A LENGTHY PROCESS FOR THE ESTABLISHMENT OF THE CONCEPT OF LIMIT

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We have set out below a part of a research project concerned with difficulties in learning the concepts of limit and infinity. Having taken account of the complexity and importance of such concepts, we consider a gradual approach is appropriate, seeking to develop an understanding from the first years at school, taking the first intuitions of primary school children as a starting point. In this context, we would emphasise the importance of approximation as a resource. Our research, as others in the field of difficulties in understanding the concept of limit, indicates the necessity for teaching development.

"And so any human knowledge begins from intuition, from there it goes on with the shaping of concepts and it ends with ideas" Kant.

1. Theoretical framework

The concepts of limit, continuity, derivative and integral of real functions are generally introduced in the last year or last two years of the secondary school cycle into all Italian high schools in a fairly formal way, enriched by technical details and the demonstration of theorems. The application of such concepts though, is often limited to routine exercises (calculations of limits, variation of functions …). As a consequence, learning is somewhat mechanical and superficial (Furinghetti Paola, 1991). The identification by many researchers of a variety of different difficulties and obstacles capable of hindering the process of the construction of the concept of limit is well known:

- Of an epistemological nature, due to reasons internal to mathematics itself (Brousseau, 1998; Sierpinska, 1985);
- Of a didactic nature, due to teaching methods which are not always effective (Brousseau, 1998; CREM, 1995; Groupe AHA, 1999; Artigue, 1998);
- Of a cognitive nature, due to the abstraction and conceptualisation processes involved (Cornu, 1991; Dubinsky, 1991; Sfard, 1992; Tall and Vinner, 1981; Tall, 1996);
- Of a meta-cognitive nature, due to the overall attitude with which students tend to approach maths generally (Zan, 2001, 2002).

Moreover there is widespread agreement among researchers in the field of maths education that, if the teaching of a concept is going to be effective, the pupil's intuitive knowledge acquired in an out-of-school context or in previous studies must not be discounted. Such knowledge may indeed be of determining importance in relation to facilitating or impeding the learning of the concept itself. In particular Fischbein (1973, 1987) refers to such intuitive ideas as "primary intuitions", emphasising the importance of sustaining and reinforcing them in such a way as to allow them to evolve into the "secondary intuitions" stage. In this way they constitute a good foundation for the acquisition and understanding of the concept involved.

Taking into account the underlined theoretical framework our group is particularly convinced on the necessity of a long term construction of the concept of limit starting from pupils' prior experience and intuition. We carried out an empirical research with the aim of investigate what kinds of intuitive ideas are present in the students of different school levels and how the teaching process can support or obstruct their development. In particular, we pointed out the presence of propitious intuitions about approximation, which are inopportunely neglected by the frequent didactical practise.

The main characteristic of our research group (named 0^0) is that all stages in schooling (primary – 6 to 10, middle school – 11 to 13 and high school – 14 to 19) are represented, both within the group itself and with respect to the pupils involved. This vertical cross section is a resource which is as rare as it is precious. It allows us, indeed, to propose similar activities at different levels and to check the evolution of the ideas, techniques and errors of the school-children over time, observing the effect of teaching methods on them. The members of the group are teachers-researchers working on the development of teaching calculus.

2. Later stages of the research

2.1 Student's "Beliefs and intuitions"

2.1.1 Students' pre-conceptions: some results

Linguistic considerations have a considerable part in mathematics teaching
(Pluvinage, 2002)

To test our initial hypothesis according to which, in the case of the concept of limit, the natural language register (to use an expression employed by Duval and Pluvinage), is unhelpful in the transition to the "mathematical register", we decided to conduct an investigation on the "linguistic" meaning given to the terms "limit" and "infinite" and, in addition to the use of a number of commonly used expressions such as "within limits" etc. Two questionnaires were drawn up based on the consideration that the terms "limit" and "infinite" are connected. The questions were open in nature in order not to influence the replies. As emphasised by Iacomella, Letizia, Marchini (1997), it is necessary to obtain information on what certain terms may evoke in schoolchildren's minds to be able to establish a link with their empirically-based background understanding. This should help to avoid the creation of that type of barrier to communication between teacher and pupil deriving from the subject itself, one of the worst of its kind.

The interviewee sample numbered a total of about 600 people including pupils (ranging from 14 to 19 years of age from different types of secondary and high schools) and adults without specialist mathematical knowledge (10% of the total). Pupils from a school specialising in the arts were also asked to provide a graphical representation of their ideas. This same question was then put to a number of third-year students from middle schools. The analysed results for the questionnaire on the term "limit" are set out below.

The open-ended nature of the questionnaire has undoubtedly complicated the interpretation of the replies. Considering in particular, the question – "explain what the term "limit" means to you" – (which to some extent incorporates the others), an attempt has been made to classify responses in accordance with the main idea expressed by the interviewee :

1. the idea of impediment, barrier, rule, restriction 44%
2. The idea of border, closing, … 30%
3. The idea of extremity … 19%
4. Other meanings (including the "mathematical" meanings) 3%
5. No reply 4%

The interpretation of the answers obtained may not be objective in character and should be considered as being more an indicator than scientific in the strict sense.

Taking the responses divided into age groups (14 to 17, 18 to 19 and >19), it can be seen that the idea of impediment (both "physical" and "moral") is more often found among the younger age groups and decreases with an increase in age.

The preceding question gave rise to the formulation of the conjecture according to which "the "strong" idea of a limit as a barrier represents a hindrance to the process of understanding of the concept of limit". It is a hindrance to the extent that it may give rise to at least two types of difficulty:

a) difficulty in relating limit to an idea of iteration process (and hence with a process that continues to infinity);
b) difficulty in accepting the possibility of an infinite limit.

The analysis of the questionnaire emphasises how a large proportion of pupils associate the term "limit" with an idea of "finite" or "finiteness" in time and space, something that leads them to consider limit as the opposite of the infinite.

In effect, the presence of a kind of opposition between the idea of limit and infinity appeared with great frequency in the response to the question "Explain what the word "infinite" means to you" in the second questionnaire: *Infinity is something that has no limits*.

2.1.2 Students' intuitions, some results

> *"It can be assumed that the teaching of a subject, in order to be effective, should be preceded by an exploration of pupil's intuitive knowledge"* (Fischbein, 1973)

The results of the research which we proposed in 2.1.1, pushed us to go deep into investigation of student's intuitions, moving from a linguistic to a mathematical context. In this perspective our group carried out a recent research (Dallanoce et al., 2000, Alberti et al. 2001) in order to explore intuitive knowledge of pupils of between 10 and 19 inclusive (300 pupils), relating to the concept of limit together with those connected to infinity, infinitesimal and continuity. The analysis of the results identified the existence of primary intuitions at all age levels considered, particularly in the younger pupils. On the other hand though, it was not possible to identify a noticeable evolution of such intuition whether in quantitative or qualitative terms. In other words, such intuition not only appears not to evolve over pupils' school career, but generally tends to degenerate. Starting from Fischbein' ideas and results, we tried to analyse the influence of frequent school practices on pupils' intuitions.

We have checked the following ideas:

- the same kind of intuitions referring to the concepts of infinity, infinitesimal and continuity are present in every age level, especially in the youngest students;
- during the school process no further development of these intuitions can be checked, on the contrary, in some cases a regression is possible.

Here we consider only two of the five cards we proposed to pupils (for more details see Dallanoce et al., 2000 and Alberti et al. 2001).

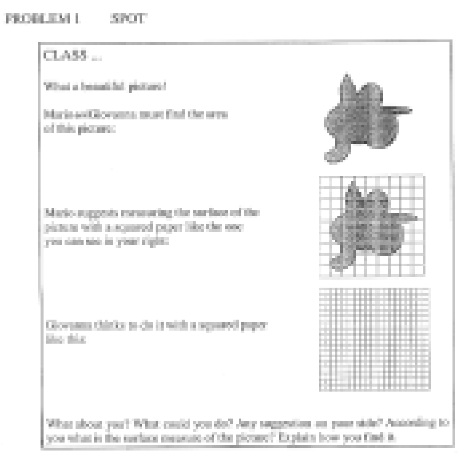


fig. 1

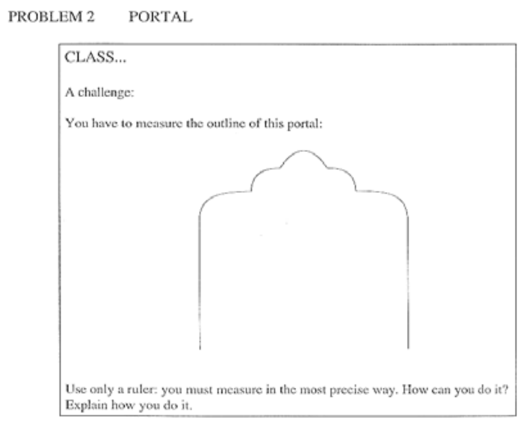


fig. 2

In those that are generically referred to as "spot card" (figure 1) and "portal card" (figure 2), the investigated subject is the approach of students to problems of measures (of either areas or lengths) than cannot be elementary solved.

Analysing the students' works, we could see that intuitions connected with the idea of "infinity" are present in any school level, particularly in the primary school. We didn't see, however, any evolution of these ideas, neither in a quantitative way (because intuitions are always present in a small percentage) nor in a qualitative way (because in any case we observe only primary intuitions, related to a concept of potential infinity). So, even if our sample can't be considered representative, we can confirm our hypothesis that these intuitions, in the didactic practice, do not find the chance to develop and to consolidate. Our results show also that a teaching frequently based on automatism can obstruct the production of "natural" and good intuitions.

For example, in the "spot card" and in the "portal card" only the youngest students took care to underline the approximation of the result: they used expressions like "the measure is about..", "approximately". In particular, Jacqueline (11 years old) realised that it is possible to find the length of the portal drawing inner and outer lines to approximate the portal (figure 3). On the contrary the oldest students gave only a number as solution (figure 4):

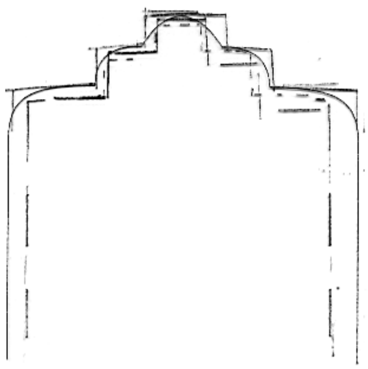


fig. 3

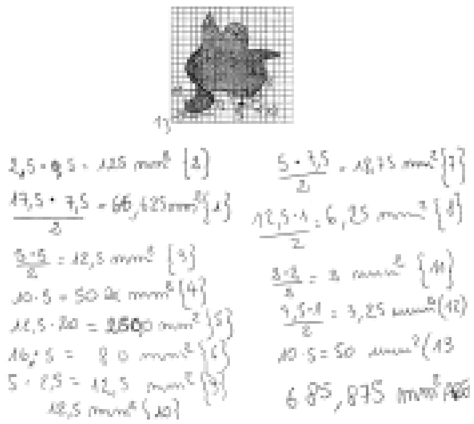
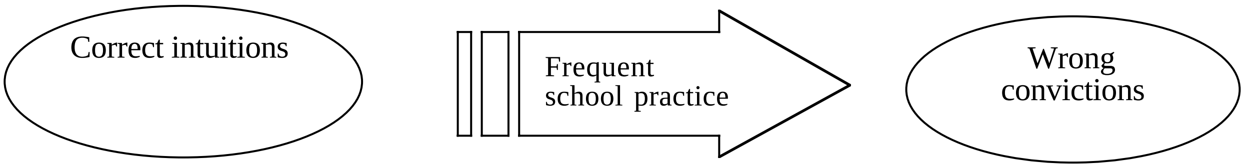


fig 4

The teaching methods used appeared to have had mainly the effect of introducing reliance on automatic mechanisms and to have switched pupils' attention to calculation and the search for the most appropriate formulae rather than concentrating on critical reasoning. We see here an obstacle of didactical nature (as considered in the theoretical paragraph).



On the basis of what we have seen above, it appears to us that the essential point is to identify teaching strategies and constructive activities capable of enriching the learning experience and stimulating evolution of intuitive understanding towards the theoretical concept of limit rather than concentrating on the formal aspect. It is hoped that pupils could use these tools to build on their own experience from their first years of school. We are aware however, that such evolution is far from easy or natural and that there are often very real conflicts between "ingenuous" ideas and intuitions and the mathematical concepts in formation with the consequent need, so far as the pupils are concerned, of a continuing re-organisation of their mental images (Mamona-Downs, 2001). We believe that the choices made by the teacher are of fundamental importance in sustaining or (involuntarily) impeding such process. The greatest risk is that of undervaluing pupils' convictions and attitudes towards themselves and their relations with mathematics which particular teaching techniques may cause to become real obstacles to learning.

2.2 Necessity for teaching development

In particular, it seems crucial to us that the work of the teacher should be concentrated on three levels as follows:

- Contents: with the need for much work designed to familiarise pupils with the concept of limit starting from the "ingenuous" mental images and then continuing with successive levels of abstraction. The basic materials for this are already present in the programmes applying to the various Italian schooling levels. The teacher should not however be too concerned to keep the problem "hidden" until it is formally defined;
- Methodology: it is appropriate to avoid "automatic" mechanisms, giving excessive importance to the calculation procedure and only proposing exercises and the application of special formulae or techniques and so fixing attention more on the result (product) than on the resolving strategy (process);
- Meta-cognition: appropriate to explain and discuss the students' attitudes and convictions; a critical awareness of their "implicit beliefs" appears to us to be an essential element in the conscious construction of their own knowledge and awareness and in their consequential ability to mobilise and re-invest them.

2.3 From mathematical theory to school practice development

So far as the first point is concerned, it is important to underline the fact above all, that when approaching the concept of limit requiring recourse to in act infinity, it is appropriate to do this through an explanation of potential infinity first. In this sense, approximation, seen as a possibility of an ever closer and testable approach to a theoretical limit, represents an excellent occasion for such an explanation.[1]

The activities we prepared in the ambit of our research, experimented in six classes at different levels within the school system, are designed to achieve the evolution of an intuitive idea of limit as a boundary or barrier towards the construction of a process which is capable of continuing improvement and *a-priori* testing.

3. Implication for school practice: Approximation as a teaching resource for the construction of the concept of limit.

[1] Another area where we have already conducted research and which we consider fertile ground in this context, is that of geometrical progressions (Grugnetti, Rizza et al. 2002).

As emphasised above, as pupils progress through the school system, approximation appears to lose its legitimacy and to become something extraneous, almost harmful, to mathematical activities. Teaching methods and a kind of implicit agreement seem to generate convictions in pupils that become increasingly wide-spread and well-rooted with increase in age.

One such conviction is that mathematics must necessarily be used explicitly when resolving any mathematical problem; in some sense the freshness of the younger school-children is lost with their readiness to have recourse to common sense or "empirical" solving strategies. In addition, for students each mathematical problem:

- a) is solved (and therefore must be solved) through a formula;
- b) has a result in round numbers (a whole or decimal number up to two decimal places).

It seems to us that in the Italian school system there is not sufficient insistence on the fact that for many mathematical problems there is no immediate formula (and sometimes never will be) and that often the result is not a round number. This does not mean that such problems are less interesting and significant. Their point lies not so much in the calculation of a "precise" result, it is rather the search for a testable and generalised approximation procedure.

Such "approximation methods", even though disdained by certain teaching techniques, constitute an essential aspect of analysis (calculus) in particular. Calculus above all, modelling the real world, cannot avoid the problems relating to approximation, connected to every measuring operation.

From a teaching point of view then, it is precisely because of approximation's empirical/experiential character that it can constitute a valid bridge for the construction of thought processes which, by analogy, proceed from the concrete to the abstract.

In particular, the definitions ε-δ of limit may be seen in terms of approximation, as the setting the degree of error one is prepared to accept a priori (ε) and choosing, as a consequence, an appropriate strategy (δ) to be sure not to exceed such a tolerance threshold.

We have identified two activities designed, on the one hand, to make pupils' convictions explicit concerning the points a) and b) (in the previous paragraph) and on the other, restoring approximation to its proper place in mathematics education. By the first activity we wanted to test if students: 1) accept to calculate the area of an irregular figure (a lake), 2) decide to use empirical methods rather than formulae and rules, 3) are conscious of the necessity of approximation for measuring the lake surface. By the second activity based on a "target game" we wanted to undermine the certitude in determinating in any case a measure precisely.

The analysis of the results of the two activities, proposed at different school levels (for more details see Falcade and Rizza, 2003), and interviews with students, indicates the necessity for teaching development in the domain of the teaching of calculus. In particular for the acceptance of methods of approximate resolution necessary for effectively solving some kinds of problems and, in the long term, for being able to perform the necessary transfer from potential infinity to infinity in act and finally for having the possibility for building the concept of limit.

Until now our empirical research had a diagnostic character: we found on the one hand some propitious intuitions (in the youngest pupils), on the other hand some wrong beliefs (in the oldest students). It means that something doesn't run in frequent school practice; it needs to start again from good intuitions and to grow them to begin a lengthy and constructive process for the establishment of the concept of limit. For this process it is important, in our opinion, to revalue the reasoning in terms of approximation as an essential and integrant part of mathematics and not as an expedient lack of exact methods.

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