PATTERNS OF FLEXIBILITY:
TEACHERS' BEHAVIOR IN MATHEMATICAL DISCUSSION

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Abstract

The significance of discussion in mathematics classes has become prominent in research literature. Different studies have stressed the importance of teachers' flexibility in discussion orchestration. We found it interesting to microanalyze situations in which teachers were flexible or not, to describe different patterns of flexibility, and to explain possible factors that influence teachers' flexibility in mathematics classes.

INTRODUCTION

In this study we focus on teachers' flexibility in the course of a whole-class mathematics discussion. We consider a teacher being flexible at a particular point of the discussion if s/he adjusts the planned learning trajectory in accordance with students' contributions that *differ from* those that s/he *expects* of them.

BACKGROUND

A child starts learning from the very first minute of his/her life. This learning is rooted in the child's interactions with the environment. School should systematize this learning process through activation of students' zone of proximal development (Vygotsky, 1978). Since communications trigger social thinking processes by which individuals construct their personal understanding (Yakel & Cobb, 1996; Lampert, 2001), discussion based on the students' need for shared meaning serves as a fruitful environment for an effective learning process. In the course of such a discussion students become aware of mathematical concepts and meta-mathematical structures, as well as evaluating their own knowledge, reorganizing their own mental models, broadening their collection of representations and problem-solving strategies, and deepening their understanding of the events under discussion.

Within the discussion on the environments in which a teacher learns to recognize students' ideas, on lesson structure, and on the teacher's role in the class, Mehan in 1979 introduced the Initiation-reply-evaluation (IRE) structure of the lesson and indicated recitation as the dominant type of teacher-student communications. In an IRE-structured lesson the teacher is the initiator and evaluator while the students are reactors. Voigt (1985) showed that in a recitation-type discussion the teacher manages students' learning *according to* his/her *expectations* by means of questioning, evaluation, voice tone, and other communicative means. In the last decade some studies demonstrated that IRE structure does not necessarily reflect every lesson structure (Forman & Ansel, 2001). It was implied that to be effective a mathematics discussion both has to be focused on important mathematical ideas and has to be of dialogical nature (Sfard et al., 1998), which promotes personal active construction of mathematical meaning through communicative processes.

Lampert (2001) describes in detail the management of a whole-class discussion in which a real dialogue between a teacher and the students occurs. In such a discussion the teacher makes significant use of students' ideas, constructs the discussion on the students' utterances, and encourages the students to explain and justify their ideas and conjectures. In this way a teacher often needs to change her/his plans and follow unexpected paths taken by the students. We consider these changes in teaching plans as an indication of a teacher's flexibility, and here we try to describe patterns of teacher's flexibility in mathematics discussions.

FLEXIBILITY: *Emerging from Simon's MATHEMATICS TEACHING CYCLE.*

Teachers' need to balance between managing the lesson according to students' ideas and questions and between focusing the discussion on particular mathematical issues complicates teachers' work in their classes. While a teacher directs his/her students towards multiple task-related communications a tension exists between his/her attempt to be flexible and reactive to the students' thoughts and needs and his/her propensity actively to manage the students' learning according to his/her plans and purposes (Simon, 1997; Cobb & Mcclain, 2001). Guiding students toward particular answers may limit development of students' individual ideas. On the other hand, if a teacher does not have an agenda, a lesson may become chaotic and the learning process will not advance (Brousseau in Simon, 1995). The teacher's task is thus to choose mathematical problems for class work, determine the way in which the students will work with the problem, and plan challenging questions to stimulate students' work on the problem and the ways of supporting students' initiative. S/he must also decide when and how to change the original plan in accordance with students' conjectures and ideas.

This tension between responding to students' ideas and managing lesson according to a teacher's plans and purposes depicted in Simon's (1997) Mathematics Teaching Cycle (see Figure 1). This cycle includes a hypothetical learning trajectory that the teacher constructs while planning a lesson with respect to (a) learning purposes, (b) learning settings and mathematical activities, and (c) possible learning processes in which the students may be involved.

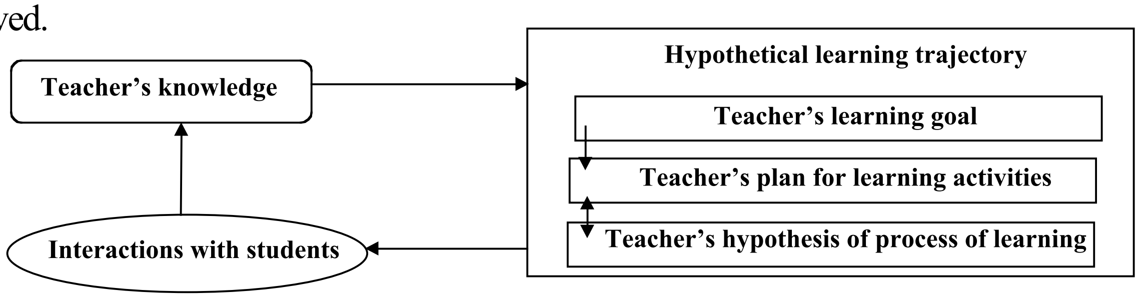


Figure 1: Mathematics Teaching Cycle (taken from Simon (1997), p.77)

Acting according to this model allows teachers being flexible while teaching. Through their interactions with the students teachers may learn about students' real needs and real learning processes in which the students involved. Teachers' flexibility is expressed in their adaptation of the initial agenda to these needs and processes. Additionally, the model includes the teacher's knowledge as one of the basic factors that affects planning and management of class mathematical discussion. In this way the teacher's knowledge develops through teacher's interactions with the students.

OPERATIONAL DEFINITION

In this study we will focus on a special kind of teachers' flexibility associated with situations in which students' replies are *unexpected by the teacher*. In terms of Simon, in this situation a teacher's hypothesis of learning process is incomplete. To describe teacher's flexibility we will compare teacher's plans regarding the lesson with the actual events and procedures that happen in the classroom. For this purpose we call hypothetical learning trajectory specified for one particular lesson "planned learning trajectory". The learning trajectory as carried out indeed including real events and procedure we call "actual learning trajectory". We consider it as mainly located in the "interactions with students" box of the Simon's model.

We consider ***a teacher being flexible*** at a particular point of the discussion if s/he adjusts the planned learning trajectory in accordance with students' replies that *differ from* those that s/he expects of them. In these cases the real learning trajectory includes elaboration of a student's contribution to the discussion and differs from the planned learning trajectory. If a teacher does not make such adjustments and/or impedes student's independent thinking we consider his/her behavior inflexible at that point.

THE PURPOSE

In the course of the class discussion a teacher has to assess the state of students' knowledge and the ways of students' reasoning. According to Simon's model this is an interactive constitution of classroom activities that joins the hypothetical learning trajectory to the teacher's assessment of students' knowledge. The first purpose of this study was to zoom in on the teacher-students interactions in the conditions of a whole-class mathematical discussion in order *to describe patterns of flexibility*. The second purpose was to analyze *how different types of teacher knowledge influence teacher flexibility,* along with Simon's argument as to the roles that different types of teacher knowledge play in the teacher's ability to assess students' knowledge. We also attempted to identify additional factors that influence teacher's flexibility in the course of the class discussion.

Space constraints of this paper allow us to present only a brief description of the patterns of flexibility.

THE STUDY

The teacher (Anat) and the teaching experiment: Anat is a secondary-school mathematics teacher, having six years of experience in junior-high and senior-high grades at different levels. Anat has a BA in mathematics and a teaching certificate. During the year of our experiment Anat taught for the first time according to the enquiry-based curriculum constructed on the basis of the "Visual Mathematics" program (CET, 1998). "Visual mathematics" is a secondary school function-based curriculum that comprises investigation procedures in a computer-based environment (Yerushalmy & Chazan, 2002). Before starting this teaching experiment Anat completed a special course for in-service mathematics teachers entitled Teaching Mathematics in an Inquiry-Based Classroom. During our intervention she taught 7^{th} grade students of middle and high ability.

Data collection and analysis: The data collection and analysis were on-going, using a qualitative approach. The theoretical concepts emerged from the complex interactions between theory and practice. We borrowed from Simon's Mathematics Teaching Cycle (Simon, 1995, 1997) the notion of hypothetical learning trajectory and teachers' flexibility. Furthermore the model influenced the research procedure according to which the data was collected in triads (as detailed bellow) correspondingly to three elements of the Mathematics Teaching Cycle (Simon, 1997, p. 77). The research findings were formulated and further revised and verified throughout the research. The analysis process was inductive all over the period of data collection, i.e., the ideas that were generated through the initial data analysis were further experimented and examined by repeating analysis of the same data and by the analysis of new data recorded.

Data collection in this study lasted eight months. The data were collected in triads (as mentioned above) of planning, teaching in the classroom, and stimulated recall. The three elements of each triad were connected by a particular lesson. All the lessons chosen for the investigation included a whole-class discussion. All the data were video-recorded and transcribed. Additionally the researcher took written field notes while collecting the data. When analyzing the data we performed multiple observations of the videotapes and careful reading of the transcripts.

At the planning stage Anat discussed with Sariga (one of the authors of this paper, who has experience in teaching the Visual mathematics curriculum) her plans for the lessons, asked questions, and answered the researcher's questions. Through analysis of the data collected at the planning stage we tried to identify Ann's planned learning trajectory, her mathematical background corresponding to the lesson, and her expectations regarding the learning processes in which the students would be involved.

The teaching-in-the-classroom stage included collection of data on the actual events and procedures that took part during the lesson in general and during the whole class discussion in particular. This stage was aimed to identify the actual learning trajectory.

The actual learning trajectory was further compared with the planned learning trajectory. Our analysis focuses on the teacher's behavior in the cases where students' replies differed from those that Anat had expected. In such cases similarities and differences between the two trajectories indicated Anat's flexibility or inflexibility. Instances that exemplified Anat's (in)flexibility were chosen for further analysis by the "stimulated recall" procedure.

At the stage of stimulated recall, based on chosen episodes, Anat was asked to discuss the lesson and to analyze how and why her plans coincided or did not with the real management of the lesson. She explained to the researcher her reasons for the decisions taken in the course of the whole-class discussion. We sought to identify factors that influence teacher (in)flexibility using the data collected at planning and stimulated recall stages.

All through the data collection and the data analysis Anat was not aware of the research purpose and the research hypothesis. The second author of this paper performed collection and the initial analysis of the data and chose the episodes for the stimulated recall procedure. These episodes were further analyzed cooperatively by the two researches. The results of this analysis were negotiated and refined.

RESULTS

We found four main patterns of flexibility: (1) Different outcomes, (2) Different strategies, (3) Different sequencing, and (4) Different scopes. Table 1 depicts and outlines each of these patterns. We realized that patterns (1), (2), and (3) rest on the mathematical nature of the problem, and the patterns' names reflect this dependence. Now we describe each pattern in detail.

(1) Different outcomes: This pattern of flexibility represents teacher-students interactions associated with an unexpected student's reply which lead to a learning trajectory that ends differently from the one planned by the teacher (let us call the expected reply *e* and the unexpected reply *u*: see Figures 2a and 2b in Table 1).

(2) Different strategies: This pattern of flexibility represents teacher-students interactions associated with an unexpected solution strategy/explanation suggested by a student (Figures 2a and 2b in Table 1). This pattern refers to different solution strategies/explanations (*e* and *u*) leading to identical results. Like the previous pattern, a teacher may expect students to use solution strategy *e* while students' replies direct the discussion to solution strategy *u*.

Both for patterns (1) and (2) there are several possibilities for unexpected replies:
(i) It may appear in cases when the mathematical task is open enough to allow variations in the solution strategies or outcomes to the problem and the teachers is not aware of the revealed variations.
(ii) It may appear in cases when the student's solution strategies or outcomes are incorrect and the teacher does not expect such mistakes.

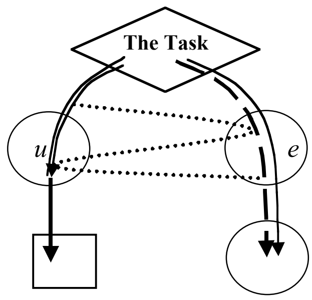
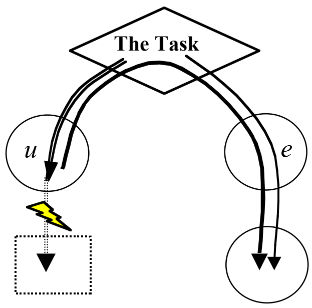
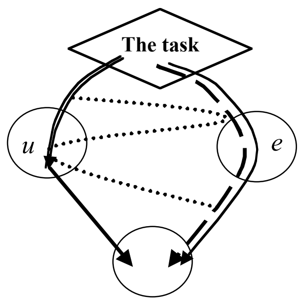
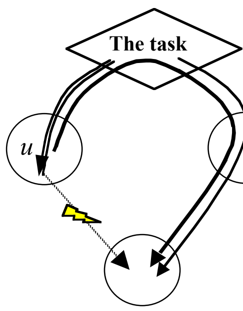
In some of these cases *e* and *u* belong or lead to the same mathematical topic. In other cases *e* and *u* belong to different mathematical topic, while *u* may open a discussion of a new topic or disclose need in a repetition of the topic learned beforehand.

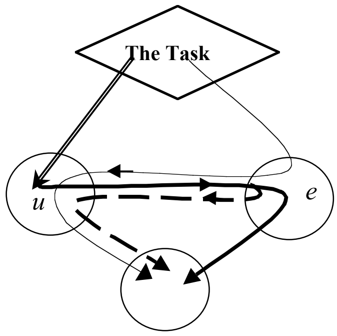
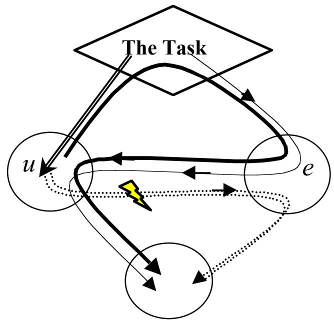
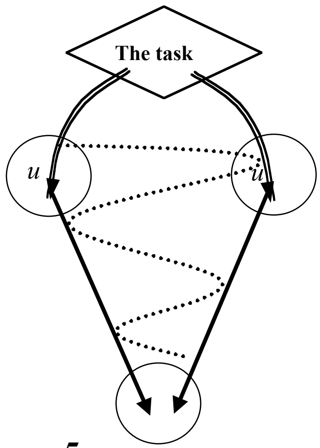
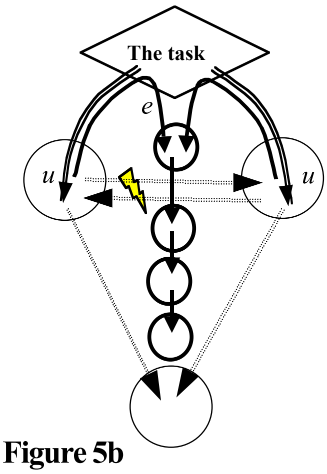
In a flexible mode, for both pattern (1) and pattern (2) (Figure 2a, 3a in Table 1), the teacher supports students' suggestions and leads the discussion in the *u* direction. In this case s/he may later return to the planned trajectory and discuss connections between the two (see dotted line in Figures 2b, 3b in Table 1). In an inflexible mode (Figures 2b, 3b) the teacher returns to the task and directs the students toward solution *e.*

(3) Different sequencing: This pattern of flexibility represents teacher-students interactions associated with a connection between equivalent properties of mathematical objects, whose direction is opposite to the one expected by the teacher (Figures 4a and 4b in Table 1). This pattern may appear for a mathematical task that focuses on connections between different mathematical properties, and the students are asked to construct logical relationships between these properties. A teacher may plan to construct connection between the properties (let us call them e and u) in a particular direction $e \Rightarrow u$ while students may consider an opposite direction $u \Rightarrow e$. In a flexible situation the teacher continues in the students' direction and helps students to construct the logical chain $u \Rightarrow e$. Later he/she may return to the planned trajectory and direct the students in the planned direction so that they will discuss equivalence of the properties $e \Leftrightarrow u$. In an inflexible mode a teacher returns to the task and directs the students toward the chain $e \Rightarrow u$.

(4) Different scopes: This pattern of flexibility represents teacher-students interactions associated with questions /conjectures that are "bigger" than those that the teacher deems possible for discussion in the particular classroom. (Figures 5a and 5b in Table 1). He/she may regard these questions as not clear enough for most of the students. The pattern may represent a students' debate over two different conjectures that have the potential for a progressive discussion. In the flexible mode the teacher gives free rein to the discussion, and the "size" of the steps by which the discussion proceeds depends on the students' reasoning only. In the inflexible mode the teacher "separates" students' queries into a chain of "smaller" questions and manages the students' discussion step by step.

Table 1: **Patterns of teacher flexibility**

Type of Behavior / **Pattern Type**	**Flexible**	**Inflexible**
(1) Different outcomes: **A.** Students' replies open opportunities of moving toward new mathematical topic **B.** Students' replies open opportunities for a solution to the problem that ends differently from the one planned by the teacher	 **Figure 2a**	 **Figure 2b**
(2) Different strategies: Students' replies open opportunities for solution strategy/explanation different from that planned by the teacher. In these cases different strategies/ explanations are leading to identical results	 **Figure 3a**	 **Figure 3b**

(3) Different sequencing: Students' replies open opportunities for a connection between equivalent properties of mathematical objects, whose direction is opposite to that planned by the teacher.	 **Figure 4a**	 **Figure 4b**
(4) Different scopes Teacher considers questions/ conjectures raised in the course of discussion to be overly "big"	 **Figure 5a**	 **Figure 5b**

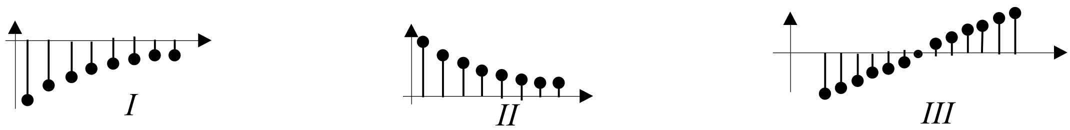
Planned trajectory — Actual trajectory — Alternative continuation trajectory

Students' reply — Possible unperformed trajectory — ***e*** – expected, ***u*** - unexpected

EXAMPLE

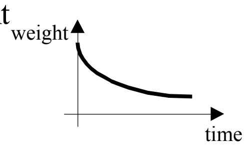
In this section we illustrate teacher's flexible behavior corresponding to one of our patterns [*Different outcomes*].

The problem: A slimming program plans to publish an ad in the journal For Women. Which of the three graphs representing the measure of change would you recommend be chosen in order to increase the number of clients registering for the program? (CET, 1996, p.52)



At the first stage of the discussion students suggested using Graph *I*, as Anat expected and planned. A student (Maya) explained:

"If the horizontal axis represents time and the vertical axis represents weight, then somebody loses weight if the measure of change is negative. First they lose weight quickly and then slower".



The corresponding graph of weight was drawn on the blackboard.

Surprisingly one of the students (Aviv) suggested using Graph *II.* Anat asked him to explain his answer.

Aviv: I chose [graph] II because you may take the rate of losing weight instead of rate of changes in weight. They [the task] did not say it should be weight.

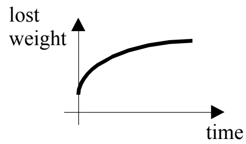
Other students (together): It [graph II] is the number kilos lost.

Anat: Just a minute. It [losing weight] depends on time… Dependence on time. Now, do you say that it is the rate of losing weight?

Anat seems to be saying this not only to the class but to herself – to be sure she understands what Aviv was saying

Aviv: It [graph II] shows how much weight he has lost.

…. It first decreased quickly and then slowly.



Anat: Okay. How would you draw the graph?

Aviv: It is concave and rising.

This stage was followed by the whole-class discussion in which other students expressed their opinions regarding different answers and expressed their preferences. During the stimulated recall meeting Anat reported that Aviv's answer surprised her, and she found it interesting to demonstrate to the other students that a different answer may be correct. Her flexibility in this episode matches Pattern (1), as depicted in Figure 6.

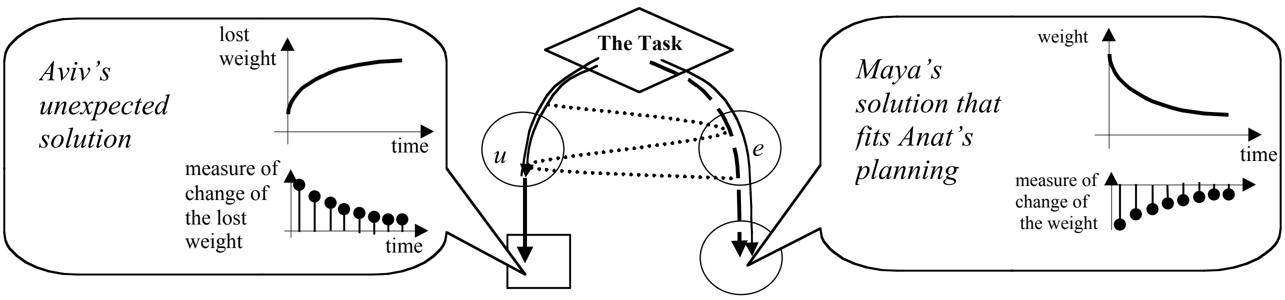


Figure 6: An Example of *Different Outcomes*

DISCUSSION AND CONCLUDING NOTES

We consider our analysis of teacher practice as an attempt to elaborate a theory of mathematics teacher flexibility initiated by Simon (1995). Simon's mathematics teaching cycle was constructed through analysis of the behavior of mathematics teacher educators. Flexibility in his model was associated with changes in the hypothetical learning trajectory based on the teacher-educator's reflection-on-action, that took place after each lesson (i.e., meeting with the teachers). We applied the idea of the mathematics teaching cycle to the micro situations in a junior-high school classroom in order to develop a theory of teacher flexibility-in-action that corresponds to teacher reflection-in action. To build our model we zoomed in on each of the components of Simon's mathematics teaching cycle (see Figure 1). Wee zoomed in on the teacher's plans and expectations to identify planned learning trajectory. The actual learning trajectory was described by zooming in on teacher-students interactions during the lesson. We detected episodes of teacher (in)flexibility by comparing between the two trajectories. Flexibility was determined by changes in the planned learning trajectory made by the teacher as almost immediate reaction to students' replies. That's why we referred to these changes as based on the teacher's reflection-in-action. Finally wee zoomed in on the teacher knowledge both during the planning stage and in the stimulated recall procedure in order to identify factors that monitor her decision-making.

Our model specifies four patterns of flexibility. We consider the first three patterns of flexibility [*Different outcomes, Different strategies, Different sequencing*] to be mainly of a mathematical nature. These patterns are hidden in the mathematical nature of the problems that allows construction of different learning trajectories. On the other hand, we consider pattern (4) [*Different scopes*] mainly pedagogic in nature as based primarily on the teacher's awareness of his/her students' knowledge. The model suggested in this paper is dynamic in the sense of interconnectedness of the four patterns. We found that continuous sequences of the patterns may appear in a particular lesson episode. Teacher inflexible behavior of one type may involve any other type of inflexibility. Teachers' flexibility of a particular type may either be supported by flexibility of another type or be broke off by inflexible behavior of any type.

We believe that these patterns describe most of the classroom situations in which a teacher behaves flexibly or not. Our study was performed in the framework of "Visual Math" curriculum, which includes inquiry-based mathematical tasks, in a classroom of middle and high ability students. Combination of these two factors ensured amount of students' replies unexpected by the teachers. However we assume (and start to see in the validation procedure right now) that our patterns of flexibility are not curricular-dependent. In any mathematical classroom a teacher may meet unexpected (correct and incorrect) students replies and behave (in)flexibly according to our patterns. Probably, based on different curricular ideas other patterns of flexibility can emerge. Still we did not identify adjustments in planned learning trajectories that could not be associated with any of the four patterns suggested herein.

Definition of flexibility given in this paper served for the research purposes of identification of patterns of flexibility. Usually one finds the teacher's behavior flexible in any situation in which the teacher works in accordance with students' contributions. In this way in our observations of the lessons we saw Anat being interactive, sensitive to the students, and working along with the students' ideas, conjectures, and questions in many situations. However, most of such episodes were not analyzed in this study since students' replies were expected by Anat. Our definition of flexibility is based on the correlation between teacher's plans and the actual events and procedures that take place during the lesson in order to understand when, how and why a teacher changes (or does not) the planned learning trajectory. For this purpose, we focus only on situations in which students replies are unexpected by the teacher and, thus, reduce the space of situations in which a teacher may be flexible in common sense. Therefore we would like to call the flexibility defined in this paper "absolute flexibility" (analogously to the notion of "absolute maximum" of a function on a particular interval). This notion helps us express the strength of the teachers' flexibility as defined in this paper. A teacher with high mathematical and pedagogical expertise will meet surprising and unexpected students answers seldom. Thus this teacher will be hardly either "absolutely flexible" or inflexible, this teacher will be flexible in the common sense.

We need to mention here that we believe that a teacher may have reasonable considerations for his/her inflexible behavior. We found that teacher's inflexible behavior may be either intentional or unintentional. We consider inflexibility unintentional if a teacher does not notice an unexpected learning trajectory hidden in the student's reply. We consider inflexibility as intentional when a teacher is aware of a learning trajectory different from one s/he performs that comes out from students' replies. In some of these cases the teacher may decide that this new learning trajectory fits better one of his/her future lessons. For example, this decision may take place when the teacher feels that the new topic/explanation is overly difficult for the particular class at the certain stage of learning. In other cases the teacher does not make changes in the planned learning trajectory, as s/he feels uncertain about teaching an unexpected learning trajectory suggested by a student. Such uncertainty may be based, for example, on the teacher's unfamiliarity with a particular problem-solving strategy suggested by a student.

Along with Simon (1995, 1997), Ball (1992), and Lampert (2001), we found (but have not described it in detail in this paper) that the teacher's knowledge of different kinds – mathematical, pedagogical, and curricular – influences his/her (absolute) flexibility. Additionally, this study realizes that even when a teacher is unfamiliar with an alternative solution to a problem there are mechanisms that help him/her behave flexibly. For example, teacher-students communications with students during the stage of students' individual (or small group) work, which result with teacher acquaintance with different students solutions before the discussion phase of the lesson, were found among the factors supporting teacher's flexibility during the whole-class discussion.

The last but not least important, we suggest that developing this theory may contribute to practice. It may endow the debate of fundamental tension of a desired outcome from the teaching versus enabling students' own directions of thinking and of an appropriate balance for effective learning. The patterns may be used by teacher educators for framing discussions of teacher flexibility in mathematics classroom.

The patterns may help teachers plan and analyse their own flexibility. "Having the patterns in mind" a teacher may try to construct various hypothetical learning trajectories when planning a lesson, to search for different solution strategies, and different solution outcomes. In this way patterns may help develop teachers' mathematical knowledge and teachers' knowledge of students. Knowledge of the patterns may enhance teachers' attention (Mason, 1998) during the lesson. The patterns may contribute to reflection-in-action process: being aware of the patterns a teacher may grasp his/her inflexibility while teaching and consciously change the planned learning trajectory. Finally the patterns may serve as an effective tool for a lesson analysis. Lesson episodes may be examined in the light of the patterns of flexibility suggested in this paper and may help teachers be reflective and creative in their work.

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