ELEMENTARY TEACHERS' MATHEMATICS CONTENT KNOWLEDGE AND CHOICE OF EXAMPLES

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The mathematics subject matter knowledge of elementary school teachers has in recent years become a high profile issue in the UK and beyond. This paper reports on one dimension of a videotape study of mathematics lessons prepared and conducted by pre-service elementary teachers. The aim was to identify ways in which their subject knowledge, or the lack of it, was evident in their teaching. One significant issue that emerged was the particular examples chosen and used within the lessons.

THEORETICAL FRAMEWORK

The complexity and multiplicity of the knowledge bases required for teaching is now acknowledged internationally. In former times, however, it was supposed that in order to teach something, it was sufficient for the teacher to know it for him/herself. In particular, conceptions of mathematics teacher knowledge consisted of understanding what teachers knew about mathematics. Aristotle wrote in antiquity: "… it is the sign of the man that knows, that he can teach…" (Barnes, 1984, p. 1553). Yet studies such as those of Begle (1968) and Eisenberg (1977) show that effective teaching in institutional settings requires more than personal mathematical competence.

The seminal work of Lee Shulman conceptualises the diversity of the knowledge required for teaching. His seven categories of teacher knowledge include three with an explicit focus on 'content' knowledge: subject matter knowledge, pedagogical content knowledge and curricular knowledge. Shulman (1986) notes that the ways of discussing subject matter knowledge (SMK) will be different for different subject matter areas , but adds to his generic account Schwab's (1978) notions of substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community). For Shulman, pedagogical content knowledge (PCK) consists of "the ways of representing the subject which makes it comprehensible to others…[it] also includes an understanding of what makes the learning of specific topics easy or difficult … (Shulman, 1986, p. 9). PCK is particularly difficult to define and characterise, but seems essentially to conceptualise the hitherto missing link between knowing something for oneself and being able to enable others to know it.

In 1998, the UK government specified for the first time a curriculum for Initial Teacher Training (ITT) in England (DfEE, 1998), setting out what was deemed to be the "knowledge and understanding of mathematics that trainees need in order to underpin effective teaching of mathematics at primary level". There is now a growing body of research on prospective primary teachers' mathematics subject knowledge, which has undeniably been facilitated by the necessity of some process of audit and remediation within ITT (e.g. Rowland, Martyn, Barber and Heal, 2002; Goulding and Suggate, 2001; Morris, 2001).

This paper is one outcome of ongoing collaborative work in this field between researchers at the universities of Cambridge, London, Durham and York under the acronym *SKIMA* (subject knowledge in mathematics). The conceptualisation of subject knowledge and its relation to teaching which informed the project has been detailed extensively elsewhere (Goulding, Rowland and Barber, 2002). The focus of the research reported in this paper is on ways that trainees' mathematics content knowledge can be observed to 'play out' in practical teaching during school-based placements.

METHOD

This study took place in the context of a one-year (three term), full-time Post-Graduate Certificate in Education course for prospective primary school teachers in a university faculty of education. The primary 'trainees' are prepared to be generalist teachers of the whole primary curriculum. In this particular course, each of the 149 trainees followed a route focusing either on the 'lower primary (LP)' years (ages 3-8) or the 'upper primary (UP)' (ages 7-11). About one month into the second term of the course, a 16-item audit instrument was administered to all the trainees, under semi-formal conditions These audits were marked by tutors, who gave individual feedback to trainees indicating any further targets for self-study.

For the purpose of this research, the total scores for each paper (maximum 64) were used to identify groups with 'high', 'medium' and 'low' scores. Two trainees from each subject knowledge category and within the two LP/UP groups were chosen for observation, although these categories do not feature in the analysis in this paper. Two mathematics lessons taught by each of the trainees were observed and videotaped i.e. 24 lessons in total. These took place approximately in the 5th and 7th weeks of the 8-week placement; school half term occurred between the two observed lessons. Trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson (usually the same day) the observer/researcher wrote a *Descriptive Synopsis* of the lesson. This was a brief (4-500 words) account of what happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser and Strauss, 1987). In his introduction to the papers of the CERME2 Working Group on social interactions in mathematical learning situations, Götz Krummheuer (2002) discusses this approach in terms of C. S. Peirce's notion of 'abduction'. Abduction is an explanatory heuristic by which hypotheses (or theories) are generated in order to account for observed phenomena, based on a "local methodology of discovery" (Pierce, cited in Krummheuer, 2002).

> In the comparative analysis such a local methodology of discovery can be seen. By comparing interpretations of different episodes on the one hand certain

> constructions of theory can be ruled out in case they do not match the interpretations. On the other hand such a comparison gives direction to a novel theoretical construction, as a confrontation of the initially employed theories shows their deficits. (*ibid*., p. 343)

In this way, we compared our interpretations of episodes from the 24 videotaped lessons. In particular, we identified aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by the trainee's knowledge of *mathematics* subject matter knowledge or *mathematics* pedagogy as opposed to other more general kinds of pedagogical awareness or expertise. Next, each of us focused in detail on about five videotapes, and elaborated the *Descriptive Synopsis* into an *Analytical Account* of each lesson. In these accounts, significant moments and episodes were identified and coded, with appropriate justification and analysis.

OVERVIEW OF FINDINGS: THE CODES

As a result of the process described above, 18 aspects of the trainees' teaching were identified, and a code assigned to each of them. The 18 codes identified *include*:

AC	anticipation of complexity	MC	making connections
AP	awareness of purpose	RCA	recognition of conceptual appropriateness
ATB	adherence to textbook	RCI	responding to children's ideas
CE	choice of examples	TU	theoretical underpinning
COP	concentration on procedures	UT	use of terminology
IE	identifying errors		

The code name should, in most cases, indicate the aspect of teaching intended to be associated with it. In principle, an event or episode in a lesson might be identified and coded as a *positive* or a *negative* instance of the any of the above aspects. Thus CE+, CE- would indicate respectively a good or poor choice of example(s). For other codes the existence of both possibilities is less apparent. The category CE has been selected for elaboration in this particular paper.

CHOICE OF EXAMPLES

The place of examples in mathematical didactics

In a theoretical analysis of the role of examples, Watson and Mason (2002) write:

> It has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques. Indeed, it is only through the examples that definitions have any meaning … (p. 379)

Reflecting on what it is that learners gain from examples, it is helpful to distinguish two rather different uses of examples in teaching. The first is essentially inductive - providing (or motivating students to provide) examples *of* something. The 'something' is *general* in character (e.g. the notion of line symmetry, or the fact that the sum of two odd integers is even); the examples are *particular* instances of the generality. The use of examples to embody abstract concepts and to general procedures is commonplace pedagogical practice. Thus, we teach a (general) procedure by a (particular) performance of that procedure. For example, if we set out to teach subtraction by decomposition, we might perform, say, 62-38 in column format. It is important to note that the 6, the 2, the 3 and the 8 were *all* chosen with care in the previous sentence. The provision of such examples is not an arbitrary matter. That is not to say that there is not usually some latitude in the choice of (good) examples. The 8 could have been a 9; on the other hand, it could not have been a 2. It *could* have been a 4, say, but arguably the choice of 4 is pedagogically less good than 8 or 9. Why? Because that would entail the learner in subtracting 4 from 12, a task that would require some pupils to engage in finger-counting, and distract them from the procedure they are meant to be learning.

In the case of concepts, the role of examples is to provoke or facilitate abstraction: once a set of examples has been unified by the formation *of* a concept, subsequent examples can be assimilated *by* the concept (Skemp, 1979). Once a concept has been formed and named by an individual, s/he is able to do something very remarkable - to entertain examples of it outside the realm of personal experience (Rowland, 1999, p. 27); Skemp (*op. cit*.) calls this psychological phenomenon 'reflective extrapolation'. A teacher's choice of examples for the purpose of abstraction will reflect his/her awareness of the nature of the concept and the category of things that it comprehends.

The second use of examples in teaching, more often called 'exercises', is not inductive, but illustrative and *practice*-oriented. We note here that exercises are examples, selected from a class of possible such examples. In the case of two-digit subtraction, 20 exercises might be chosen from the class of some 4000 possible examples. Why choose one subset in preference to another? Characteristically, having learned a procedure (e.g. to add 9, to find equivalent fractions, to find the 'difference' of two integers), the student rehearses it on several such 'exercise' examples. This is initially to assist retention of the procedure by repetition, then to develop fluency with it. Such exercises are also, invariably, an instrument for assessment, from the teacher's perspective. We recognise that such 'mere' practice might also lead to different kinds of awareness and comprehension (just as repeated rehearsal of the notes of a violin concerto might awaken new constructions of the 'meaning' of the piece). Again, the selection of such examples by teachers is neither trivial nor arbitrary. The argument for examples to be 'graded' is generally well understood, so that students experience success with routine examples before trying more challenging ones. Exercise examples 'for practice' will also ideally expose the learner to the range of types of problem that s/he might encounter from time to time.

For instance, practice examples on subtraction by decomposition (if we were to insist on teaching it) ought to include some possibilities for zeros in the minuend e.g. 205-87. Bierhoff (1996) has commented that English primary textbooks are poor examples of pedagogy in their provision of examples, compared with those in Germany and Switzerland, whose authors demonstrate far greater didactic awareness.

In both senses of the word, we suggest that the examples provided by a teacher ought, ideally, to be the outcome of a reflective process of *choice*, a *deliberate* and informed selection from the available options, some 'better' than others in the sense illustrated above. While we do not pretend to be able to infer such a process of choice, or the lack of it, from the evidence of the videotapes, we can comment on the examples actually chosen by trainees, and how they compare with available alternatives.

The Trainees' Choice of Examples

Whilst we looked for instances of both good and poor choices of examples, the latter seemed to be more prevalent. To redress the balance of this section, therefore, we begin by citing a somewhat isolated (though not unique) instance that seems to us to draw on some key aspects of mathematics content knowledge.

Naomi's lesson was with a Year 1 class. The main activity[1] was about the meaning of the 'difference' of two numbers within 20. Naomi had achieved the maximum score on the audit, yet there is little evidence of overt SMK in the lesson. The lesson would be rich material for a case study, but this is not our purpose here. Suffice it to note the following episode from the mental and oral starter, where the children practised bonds to 10. They sat in a circle, and Naomi chose particular individuals to answer questions such as "If we have nine, how many more to make 10?" The sequence of starting numbers was 8, 5, 7, 4, 10, 8, 2, 1, 7, 3. This seems to us to be a good sequence, for the following reason. The first and third numbers are themselves close to 10, and require little or no counting. 5 evokes a well-known double - doubling being an explicit NNS strategy. The choice of 4 seemed (from the videotape) to be tailored to one of the more fluent children. The degenerate case 10+0 merits the children's attention. One wonders, at first, why Naomi then returned to 8. The child (Bill) rapidly answers '2'. The answer to our question becomes apparent when Naomi asks the next child, Owen, what he must add to 2 to make 10. Owen counts from 2 on his fingers, and declares '8'. This somewhat drawn-out process proceeds as follows.

Naomi: Owen. Two.

(12 second pause while Owen counts his fingers)

[1] The National Numeracy Strategy *Framework* (DfEE, 1999) guidance effectively segments each mathematics lesson into three distinctive and readily-identifiable phases: the *mental and oral starter*, the *main activity* (an *introduction* followed by *groupwork*) and the concluding *plenary*.

Naomi: I've got two. How many more to make ten?

Owen: (six seconds later) Eight.

Naomi: Good boy. (Addressing the next child). One.

Child: (after 7 seconds of fluent finger counting) Nine.

Naomi: Good. Owen, what did you notice … what did you say makes ten?

Owen: Um … four …

Naomi: You said two add eight. Bill, what did you say? I gave you eight.

Bill: (inaudible)

Naomi: Eight and two, two and eight, it's the same thing.

Naomi's reason for asking the child after Owen for the complement of 1 in 10 (Naomi's third 'turn' above) immediately after Owen's question (about '2') is not apparent. It could be justified in terms of one less, one more, but Naomi does not draw out this relationship. Instead, Naomi returns to Owen, to ask whether he had noticed the last-but-one question and Bill's answer, adding "eight add two, two add eight, it's the same thing". Admittedly, the significance of Owen's example is lost on him, or has escaped his memory. Nevertheless, there seems to be some conscious design in Naomi's sequence. Her choice of examples (a) was at first 'graded' (b) included later an unusual/degenerate case, and (c) finally highlighted a key structural property of addition i.e. commutativity. She draws attention to this relationship yet again in her final choice of 7, then 3, and in her comments on this pair of examples.

Certain key categories of the trainees' choice of examples is beginning to emerge. One is the choice of examples that *obscure the role of the variables* within it. One such case concerns Michael, in a lesson with a Year 4 class. The main activity was about telling the time with analogue and digital clocks. One group was having difficulty with analogue quarter past, half past and quarter to. Michael intervened with this group, showing them first an analogue clock set at six o'clock. He then showed them a quarter past six and half past six. When asked to show half past seven on their clocks, one child put both hands on the 7. The child's inference from Michael's demonstration example (half past six) is clear. Of the twelve possible examples available to exemplify half-past, half past six is arguably the most unhelpful.

Another instance took place in a Year 6 lesson which began with work on co-ordinates. Kirsty began by asking the children for a definition of co-ordinates. (The place of definitions as opposed to examples is a topic in its own right, but not for consideration here). One child volunteered that "the horizontal line is first and then the vertical line." Kirsty then asked children to identify the co-ordinates of points as she marked them on a grid. She reminds them that "the x-axis goes first". Her first example is the point (1, 1), which is clearly ineffective in assessing the children's grasp of the significance of the order.

Several other similar examples were readily identified in the videotaped lessons, where the role of a particular variable in a calculation is obscured by the presence of another variable with the *same* value. Chloe is teaching a Year 1/2 class a strategy for adding and subtracting 9, 11, 19 and 21 i.e. by a suitable adjustment of the tens digit and then by adding or subtracting 1 from the units. She asks one child to demonstrate on a number (1 to 100) square by adding 9 to … 9. To criticise her choice of starting number (9) may seem somewhat churlish. But it was the *first* example offered in the lesson, and she had some 90 starting numbers to choose from (some of which would be unsuitable for a different reason: we return to this episode in a moment). Before moving on from this category of obscuring the role of variables, we mention Colin (Reception class), who selected 10-5 as the first example of subtracting numbers from 10, and also Naomi (mentioned earlier) whose first example of 'difference' was the difference between 4 and 2. Each of these examples inadvertently invites children to construe that the 'answer' is to be found within the original question.

The more general issue that some examples are pedagogically preferable to others is again illustrated within Chloe's lesson (above) when she demonstrates a strategy for subtracting 19 (i.e.. subtracting 20 and adding 1) on a number square. The usual visual representation would be 'up two, right one', like a knight's move. This is good pedagogy, akin to the identification of diagonal lines in the multiple of 11 and 9 which relate to the place value system of numeration. But Chloe chooses 70 for the starting number in her first example, on the extreme right boundary of the 1 to 100 square. After moving up two squares to 50, there is no 'right one' square: it is then necessary to move down and to the extreme left of the next row.

A second category of poor choices of examples arises from the selection of calculations to illustrate a particular procedure, when *another procedure would be more sensible* to perform those particular calculations. A minor instance occurred in Naomi's lesson on 'difference', where she asked (on a worksheet) for the difference between 11 and 10, expecting them to 'count on' from the lesser of the two numbers. This is akin to giving e.g. 302-299 in a set of exercises on subtraction by decomposition. A more worrisome case concerned Laura's choice of demonstration examples in her first videotaped lesson, on column multiplication (the standard 2-digit by 1-digit algorithm) with a Year 5 class. Her first example (37×9) is not a bad one (though not the best either), but she then goes on to work through 49×4, 49×8 and 19×4. Now, the NNS emphasises the importance of *mental* methods, where possible, and also the importance of choosing the most suitable strategy for any particular calculation. 49×4, 49×8 and 19×4 can all be more efficiently performed by rounding up, multiplication and compensation e.g. 49×4 = (50×4)-4. For that matter, 49×8 is readily found by doubling the answer to 49×4. As we mentioned earlier, the NNS makes much of doubling strategies, and 19×4 could be a double-double. In any case, to carry out these calculations by column multiplication flies in the face of any messages about selecting 'sensible' strategies.

Finally, we note that the videotapes offer copious instances of *examples being randomly generated,* typically by dice. This may have a limited but useful place in the generation of practice exercises, but it is pedagogically perilous in the teaching of procedures or concepts, when, as we have argued, it is simply not the case that any example is as good as any other. The example of subtracting 5 from 10 (in Colin's lesson mentioned earlier) was generated in this way, using specially modified dice. Colin went on to generate further expository examples - 5, 3, 8, in that order - with the dice. This contrasts with Naomi's skilful control of the examples in an episode with a closely-related learning objective (bonds to 10) described above. There seems to be some confusion in the minds of many trainees between the legitimate random choice of examples to enhance conviction about the truth of some principle or the efficacy of some established procedure on the one hand, and the choice of examples to inculcate awareness of a procedure or concept in the first place on the other. The latter is often better controlled and determined by the teacher, and random selection of examples in this case is effectively an abdication of responsibility.

CONCLUSIONS

Our grounded approach to the analysis of the lessons has highlighted several normative notions of teaching practice, one of which we have elaborated in this paper. This code - the teacher's *choice of examples* - was noticeably prevalent in the *Analytic Accounts* of the 24 lessons. One benefit of observing so many lessons was the possibility of comparing interpretations, and the realisation that the significance of some aspects of mathematics teaching could be observed in most of them. In particular, the evidence from our research has greatly enhanced our own awareness that novice teachers need guidance and help in appreciating the different roles of examples in mathematics teaching. The extent to which trainees choose examples wisely, or otherwise, seems to us to be a significant indicator of their mathematics content knowledge for teaching.

Clearly, our work has not simply confirmed the pedagogic importance of examples, which is already upheld in the literature. More importantly, it refines and illuminates this category by reference to the classroom practices of novice teachers. So whilst formerly we might have spoken about the importance of choosing examples with care, we are now able to give a more analytical account of the place of examples in mathematics teaching and learning, and to give examples. In this regard, our own appreciation of the significance of this dimension has been substantially enhanced.

Whilst our focus in this paper has been on the significance of the teacher's choice of examples, we mention here the emergence of four broad categories in which the 18 codes mentioned earlier can be grouped within more comprehensive, higher-order concepts (Strauss and Corbin, 1998, p. 113). The statement here is necessarily brief, but is intended to indicate the broader framework of the emerging theory. The first of these categories concerns the teacher's personal system of beliefs about mathematics and how it is learned, together with their 'theoretical' knowledge of subject matter and mathematics pedagogy. The second is about the transformation of the teacher's own knowledge into (re)presentations and explanations for the classroom. The third concerns the coherence of the mathematics presented and experienced, including the way it is connected and sequenced. The fourth category relates to the teacher's contingent action in the lesson, how they respond to situations and opportunities that have not been planned or anticipated.

Finally, it is important to emphasise our narrow attention to mathematics *content* knowledge, both SMK and PCK, in our scrutiny of the lessons. Whilst our research has not provided a direct mapping between mathematical knowledge and competence in teaching the subject, it does throw considerable light on this link by way of reference to particular moments and episodes. The grounded approach that we chose to take has the advantage that 'theory' can always, of necessity, be exemplified in this way.

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