ON RELATIONSHIPS BETWEEN BELIEFS AND KNOWLEDGE IN MATHEMATICS EDUCATION

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Summary

The relationship between beliefs and knowledge is usually left undefined in educational papers. Furinghetti & Pehkonen (2002) point out that when discussing on beliefs it is better distinguish between objective (formal, public) knowledge and subjective (informal, personal) knowledge. Here we understand beliefs as an individual's subjective knowledge and emotions concerning objects and their relationships, and they are based usually on his personal experience. Therefore, beliefs are not necessarily reasoned in a generally accepted way, and they represent some kind of tacit knowledge.

The affective domain has been a neglected area in research of mathematics education, until McLeod and Adams highlighted it with their famous book (McLeod & Adams 1989). The main components of affect are often considered to be beliefs, attitudes, and emotions (cf. McLeod 1992). Two decades ago an individual's attitude toward mathematics formed one of the central research topics in the affective domain of mathematics education; the well-known Fennema-Sherman attitude scale (Fennema & Sherman 1976) represents this phase. In recent research the focus has changed to beliefs. Another direction of change in research has been in the concept of mathematics itself. Today we are no more considering attitudes toward mathematics as an entity, but researchers distinguish e.g. attitudes or beliefs on geometry or problem solving.

There are several difficulties in defining concepts related to beliefs. Some researchers consider beliefs to be part of knowledge (e.g. Pajares, 1992; Furinghetti, 1996), some think beliefs are part of attitudes (e.g. Grigutsch, 1998), and some consider they are part of conceptions (e.g. Thompson, 1992). There can be differences also depending on the discipline. For example emotions can have different meaning in psychology than in mathematics education (e.g. McLeod, 1992). In addition it is possible that researchers use same terminology although they study different phenomena. This all makes it hard to understand studies and compare them to each other (e.g. Ruffell, Mason & Allen, 1998).

The focus of this paper is, to analyze and sharpen the concept 'belief' and related concepts such as knowledge, since their characterization seem to be in the literature somewhat fuzzy (cf. Furinghetti & Pehkonen 2002). In this paper we concentrate on beliefs in mathematics teaching.

What are beliefs?

In the literature, several overviews on results of mathematical belief research are published recently (e.g. Underhill 1988, Schoenfeld 1992, Thompson 1992, Pehkonen 1994, Op 't Eynde, de Corte & Verschaffel 2002) which have tried to clarify fuzzy concepts. For some, beliefs can be thought to form one part of an individual's meta-cognition (Schoenfeld 1987). Others have tried to define beliefs i.a. through attitudes (e.g. Törner & Grigutsch 1994).

Furinghetti & Pehkonen (2002) have recently tried to clarify the problems of the concept 'belief' with the help of specialists' evaluations of mathematical belief research, and they conclude with the following suggestions. "When dealing with beliefs and related terms, it is advisable

• to consider two types of knowledge (objective knowledge and subjective knowledge)

• to consider beliefs as belonging to subjective knowledge

• to include affective factors in the belief systems, and distinguish affective and cognitive beliefs, if needed

• to consider degrees of stability, and to acknowledge that beliefs are open to change

• to take care of the context (e.g. population, subject, etc) and the research goal within which beliefs are considered."

Therefore, it is important also here to describe how we understand beliefs. An individual's *beliefs* are understood as his subjective, experience-based, often implicit knowledge and emotions on some matter or state of art. Such a characterization is very near the one given in the published paper of Lester, Garofalo & Kroll (1989). In the literature, the term conception is often used parallel to beliefs. Here we define *conceptions* according to Saari (1983) as conscious beliefs, i.e. they form a subgroup of beliefs. In the case of conceptions, the cognitive component of beliefs is stressed, whereas in subconscious beliefs the affective component is emphasized.

The spectrum of an individual's beliefs is very wide, and they are usually grouped into clusters of beliefs. Some beliefs depend on other ones, for the individual more important beliefs. Here Green (1971) uses the term 'the quasi-logical structure of beliefs'. Thus, beliefs form *belief systems* that might be in connection with other belief systems or might not. The affective dimension of beliefs influences the role and meaning of each belief in the individual's belief system. Beliefs represent some kind of tacit knowledge. Every individual has his own tacit knowledge which is connected with learning and teaching situations, but which rarely will be made public. Beliefs differ from scientific knowledge (objective knowledge) that can be expressed with logical sentences, and on which we can discuss.

Objective knowledge vs. subjective knowledge

When discussing on beliefs, it is advisable to distinguish two parts in knowledge (cf. Furinghetti & Pehkonen 2002): objective knowledge (formal knowledge, official knowledge, public knowledge) and subjective knowledge (informal knowledge, personal knowledge, private knowledge). This dichotomy helps us to situate and understand beliefs and knowledge together, and at the same time to distinguish them from each other.

Objective knowledge in mathematics means the generally accepted structure of mathematics that is a compound of mathematicians' work during more than 2000 years. The mathematical knowledge structure is today so huge that it is beyond human ability of comprehension. The last mathematician who is said to have an overview of ”all mathematics” was Poincaré who lived about hundred years ago (cf. Boyer 1985, 650)._And today our knowledge is said to double in seven years. When we study mathematics we can learn only a small part of it – and usually in our characteristic way, i.e. we form our own conceptions on the topics to be learned.

One main feature of mathematical knowledge is its pure logic. An assumption of objective knowledge is that all the beliefs that form its reasoning ground, must be logically justified and generally accepted in the way that all other facts in phenomena world support it. An individual's *subjective knowledge* is something unique which is usually possessed only by the individual self, since it is based on his personal experiences and understanding. According to our definition, beliefs belong to subjective knowledge.

The difference between these two types of knowledge can be presented, as follows: Objective knowledge is accepted by the research community, whereas subjective knowledge is not necessarily evaluated by anybody from outside. Mathematical beliefs pertain to an individual's subjective knowledge, and if they are presented as statements, they may (or may not) be logically true. But knowledge has always this truth property (e.g. Lester & al. 1989). The truth property can be described with the help of probability: Objective knowledge is true with a probability of 100 %, whereas in the case of belief the corresponding probability is usually smaller than 100 %. Therefore, this is one of the distinguishing properties between knowledge and belief. When we speak on knowledge in the case of an individual, this means that the individual is 100 % sure of that belief.

In science we strive towards permanent and unchangeable objective knowledge, but we are not always successful (not even in mathematics). Sometimes also the research community of mathematicians has been compelled to admit that they have accepted an untrue belief as a mathematical truth. For example, in the 1700s one generally accepted piece of knowledge among the mathematicians was that all infinite series, with the limit zero of the general term, are convergent. This conception was rejected as knowledge when the well-known counterexample, a harmonic series $\sum(1/n)$, was introduced at the end of that century, and, in consequence, the theory of infinite series was developed.

Connections between objective and subjective knowledge

There are many connections between an individual's subjective knowledge and objective knowledge. On one hand, the individual can study mathematics (objective knowledge), and thus enlarge his own subjective knowledge structure. The topic of his study might be, for example, the concept function from the objective knowledge. Knowledge that he adapts on function belongs all the time to his subjective knowledge structure, although his conception on function may asymptotically get closer and closer to the ”official” concept of function that pertain to objective knowledge (cf. the idea of knowledge building in the paper Bereiter & Scardamalia 1996). Thus his subjective knowledge in mathematics contains, among others, his conception on mathematics as a whole, and also conceptions on detailed mathematical knowledge. On the other hand, the individual's subjective knowledge can enrich common objective knowledge, when some part of his subjective knowledge has been presented in public, is justified, has been discussed, and is socially accepted (e.g. in the form of a scientific paper).

The same idea on connections between beliefs and knowledge are explained by Anna Sfard (1991, 3) as follows. She considers conceptions as the subjective (private) side of the term 'concept': "*The word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form as a theoretical construct within "the formal universe of ideal knowledge"*". The distinction between conception and objective knowledge is complicated by the fact that an individual's conception of a certain concept can be considered as a "picture" of that concept. Since a picture and its object are not the same, and usually the picture shows only one view on the object, similarly a conception represents only partly its object (concept).

Beliefs, attitudes, and emotions

An individual observes all the time the world around. Based on his experiences and observations, he concludes statements on different phenomena and their nature (cf. Malinen 2000). A person's *view of mathematics* is formed based on this – cf. with the concept "mathematical world view" used by Schoenfeld (1985). Thus, an individual's view of mathematics is a compound of knowledge, beliefs, conceptions, attitudes, and feelings. It is the filter that regulates his thinking and actions in mathematics-related situations.

The mechanism in beliefs seems, according to our experience, to function, as follows: A person compares his beliefs with new experiences and with other individuals' beliefs. Thus his beliefs are under continuous evaluation and change. When he adapts a new belief, it will be situated automatically into the structure of other beliefs, since beliefs do not exist fully independently. Thus the individual's belief system is a composition of his conscious and subconscious beliefs as well as of his hypothesis and expectations, and their combinations. (cf. Green 1971)

In order to better understand relationships between beliefs and knowledge, we try to situate them with other main concepts – attitudes and emotions – of the affective domain. McLeod (1992, 578) divides affective domain into emotions, attitudes and beliefs. These terms vary in stability, intensity, in cognitive involvement and in how long their development takes. Emotions, negative or positive feelings, are most intense and least stable. They involve least cognitive appraisal and may appear and disappear rather quickly. (ibid, 579) Examples of intensive negative feelings that are connected to mathematics are fear, anger, horror or even panic when pupil cannot solve a mathematical problem (e.g. Op 't Eynde & al., 2002). The "Aha!" experience during problem solving is a very short-term positive emotion. On the other hand, satisfaction and joy experienced after completing a challenging mathematical task are slightly longer-term positive emotions. (e.g. Malmivuori, 2001)

Attitudes are affective reactions that contain relatively intense and long-term positive or negative feelings (McLeod 1992, 581). They are therefore rather stable and in them affective and cognitive sides get balanced (Goldin 2002). Attitudes related to mathematics include liking, enjoying, and interest in mathematics, or the opposite, and at worst math phobia (Ernest, 1989). Attitudes also include pupils' reactions to the easiness or difficulty of mathematics (Ma & Kishor 1997). There are different attitudes toward mathematics: "I'm interested in percentage calculation" and "fractions are boring". It is important to note that mathematics is a wide area and that pupils can have different attitudes to different parts of mathematics (McLeod, 1992). Attitudes can form in two different ways: repeating emotional reactions can stabilize into an attitude. For example, if a pupil has many negative experiences in doing geometrical tasks, the reaction to similar tasks can become more automatic and stabile. An attitude can be formed also so that an already existing attitude is assigned to a new but related task. It is for example possible that pupil who has a negative attitude toward proofs in algebra may attach the same attitude to proofs in geometry. (ibid, 581)

Beliefs are according to McLeod (1992, 579) cognitive, and they are formed rather slowly. The difference between these concepts can be characterized in a symbolic way: emotions are "hot", attitudes "cool" and beliefs "cold" (ibid, 578).

View of mathematics as a schema

With the help of mentioned theoretical considerations we can sketch a schematic picture of an individual's view of mathematics, i.e. the relationships between the main concepts (objective knowledge, subjective knowledge, emotions, beliefs, attitudes) in beliefs (Figure 1).

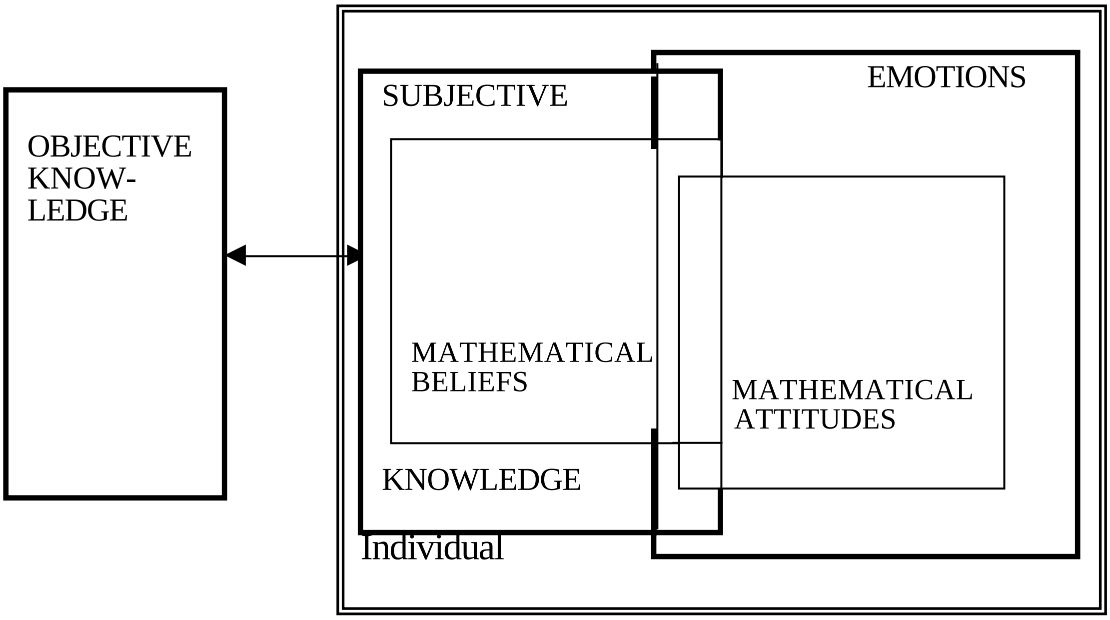

Figure 1. Relationships between main concepts in beliefs.

In knowledge, we distinguish between its objective and subjective share, and the former is situated outside of the individual (cf. Sfard 1991). However, objective and subjective knowledge are thought to be in interaction with each other. Since the individual's subjective knowledge contains also some part of his emotions, these two areas intersect each other. It could be thought that a pupil has knowledge on his emotions. The pupil recognizes, for example, that when he has solved a difficult task, he feels joy and satisfaction.

Mathematical beliefs pertain to subjective knowledge, and mathematical attitudes to emotions. But these two sub-domains intersect, since one can imagine statements that can be understood at the same time as beliefs and attitudes. For example, the statement "*I am not good in mental calculations*" can be understood as a belief concerning oneself, but also as an attitude toward mathematics. The schema of Figure 1 has been dealt with in detail in the published dissertation (Pietilä 2002).

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