diversity in the acceptance of errors.

Some general questions arise:

Question 1. How are the objectives of the middle school different from those of the high school? (cf. Furinghetti *et al*. 2002)

Question 2. How do the teaching activities of middle school teachers differ from a methodological point of view from those working in a high school?

Question 3. Is it methodological difference that influences the conceptions of the pupils?

Question 4. To what extent can van Hiele's theory help middle school teachers to clarify their objectives, in concordance with the activities to be carried out?

Then there are two question that regard this work in particular:

Question 5. To what extent do teachers used to working with younger pupils and to basing the teaching of geometry on concrete artefacts consider it possible for pupils to conquer an inclusive-set theory vision of the quadrilaterals?

Question 6. Is the activity proposed by us in the high school to be considered a recapitulation of things already seen in the middle school, or is it something different?

Differently from the situation outlined in the previous paragraph, this specific research arises to answer some of our questions on teachers' conceptions and professional habits, rather then from teachers' questions on their own teaching modes.

We involved a particular category of teachers: precisely (six) teachers who have graduated in mathematics and who were leaving middle schools to work in high schools. These teachers manifested their will to compare their previous middle school methodology with the requirements set by secondary school.The theme, anyway, did not arise from teachers themselves: we proposed (almost imposed) the topic (elementary geometric figures), the worksheets for students and the materials on van Hiele's theory.

Teachers have been invited, through repeated collective discussion sessions, to go backwards and forward from their long past experience in the middle school to their actual experience, to reflect on their own practice, on their curricular activity and on their teaching aims.

These teachers were and are therefore well aware of the problems of teaching in middle schools; they were and are looking at the problem of what and how to teach in high schools. The group of teachers cannot be said to be representative of middle school teachers because their degrees in mathematics make them different from the majority of their colleagues at Italian middle schools, which are usually holders of a degree in biology or in natural sciences. In their opinion this diversity has a great influence on the curriculum carried out in middle schools in terms of both content and methodology.

Methodology

As in the case of the previous research, we called the teachers to some collective interviews through which we asked them to specify their observations on the worksheets produced by the previous group, on the way to use them, on the materials to back these up, on the expectations regarding the results, on the teaching style in their current schools (high schools) and on van Hiele's theory of levels. The interview takes as a basis some of the materials we had supplied to them: articles relating to van Hiele's theory and to the introduction of definitions in schools and the worksheets prepared in the group of high school teachers.

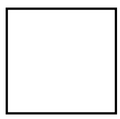
We show a pair of examples of exercises contained in the worksheets, remembering that the worksheets are not the core of this research.

3.1) The figure drawn is a rectangle.

Look at it carefully; are the angles all right angles? *Yes/no*

The opposite sides *are equal/not equal*

3.2) The figure drawn is a square. Look at it carefully.



Both the square and the rectangle have four right angles and opposite sides equal.

What distinguishes a square from any rectangle? The sides of the square are all…..

In conclusion:

The square has four ……… angles and four ………sides

The rectangle has four ………angles and the opposite sides are ....…

Is the square always a rectangle? *Yes/no*

Is the rectangle always a square? *Yes/no*

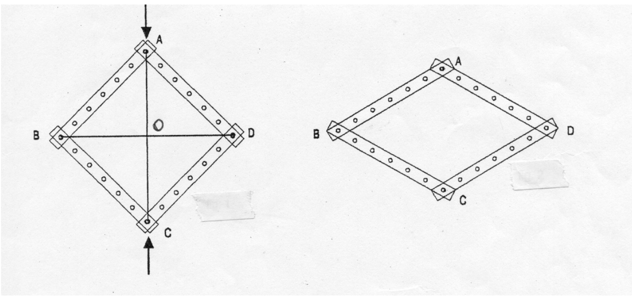
4.1)

You can see a square ABCD made by the pieces of meccano:

Using the same pressure, push the square in the vertexes A and C in direction AC, the segments AO and OC will shorten, but will remain equal.

Are the diagonals AC and BD still equal? *Yes/no*.

The sides of fig. 2 are equal / not equal.



Is fig. 2 a rhombus? *Yes/no*.

Can you say: "The rhombus is a quadrilater where all the sides are equal"? *Yes/no*.

The square has all its sides equal: can we state that it is always a rhombus? *Yes/no*.

Can we say that the rhombus is always a square? *Yes/no*.

From the construction can you deduct that the diagonals are perpendicular to each other? *Yes/no*.

The interviews have been conducted following a list of questions proposed to theachers as a basis for conversation:

- Which of the proposed activities did you normally use in the middle school?
- Over what time periods?
- Which were the main objectives of each activity?
- Which materials did you use as a support?
- Did you carry out an analogous series of activities in the middle school in order to arrive at the definitions?
- Would you have used some or all of the worksheets in the middle school?
- Would you use them in the high school?
- In your opinion, which levels in Van Hiele’s theory correspond to objectives in the middle school?

First results

We cannot refer of results because the work has just started; we have however already reconstructed and recorded "scenarios" corresponding to different phases of the interviews, which simultaneously confirm the good sense of some of the general questions and begin to outline some possible answers.

First scenario (in relation to questions 4 and 5 in the previous section)

The teachers consider that van Hiele’s theory is entirely appropriate for describing what they do, and that the levels from 1 to 3 are objectives that are actually achieved. In truth they refer to a “blander” first version, which unites levels 2 and 3 in a "descriptive" level, which can be reformulated as follows: The figures are recognized by their properties. These properties will prove to be firmly fixed if they are compared with the analogous properties of other figures. At such a level the pupil is able to manipulate the known characteristics of a model familiar to him or her. However, their properties are as yet not in order. At this phase a square is not yet recognized as a special rectangle.

In fact, only one of the teachers, Loretta, completely subscribes to such a formulation of the levels. For all colleagues, accustomed to working in class using the manipulative tools of Emma Castelnuovo (cf., for example, Castelnuovo & Barra 2000, see also Bartolini Bussi & Mariotti 1996), it seems that the use of specifically aimed materials can favour the rendering explicit of the inclusive relationships between quadrilaterals, and therefore the pupils recognize the square as a special rectangle. In a further fragment of the interview and discussion, the confidence in a generally positive result vanishes, also amongst the other teachers. They agree that the pupils probably accept the inclusive relationship between individual pairs of quadrilaterals, those linked with particular artefacts used: rectangle-square, rhombus - square, parallelogram-rectangle, etc.

Second scenario (regarding questions 1, 2, 3 and 4)

When we try to turn the discussion to the emergence of the "significant signal" as the moment of birth of the definition, the teachers recognize the foundation of this point, but this does not seem to be so illuminating as it had been for high school teachers. They clarify the fact that in middle schools the properties of a figure and the relationships between these are not highlighted much. A lot of work is done through problems, using the properties needed from time to time, but without going more deeply into them.

Third scenario (in relation to questions 2 and 3)

With regard to the worksheets, the teachers maintain that they would not use an approach with worksheets in middle schools (in effect the worksheet foresees autonomous elaboration, a form of personal mathematical understanding, and is more appropriate at high school level). The teachers judge, besides, some worksheets to be too difficult in that they require constructions with a rule and compass which they consider more appropriate for high school level. In effect the first worksheet required, albeit not a fully-fledged construction, the drawing of the reflection of several triangles with respect to a vertical line. Worksheets 2 and 3 are also deemed inappropriate for middle schools because essentially they are centred on aspects of terminology and language (they ask, for instance, if a triangle with two given properties exists); they do not seem to be adequate at least in the given order.

Fourth scenario (relating to questions 1, 2 and 3)

Excluding the aforementioned worksheets, at the beginning of the secondary school the material is judged to be useful for the activity of recapitulation for "their own" former pupils. As, for example, the worksheets on quadrilaterals, which push students to think of the characteristic properties specific to the various figures, underling the differences between them with respect to the properties of their sides, their angles, their diagonals. However, they maintain, these are certainly not appropriate for the majority of middle school pupils. For these, the worksheets should be used to recover all the phases leading from one level to the other. They then provide a description of the typical activities at middle school, which is very similar to that of Clements & Battista's, mentioned above.

Fifth scenario (relating to question 1).

The teachers also reacted very badly when they saw that the worksheets suggest, in order to compare the diagonals of squares and rectangles, calculating them using Pythagoras' theorem once their *measurements* are given: they have already had their fill of measuring and in the high school it is a good thing to return to synthetic geometry!

Sixth scenario (relating to question 5).

Not everything proceeded smoothly! … It emerged that one of the principal differences between them and the other teachers is that they suggest the set theory classification of the figures, using Venn's diagrams, so that their pupils are able to see, for example, a square as a special rectangle.

At this stage, through action taken by us the group becomes a virtual group expanded to include the high school teachers involved in the previous experience. We refer the lively discussions of the other group and we stress that together with the high school teachers, we have learnt to see the inclusive definitions, and hence the set theory classification, as the endpoint of the stages regarding definitions. This is where implication, and thus deduction, starts from.

The perplexity is tangible. They reflect, and clarify that in reality the goal of the middle school is not to arrive at proofs. However, in their opinion, classifying the various geometrical figures is useful for getting the pupils used to appropriate use of language; the set theory representation is used in many contexts. In this way the pupils get used to translating the set theory scheme into phrases ("if it is in the boundaries of the squares then it is also in the boundaries of the rectangles", "it is necessary to be 'in the rectangles' in order to be 'in the squares'" etc).

We do not take the point too far, but it seems to us that they do not grasp the concept of deduction hidden in their activity.

Synthesis and coming developments

We can affirm that the secondary school teachers we refer to in the 'prologue' phase, fully recognize level 4 as a goal of their teaching and as a result the previous levels too take on special significance, they map out a way forward. Without the final level, the previous ones perhaps lose a little meaning.

The teachers we interviewed certainly recognized level 4 as an objective for secondary schools, but they do not see it as completion of the work done in the middle school. They have a different route in mind, which does not consist in the recovery of levels 1, 2 and 3; at least not in a series of steps based on the analysis of the properties of the figures. They prefer to begin with the geometric transformations (introduced through the Cartesian plane), continuing with the criteria of congruency and going on then to demonstrating some Euclidean propositions. In the first instance two of them saw our worksheets as *verification* for the propositions regarding rectangles and parallelograms. Only after reflection with the other colleagues did they realize that these worksheets should be handed out at the beginning and that they are a preparation for proof. They declared their interest in using them, but in their own way.

We are now collecting teachers' notes on the how worksheets have been used and their classroom notes on their use.

Up to now, it seems reasonable that these teachers' expertise, very much near to the spirit and the actions described for the middle school by van Hiele in 1958, might really facilitate the acquisition of level 1, 2, 3 as set out by van Hiele and Clements and Battista.

On the other hand, some teachers' observation confirms de Villiers (1994) comments regarding the teaching of a hierarchical classification of the quadrilaterals. Teachers affirm to have appreciated and actually appreciate Venn Diagrams in order to develop students' care and precision in using a hierarchical classification. They seem not to be conscious of the functional aspects offered by different kinds of classifications nor they assume the aim of developing such awareness in their students. Moreover, teachers (explicitly and clearly) state that the focus of middle school is not on properties; students are engaged in observing figures, in comparing different figures' properties but rarely are engaged in deducing properties from other properties or in identifying clearly a 'significant signal'.

Starting to use our worksheets they come back to their middle school way of teaching: starting with practical activities using meccano, or Castelnuovo equipments, or dynamic software Cabrì. Exactly as those secondary school teachers we referred to in the 'prologue' did during the previous research. But differently from those they conceive non-hierarchical classification very strange, almost dangerous, not apt for secondary teaching (nor for middle school).

We are, now, waiting that all the teachers have administered the worksheets, in order to analyse the protocols of the pupils and to discuss results and teachers impressions, so as to compare their methodology with that of their colleagues who have been teaching in high schools for many years and so as to look again at objectives. We plane to involve teachers (of both the two experiences together) in a two-sessions workshop for discovering or discovering again the value of a specific non-inclusive classification. A first session will be devoted to work theoretically; a second session will be devoted to come back to a practice level for projecting the general line of a curriculum.

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