CHANGING TEACHERS' BELIEFS ABOUT STUDENTS' HEURISTICS IN PROBLEM SOLVING

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Abstract

We report on transformations of teachers' beliefs about students' heuristic strategies in problem solving. Twenty in-service teachers responded to questions on their and their students' heuristic experience. Then two of them took part in a six-month teaching experiment focused on heuristic training of their 8-graders. We found that the teachers' considerations on usefulness of particular heuristics and their pedagogical applicability were changed while learning through teaching. They also developed and used in their instructional practices an empirical scheme of students' problem solving behavior.

Introduction

Problem solving traditionally plays a prominent role in mathematics and mathematical education. According to NCTM (2000), solving problems is not only a goal of learning mathematics, but also a major means of acquiring effective ways of thinking in unfamiliar situations, i.e., heuristics.

Teaching from problem-solving perspectives means either a personal competence in mathematical problem solving or using various know-how, which enable students to think for themselves (Lester, 1985).

Teachers' beliefs about students' ways of thinking greatly influence their instructional practices (e.g., Chapman, 1999; Nathan and Koedinger, 2000; Leikin, Berman and Zaslavsky, 2000; Gates, 2001).

Chapman (1999) noted that there is ongoing under-representation in the literature of studies on teachers' development in problem-solving instructions.

To carry out the study reported here, we have been influenced by the following consideration of Nathan et al (2000):

"Any improvement in our understanding of teachers' views of the development of students' knowledge and problem-solving abilities strengthens our picture of the complexities of teaching and may ultimately enhance programs for teacher preparation " (p. 168).

In this paper, we examine changes in teachers' beliefs about students' heuristic strategies in problem solving. The study focuses on two main questions:

What, in the opinion of experienced in-service teachers, are the heuristics commonly used by high school students with different mathematical abilities?

How do the teachers' beliefs about students' heuristics change while engaged in research on teaching heuristic strategies?

Heuristics

Verschaffel (1999) considers heuristics as "…systematic search strategies for problem analysis and transformation" (p. 217).

Many heuristic strategies mentioned in the literature (e.g., Polya, 1973; Larsen, 1983; Schoenfeld, 1985; Verschaffel, 1999; NCTM, 2000) are described empirically or accompanied by mathematical examples. In order to ask teachers with different problem-solving experience understandable questions, it was important to use unified and as much as possible free-of-context names and descriptions of strategies.

Table 1 contains slightly reduced descriptions of heuristics used in the current inquiry. The formulations had been validated by two mathematical educators and tried out in analysis of the protocols of thinking-aloud interviews in the framework of the larger study (Koichu, Berman & Moore, in progress - a).

The strategy	Explanation
1. Search for special cases, patterns, or symmetry	Attempts to solve an easier problem than a given one, i.e. examining special cases, patterns or exploration of symmetry.
2. Drawing a picture	Pictorial description of a problem by means of a figure, a diagram, or a graph.
3. Formulating an equivalent problem	Attempts to reformulate the problem into an equivalent but simpler form, i.e., choice of effective notation.
4. Finding what is easy to find	Search for additional data, which can be derived easily from the given.
5. Planning: thinking forward, including division into sub problems from given data to conclusion or vice versa	Evaluating if it is worthwhile to use a particular problem solving method before doing it, including dividing problem into a small number of sub problems either assuming a conclusion or starting from the given.
6. Arguing by contradiction	The assuming that the conclusion is not true and then drawing deductions until arriving at something that is contradictory either to what is given or to what is known to be true.

7. Ignoring a particular given datum	Attempts to think on the problem ignoring a particular given datum in order to highlight its role in a problem.
8. Introducing an auxiliary element	Attempts to deal with the problem by bringing in an auxiliary element that wasn't mentioned as "given", i.e., change of variable, auxiliary construction etc.
9. Generalization	Attempts to solve a more general problem than a given one, when it may simplify the solution.
10. Reasoning by analogy	Using the structure of a known problem to help solve a new one.
11. Self-feedback: thinking backward	Evaluating if it was worthwhile to use a particular problem solving method after doing it, work check-up.

Table 1: The heuristics used in the inquiry

The study

Heuristic questionnaire

Twenty in-service high-school teachers (on average, 19 years of pedagogical experience, M.Sc. degree) took part in a 120-minute workshop dealing with heuristic strategies. One of the authors acquainted the teachers with the formulations, similar to those presented in Table 1. Then the teachers were divided into small groups and encouraged to formulate mathematical problems that provide examples of using a particular strategy. All the examples were presented in front of the entire group and discussed from the heuristic point of view. When the teachers reached an agreement on what every strategy means, they individually filled in the Heuristic questionnaire. In the questionnaire, the participants were asked:

A. To add their own heuristic strategies, which probably weren't mentioned among the eleven listed in Table 1.

B. To divide 11 or 11+ strategies into three categories: from "*very useful*" marked "1" to "*less useful*" marked "3", based only on personal problem-solving experience. Each category had to include at least three strategies.

We have chosen such a structure of this question in order to avoid a situation when the participants, evaluating themselves in the field of their expertise, could state that they "use everything".

C. To mark each of the strategies by one of the five numbers: from "4" for the most useful strategies to "0" for the useless ones, based on taken separately problem solving experience of their "strong", "ordinary" and "weak" students.

D. To answer the following questions:

Is it possible to improve mathematical achievements of your "strong", "ordinary" and "weak" students by purposeful (not only intuitive) teaching of heuristic strategies? If yes, how can one teach them?

Observations and interviews

Two of the teachers and their 8-graders volunteered to take part in a six-month teaching experiment. In its framework, a regular mathematical curriculum was taught in two classes with emphasis on heuristic aspects of problem solving. Along with Anna and Larisa (the teachers), we chose to concentrate mostly on "*Planning…*" and "*Self-feedback…*" strategies, while teaching the following topics: "Quadrilaterals", "Pythagorean theorem" "Abridged multiplication formulas", "Quadratic equations", "Identical expressions including fractions" and "Word problems".

Once a week, along with the teachers we carefully planned their classroom activities for next week term. We also conducted non-participant observations in the two classes, and afterwards discussed the lessons – with each teacher separately. The teachers also were encouraged to attend thinking-aloud interviews with their students. At the end of the experiment, a videotaped interview with Anna and Larisa was carried out.

Findings

Only one additional strategy was suggested as a response to question A of the Heuristic questionnaire, namely, "*Guess how a problem was composed*". The teacher considered this strategy to be useful for her "strong" students in mathematical Olympiads. The rest of the teachers considered that the eleven heuristics of Table 1 cover their and their students' heuristic arsenal. In four questionnaires we found some written responses about possible fields of application of particular strategies. Commonly, all the heuristics were treated as universal, useful either in algebra, or in geometry.

In Table 2, we present a distribution of 20 teachers' responses to questions B and C by means of means and standard deviations (SD). Averages in the second column (teachers as problem solvers (PS)) were computed for the scale 1-2-3. Here smaller numbers point on more popular strategies. Averages in the third, forth and fifth columns (teachers' beliefs about their students' strategies) were computed for the 0-1-2-3-4 scale. Here higher numbers point to more useful strategies for students, from the teachers' point of view.

The strategy	**Teachers as PS** **Mean** (SD)	**Students as PS** **Mean** (SD)		
		"Strong"	**"Ordinary"**	**"Weak"**
1. Search for special cases,	**1.42** (0.59)	**3.21** (1.27)	**2.74** (1.24)	**2.16** (1.71)

patterns, or symmetry				
2. Drawing a picture	**1.45** (0.59)	**3.55** (0.76)	**3.25** (0.91)	**2.50** (1.64)
3. Formulating an equivalent problem	**1.80** (0.68)	**3.16** (0.90)	**2.33** (1.33)	**1.32** (1.63)
4. Finding what is easy to find	**2.05** (0.67)	**2.58** (1.26)	**2.21** (1.27)	**2.05** (1.76)
5. Planning: thinking forward	**1.30** (0.46)	**3.36** (0.50)	**2.50** (1.25)	**0.95** (1.23)
6. Arguing by contradiction	**2.75** (0.43)	**2.58** (0.90)	**1.05** (1.13)	**0.32** (0.82)
7. Ignoring a particular given datum	**2.47** (0.75)	**2.06** (1.30)	**1.11** (1.18)	**0.44** (0.62)
8. Introducing an auxiliary element	**1.85** (0.87)	**2.68** (1.07)	**1.58** (1.02)	**0.58** (0.69)
9. Generalization	**2.05** (0.80)	**3.37** (0.50)	**1.68** (1.20)	**0.63** (0.68)
10. Reasoning by analogy	**1.35** (0,65)	**3.45** (1.00)	**3.11** (0.99)	**2.47** (1.39)
11. Self-feedback: thinking backward	**2.05** (0.67)	**2.47** (1.12)	**1.74** (0.99)	**0.84** (0.96)

Table 2: Distribution of teachers' beliefs about heuristics of themselves and of their students

Relatively larger values of SD point to a variety of teachers' problem solving preferences, heuristic styles and beliefs.

One can see that teachers, as problem solvers, believe in the usefulness of "*Planning...*", "*Reasoning by analogy*", "*Search for special cases…*", and "*Drawing a picture*". The heuristics "*Formulating an equivalent problem*" and "*Introducing an auxiliary element*" are still useful. "*Arguing by contradiction*" is the least popular strategy.

In the teachers' point of view, their "strong" students are better in using all the strategies than "ordinary" ones, and "ordinary" are better than "weak" students, but all of them mainly use "*Drawing a picture*", "*Reasoning by analogy*" and "*Search for special cases…*". In addition, "strong" students think of a problem by means of "*Generalization*", "*Planning...*" and "*Formulating an equivalent problem*". "*Arguing by contradiction*" and "*Ignoring a particular given datum*" are marked as the least useful strategies for all the categories of students.

The largest differences between "strong" and other students are in "*Generalization*", "*Arguing by contradiction*" and "*Planning...*". Teachers also believe that "weak" differ from "ordinary" students mainly in "*Planning…*", "*Generalization*" and "*Formulating an equivalent problem"* strategies.

Twelve of the teachers answered question D. All of them considered that the planned (not only intuitive) teaching of heuristic strategies is a worthwhile tool of improving students' mathematical achievements. A few teachers (including both who participated in the teaching experiment) believed that heuristic training might mainly help their "strong" and "weak" students. They suggested to teach heuristic "*by examples*", "*in problem solving*", "*by thinking aloud in front of the class*" and "*as integral part of teaching mathematics*".

The following body of data was gathered from the teachers who took part in the teaching experiment.

At the beginning of our cooperation, Anna and Larisa did their best following the plans we developed for them. They began to suggest some mathematical tasks for heuristic-oriented activities from the fourth week of the experiment. After three months, Anna and Larisa prepared about 50% of experimental lessons by themselves. Commonly, a particular idea or a task discussed in a heuristic-oriented lesson was expanded to the next lessons, and then our once-a-week interventions influenced the entire teaching during the experiment.

In Larisa's words from the interview:

"*We started to prepare the lessons, only you at first, afterwards together… At the beginning it works sometimes better, sometimes worse, but after a short period of time there were many very successful activities in succession… I often started some activity in your presence, and continued it for one or two additional lessons. At the beginning, my students had distinguished between "your" and "my" activities, but after a month or two it wasn't important for them, it did not work like "*[with] *heuristics"* [in your presence] *and "without heuristics"* [in your absence]*, not at all*".

Sometimes an activity planned to emphasize the role of a particular strategy turned into a lesson on another one. Analyzing the protocols of non-participant observation and memory records of our discussions with Anna and Larisa, we found that both teachers intended to carry out the messages "*Think by analogy*" and "*Find what you can find*" along with "*Planning...*" and "*Self-feedback...*" in their explanations. They also learned to distinguish many strategies in students' problem solving, reflecting either on classroom activities, or on thinking-aloud interviews with the students. In the words of Anna,

"*When we started to work with you, I activated the strategies in my head, I understood them better*".

An appendix contains three fragments of the interview with Anna and Larisa that highlight their beliefs about students' heuristics at the end of the teaching experiment.

In the terms of the Heuristic questionnaire, they consider (in order of preference) the prominent role of "*Reasoning by analogy*", "*Finding what is easy to find*" and "*Planning...*" in students' problem solving. Larisa also states that either "strong" or "ordinary" and "weak" students use the same strategies, but the analogies of stronger students may be more non-direct and sophisticated. Both teachers feel now that heuristic training helped mostly their "ordinary" students.

Summary

It is not surprising that experienced in-service teachers believe that they and their "strong" students use about the same heuristic arsenal. In the teachers' point of view, "*Planning...*", "*Reasoning by analogy*", "*Search for special cases…*", "*Drawing a picture*" and "*Formulating an equivalent problem*" are useful either for them or for their A-students. The strategies less useful for teachers were marked as less useful for students. That supports a consideration that many teachers intend to take into account pedagogical perspectives of problem solving, "…even though they were asked to respond from their personal perspectives, not necessarily as teachers" (Shir and Zaslavsky, 2001).

Learning through teaching (Leikin, Berman and Zaslavsky, 2000) leads to the more subtle understanding of students' cognition in problem solving. The empirical scheme "*Reasoning by analogy – Finding what is easy to find – Planning... – Another strategy"* was considered as acceptable for all the categories of students (see Appendix). The teachers found differences between problem solvers in the use of particular strategies. This gradually growing understanding was expressed in heuristic-oriented classroom instructions, in part developed by teachers without our assistance.

It is interesting to note that in spite of the fact that the teachers had appreciated the contribution of heuristic training mostly for the "ordinary" students, their "weak" ones got the greatest benefits. This was evident from the pre-post testing carried out in the framework of the larger study (Koichu, Berman and Moore, in progress - b).

In closing, we would like to turn to the question posed in NCTM (2000):

"...How should these [heuristic] *strategies be taught? Should they receive explicit attention, and how should they be integrated with the mathematical curriculum?" (p. 54).*

We believe that our study provides one of the possible answers: heuristic training of students may be an effective tool in combination with heuristic training of their teachers, induced either by personal problem solving experience or by learning through teaching of regular curriculum with deliberate emphasis on heuristic approach in problem solving.

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Appendix

Interviewer: *You said that during our common work you started to think what happens in the heads of your students during problem solving. So let us imagine a very motivated student who knows the material and is given a difficult mathematical problem. How is he* [or she] *thinking?*

Anna: *He* [or she] *is looking for theorems, which can be useful.*

Larisa: *A student has some associations with a particular drawing, solving a geometry problem. In algebra, he has associations with a known structure, or word-combination… He is looking for analogies that helped him in the past.*

Interviewer: *And what is he doing if his analogies do not help?*

Anna: *He will try something. Trial and error...*

Larisa: *And then he will try to use our strategies* [the strategies we taught].

Interviewer: *It is interesting, in what order…*

Larisa: *I think he will try to divide the problem into smaller ones, and to start with the easiest…*

Anna: *In order to do that he must understand how to start…*

Larisa: *No, no, may be he doesn't understand, and may be his choice* [of a starting point] *leads to nothing. Then he will try to handle another small problem… What is really important, the students began to make some stops and to think before doing. They didn't try something without evaluating if it is a worthwhile way. It was not the same before* [the experiment]……..

Interviewer: *And what are the differences between "weak" and "strong" students?*

Anna: *A "weak" student is looking for something similar* [done in the past], *and if he can't find, he can't solve.*

Interviewer: *Aren't "strong" students looking for something similar?*

Larisa: *They are, I suppose they use the same strategies, but for "strong" and "weak" students "something similar" means different things. For a "weak" student "similar things" are, for example, the same exercises with different numbers. For a "strong"* [one] *it is a direction, or a topic, that is, he is fluent in entire topic, and a "weak"* [student is fluent] *in few exercises…………*

Interviewer: *You said that heuristic training helped your students. In your opinion, who acquired more, your "strong" students or "weak" ones?*

Larisa: *I think my "ordinary" students gained the most. The strongest… I don't think they changed a lot, and "weak"* [students]*… the heuristic training is only a part of the treatment that can help them.*

Anna: *Yes, I agree. You know, there were no really "strong" students in my class. But for the most of students, not for the "worst", it* [heuristic training] *was a great deal.*

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