MATHEMATICAL THINKING STYLES – AN EMPIRICAL STUDY

Rita Borromeo Ferri

Department of Education, University of Hamburg

Abstract: *In the study described in this paper, mathematical thinking styles of 15 and 16 year old pupils shall be reconstructed. In the actual discussion on mathematics there already exist classifications of thinking styles: F. Klein (1892, quoted by Tobies 1987) for example, distinguishes the thinking styles of the "analyst", the "geometer" and the "philosopher", while Burton (1995) describes a visual, an analytic and a conceptual thinking style. Some of these classifications were developed intuitively or through empirical examinations, and the study only concluded practising mathematicians but no pupils learning mathematics. In this paper it will be shown among others, how mathematical thinking styles have been reconstructed in the study until present.*

1. Introduction and overview

From our experiences we learned, that there are many ways to explain mathematical facts and that there are as many ways to understand and to think them through. Some people for example easier understand mathematical facts by drawing sketches or other kinds of graphics, while others are tending more to search for structures, patterns or formulas and it's application. This means that people may have preferences for the so-called visual or the so-called analytic or so-called conceptual way of thinking, or they show preferences for two or three of the thinking styles simultaneously (mixed types).[1] Already in 1892 F. Klein (quoted by Tobies 1987) distinguished - on an intuitive base – the styles "analyst", "geometer" and "philosopher". Empirical examinations (Burton, 1995) pointed out, that one may classify a visual, analytic and conceptual thinking style. Since these classification are limited to practising mathematicians and their results, they cannot applied directly to pupils. This is the reason why a special study was carried out which started from the following research questions:

(a) Can these thinking styles also be reconstructed with 15 and 16 year old teenagers, who are still in the phase of learning mathematical concepts and methods, but, compared with practising mathematicians, have much less experience in working with mathematics?

(b) If so, how can these thinking styles be described ?

[1] Mixed type can have two meanings: 1. Mixed type as own style, which concludes characteristics of the visual and analytic thinking style; 2. Mixed type if ,depending on the situation, one style is chosen.

(c) If these thinking styles can be reconstructed, are there thinking styles which exclude each other, meaning "polar types" or do "mixed types" also exist, meaning thinking styles, which conclude characteristics from various thinking styles?

(d) What does it mean to pursue a visual, analytic or conceptual thinking style ? How can these age-related thinking styles be described?

Research in the area of mathematical thinking processes and cognitive psychology have shown that there are various ways to reconstruct individual ways of thinking. In this paper a possible way of reconstructing mathematical thinking styles (analytic, visual, conceptual and mixed type) will be described.

2. Theoretical foundations and considerations

The theoretical considerations of this study conclude findings of the cognitive psychology and mathematics didactics. Up to now the concept of 'style' and the concept of 'thinking style' has been used only occasionally within the mathematics didactic discussion. Thus, at present the construct of 'style' is discussed mainly in the field of cognitive psychology (see Sternberg 1996, 1997, 2001). An more ability-oriented understanding of styles from earlier discussions has been replaced by conceptions which are emphasising choice and the independence of performance. Sternberg & Grigorenko (2001) support the following characterisation of style which is largely agreed at the moment: "reference to habitual patterns or preferred ways of doing something […] that are consistent over long periods of time and across many areas of activity". For research on teaching and learning it is of central meaning to distinguish learning styles, thinking styles and cognitive styles, although the underlying conceptions are often not clear and an overlapping of styles is therefore unpreventable. This study is focussing on thinking styles and is aimed to develop an adequate characterization of the construct 'mathematical thinking style' for mathematics education. Sternberg (1997: 19) takes the construct thinking style as a "preferred way of thinking" or " preference in the use of abilities we have". There consists the possibilities of changing thinking styles but they may change depending on time, environment and life demands. Sternberg states that thinking styles are acquired at least partly through socialisation. There is almost no study on the theoretical construct of mathematical thinking styles, especially no empirical one. However, one can find quite a lot of research which refers to the concept of mathematical thinking. Schoenfeld (1994) for example, he worked extensively, theoretically as well as empirically, on the learning of mathematical thinking and its necessary pre-conditions. Saxe et al. (1996) give special emphasize to the construct's reliance on culture and context, Dreyfus & Eisenberg (1996) clarify affective aspects, such as self-confidence or mathematical creativity.

From the mathematics didactics discussion there are known classifications and typologies of thinking styles (Klein, 1892 (quoted by Tobies, 1987); Ribot, 1909; Burton, 1995) and of cognitive structures (Schwank, 1996) as well. These classifications and typologies sometimes offer quite helpful approaches for describing mathematical thinking styles, for example the empirically verified classifications for research doing mathematicians from Burton (1999: 95):

Style A: Visual (or thinking in pictures, often dynamic),

Style B: Analytic (or thinking symbolically, formalistically) and

Style C: Conceptual (thinking in ideas, classifying)

Besides the clarification of the theoretical construct 'mathematical thinking style' an adequate, age-dependend description of the visual, analytic and conceptual thinking style thinking style shall be generated from the data of this study.

3. Methodology and design of the study

This study is quality-oriented, and the analytic method shall lead to results and theories. The aim of the study is to generate hypotheses, but because of its case-study-like character generalisation going beyond the sample can be done only to a limited degree.

In its qualitative research this study is applying the Grounded Theory (Strauss & Corbin, 1996), in which's framework various research methods will be described in order to develop an inductively derived grounded theory about a phenomenon (see Strauss & Corbin, 1996: 8). By systematically collecting and analysing data on an examined phenomenon, data gathering, analysis and theory are mutually connected. "The aim of Grounded Theory is to create a theory, which is fair to the examined object and illuminate it." (Strauss & Corbin, 1996: 9). Nevertheless Grounded Theory does not claim, that the researcher starts one's approach to a research object from a "tabula rasa" situation. Therefore Strauss & Corbin are emphasising the importance of the theoretical sensibility which allows them "to develop a grounded, conceptual dense and well integrated theory – much faster than if this theoretical sensibility is missing." (ibid 1996: 25). Concerning the systematic data collection, in this study the following methods were used: video-taping of problem-solving processes, stimulated recall (Wagner, Weidle, Uttendorfer-Marek, 1977; Schoenfeld, 1985) and focussing interviews (Flick, 1999).

Altogether 12 pupils, 6 boys and 6 girls in years 9 and 10, who that time were 15 and 16 years old participated in the study. Six pupils from each year group were arranged in pairs: a pair of boys, a pair of girls and a mixed one. It was paid attention for the chosen students were accustomed to work together in their lessons. Of course, this study is aimed to reconstruct individual thinking styles, but the reason for the methodical decision for problem-solving in pairs (see Goos, 1994) is, that in this way there is much more verbalisation during the problem-solving process and more questions arise. There were two sessions for each pair. In each session 4 non-routine problems were to be solved. In order to get as much as possible information about the pupils' reflections on problem-solving processes and on their way of thinking the following 3-steps design of has been developed in accordance with Busse (2001). The procedure of each session was the following:

Step 1: Problem-solving process. Each pair of students solved 4 problems, one after another. They were free to decide how far to work together. The working process was video-taped.

Step 2: Individually stimulated recall. Afterwards, each student was shown individually a video-recording of Step 1. Beforehand, the pupils were asked to stop the video to give them the chance to express their ideas, explanations or difficulties they had while working on the problem. Additionally, I stopped the video at positions I wanted to know what they where thinking. This grade of intervention (see Schoenfeld, 1985) could be justified by the fact that especially through these enquiring questions I received quite a lot of important information which otherways I would not have got. This step of stimulated recall was tape-recorded.

Step 3: Individual interview. Each student was interviewed directly after the stimulated recall. The individual interview was devided into two parts: In part one they were asked about the problem-solving process and their judgements concerning the items to be solved. In part two the questions focussed on their image of mathematics: For example what the pupils' understanding of mathematics is, what are their preferences or aversions of mathematical topics in school. Data on the image of mathematics were collected in order to investigate, how stable the reconstructed thinking style preferences of an individual are. These interviews were tape-recorded too.

4. Analysing the data

The exclusively verbal data and the pupils' products from this study are analysed and encoded according to the Grounded Theory: at first by open coding, then by axial coding and finally by a selective coding. Encoding the data is the basic strategy for decomposing – or breaking off – the data and then recomposing them in a new way. Therefore, it is the central process through which theories are developed out of the data. Before starting the encoding procedure the problem-solving processes of all pupils must be reconstructed in a sequential way in order to get a better understanding of the thinking processes. This first step enabled me to divide the solving process into the following 5 phases: (1) Reading and understanding a problem (2) First ideas and impressions (3) Searching for ideas (4) Creating solutions (5) Results and checking. Through these phases the pupils could be compared, in general and within each phase of the solving processes. By reconstructing the problem-solving processes I received 4 dimensions which then were used to develop codes:

1. **Internal imagination** of a person ("in the brain", "inner eye") while trying and solving a problem
2. **External representation** of mathematical facts through a person
3. **Wholist – Analyst** way of thinking and procedure

4. **Image of mathematics** as confirmation of stabilized preferences of thinking styles

These phases of the problem-solving process together with the above listed 4 dimensions served as codes during the open coding phase. The open coding phase is that part of the analysis which especially refers to the naming and categorising of phenomena. (see Strauss & Corbin, 1996: 44). In this study the data were decomposed and categorised by a line-per-line analysis and by comparing single events, which in the following were categorised together as one similar phenomenon, for instance as phenomenon which describes an individual's internal imagination which was often carried out with the help of stimulated recall. Here an example of Sylvia, 15 years, grade 9 with her first idea to solve one item of the examination with her internal pictorial imagination

S:" I had directly an pictorial imagination, the numbers were not important for me, I must have an pictorial imagination."

During the axial coding procedure the data will be put together in a new way, in which the connections of one code to a sub-code will be investigated. Actually, sub-codes are codes too, but they specify the main codes more exactly.

Again, Strauss & Corbin emphazise the connection between open and axial coding: "Although open and axial coding are separate ways of analysis, the researcher is changing between these two modi during the analysis." (ibid 1996: 77). Referring to this study, here an example: the internal imaginations of a person find their expression in various ways in the data. Pupils told that they had strong pictorial imaginations or that they used more mathematical symbols or terms in their imaginations to solve a problem such as Jenny, 15 years, grade 9:

J: " I had numbers in my mind, no pictures."

These sub-codes can be 'dimensionalised' further during the researcher's analysing process, for example into static and dynamic imaginations of the pupils. This putting-into-relation of a sub-code to a code is done by using questions which describe a form of relationship. Therefore axial coding represents a complex process of inductive and deductive thinking.

Finally, selective coding is the process of choosing the central category as well as the systematically putting-into-relation of the central category to other categories, the validation of the received relations and the refilling of the categories which then need to be refined and developed further (see Strauss & Corbin, 1996: 94).

All data of this study are coded with the help of the softare-tool ATLAS.ti, so that an overview about fixed codes, sub-codes and the creation of code families will make the analysis comprehensible easily.

5. Reconstructing mathematical thinking styles

As mentioned above and illuminated under methodological aspects in the chapter before, there are 4 dimensions serving as base for developing codes. Through the encoding process these dimensions shall help to clarify more deeply the question what does it mean to practice an analytic or a mixed style of thinking. However, one should not decide too quickly in classifying a pupil as being exclusively a visual or analytic thinker just because he or she uses a pictorial demonstration while solving a problem. From my analyses I learned, that in this regard it must be distinguished more exactly, if, for instance, pupils put down a graphical demonstration only because teachers told them to do so and not because that moment they tried to visualize. For this reason it is important to look at the internal imagination and at the externalised presentation as well. Different internal imaginations are already mentioned above in chapter 4. In this context Skemp (1987) is cited, who distinguishes verbal-algebraic and visual symbols.

The 3rd dimensions is about the way pupils are structuring their thinking and the information during the problem-solving process. This can either be a done in a wholistic or an analytical way, or if settled between these two extremes, in a mixed way. This wholist-analytic dimension is more related to an individual's imagination and will be reconstructed through the problem-solving process in which internal imaginations are taken into account. By this it shall be found out, if for example, a person prefers to adapt a visual thinking style, but nevertheless, simultaneously follows the analytic way. Riding (2001) also distinguishes the dimensions 'wholist' and 'analytic' which I refer to in my descriptions:

„Wholists see a situation as a whole and are able to have an overall perspective, and to appreciate its total context. By contrast, analytics see a situation as a collection of parts and often focus on one or two aspects of the situation at a time to exclusion of the others. Intermediates are able to have a view between the extremes, which should allow some of the advantages of both."(Riding 2001: 55-56)

The 4th dimension is about the pupils' image of mathematics. It is aimed to stabilise the found out preferences for one (ore more) thinking style.

6. Results up to now

The results show that distinct preferences for a visual and analytic thinking style, the so-called "polar-types" can be reconstructed with 15 and 16 year old pupils, but not distinct preferences for the conceptual thinking style (following Burton "Conceptual", following Klein "philosopher"). The conceptual thinking style could only be reconstructed in connection with the other two thinking style, as a so-called "mixed type". Besides this, other "mixed-types" with only two thinking styles could be reconstructed. (see Borromeo Ferri, 2002 and 2003)

From my 12 participants 3 girls are "polar types" of the visual thinking style, one girl and one boy are "polar types" of the analytic thinking style. There's one boy who is a "mixed type" of the three thinking styles and the remaining pupils are "mixed-types" of the analytic and visual thinking style. The "mixed types" show a higher degree of flexibility in solving problems during the examination, as well as in their descriptions in school about their approach to an item and on how to work on it. The results show that a learner's internal imaginations must not correspond with his external representations. Following two quotations of two girls of my research work who are "polar types" of the analytic and visual thinking style:

Saskia, 16 years, grade 10, "polar type" of the analytic thinking style":

"*Yes, one always must think in formulae, I think, because somehow mathematics always has to do with formulae, even if a teacher does not say so at the beginning, it always has to do with mathematical formulae!"*

Sylvia, 15 years, grade 10, "polar type" of the visual thinking style:

"I only memorize formulae and mostly I didn't understand them, because it does't help me much, but I know how it pictorial belongs together, but by formulae, no, I can't cope with that."

With reference to the present results of the study I can give the following description of the concept of "mathematical thinking style":

A mathematical thinking style is an individual's preferred way of thinking and understanding mathematical facts and connections by using various internal imaginations and externalised representations.

Due to one's mathematical socialisation, an individual's mathematical thinking style finds its expression more or less clearly in certain mathematical topic areas, in which a dependence on items an context is concluded.

Mathematical thinking styles are not the same as structures, but they can help to build structures in knowledge. Each individual gives preference to one's own thinking style by which he or she is able to understand mathematical facts and contexts. These individual preferences help to establish structures within one's knowledge. The structures are created gradually by the fact that young individuals think through again mathematical facts, so that the structure of knowledge extends continuously.

8. Conclusions

This paper is aimed to show one way how different mathematical thinking styles can be reconstructed by applying the methodology of the Grounded Theory and what are the underlying theoretical approaches of this study. Furthermore, these results obtained up to present indicate a highly didactical relevance of this kind of study: Its significance for mathematics lessons is obvious. Pupils who are not sharing the mathematical thinking style with their teacher may have problems of understanding, but if the teacher is conscious of his own style and arranges mathematical facts in different ways, problems of understanding could be prevented.

"My previous teacher explained fast and much and did not make any drawings and then one time I got a six[2] for a maths test and then I only got a four and then I thought I don't know maths. [...] And that was I couldn't cope with. My new teacher always makes a drawing and now I understand how to come to the result, not like only by formulae and calculation and for the first and third test I got a one." Vera, 15 years, grade 9

These results correlate with results from other empirical studies, among others that of Zhang & Sternberg (2001), who pointed out:" Findings from a third study indicated that teachers inadvertently favoured those students whose thinking styles were similar to their own" (2001: 204). Therefore, it is necessary that teachers become conscious about their own thinking style, on the one hand in order to guarantee equality of chances among pupils, and on the other hand to develop their own mathematical potentials.

References

Borromeo Ferri, Rita. Erste Ergebnisse einer empirischen Studie zu mathematischen Denkstilen von Schülerinnen und Schülern der 9. und 10. Klasse. In Beiträge zum Mathematikunterricht, p. 123-126, Hildesheim: Franzbecker, 2002.

Borromeo Ferri, Rita and Gabriele Kaiser. First Results of a Study of Different Mathematical Thinking Styles of Schoolchildren. In Leone Burton (Ed.) Which Way in Social Justice in Mathematics Education ?, p. 209-239, London: Greenwood, 2003

Burton, Leone. Moving towards a Feminist Epistemology of Mathematics. Educational Studies in Mathematics, 28, 3, p. 275-291, 1995.

Burton, Leone. Mathematicians and their Epistemologies – and the Learning of Mathematics. In Inge Schwank, (Ed.), European Research in Mathematics Education Vol. I, Osnabrück: Forschungsinstitut für Mathematikdidaktik, p. 87-102, 1999.

Busse, Andreas. Zur Rolle des Sachkontextes bei realitätsbezogenen Mathematikaufgaben. In Beiträge zum Mathematikunterricht, p. 141-144, Hildesheim: Franzbecker, 2001.

Dreyfus, Tommy and, Theodore Eisenberg. On different facets of Mathematical Thinking. In Robert Sternberg, and Talia Ben-Zeev, (Eds.). The Nature of Mathematical Thinking (pp.253-284) Mahwah NJ: Lawrence Erlbaum, 1996.

Flick, Uwe. Qualitative Forschung. Reibek: Rowohlt. 1999.

Goos, Marilyn. Metacognitive Decision Making and Social Interaction During Paired Problem Solving. Mathematics Education Research Journal, 6, 2, p. 144-165, 1994.

Ribot, Théodule. L'evolution des Idées Générales, 3.Édition-Paris, 1909.

Riding, Richard. The Nature and Effects of Cognitive Style. In Robert Sternberg, and Li-Fang

[2] Six is the worst mark in a one-to-six assessment scale usual in German teaching.

Zhang (Eds.), Perspectives on Thinking, Learning and Cognitive Styles (pp.47-72), London: Erlbaum, 2002.

Saxe, Geoffrey, Venus Dawson, Randy Fall, and Sharon Howard. Culture and Children's Mathematical Thinking. In Robert Sternberg, and Talia Ben-Zeev (Eds.), The Nature of Mathematical Thinking (pp.119-144) Mahwah, NJ: Lawrence Erlbaum, 1996.

Schoenfeld, Alan H. Making Sense of „Out Loud“ Problem-Solving Protocols. In The Journal of Mathematical Behaviour 4, p. 171-191, 1985.

Schoenfeld, Alan H. Mathematical Thinking and Problem Solving. Hillsdale: Erlbaum, 1994.

Schwank, Inge. Zur Konzeption prädikativer versus funktionaler kognitiver Strukturen und ihrer Anwendungen. In Zentralblatt für Didaktik der Mathematik 6, p. 168-183, 1996.

Skemp, Richard. The Psychology of Learning Mathematics. Hillsdale NJ: Lawrence Erlbaum, 1987.

Sternberg, Robert. Thinking Styles. New York: Cambridge University Press, 1997.

Sternberg, Robert, and Talia Ben-Zeev (Ed.) The Nature of Mathematical Thinking. Mahwah: Erlbaum, 1996.

Sternberg, Robert, and Li-Fang Zhang (Ed.) Perspectives on Thinking Learning, and Cognitive Styles. London: Erlbaum, 2001.

Sternberg, Robert, and Elena Grigorenko. A Capsule History of Theory and Research on Styles. In Robert Sternberg, and Li-Fang Zhang (Eds.). Perspectives on Thinking, Learning, and Cognitive Styles. London: Erlbaum. p. 1-21, 2001.

Strauss, Anselm L. and Juliet M. Corbin. Grounded Theory, Grundlagen Qualitativer Sozialforschung. Beltz, 1996.

Tobies, Renate. Zur Berufungspolitik Felix Kleins. Grundsätzliche Ansichten. In NTM-Schriftenreihe Geschichte, Naturwissenschaft, Technik, Medizin, 24, 2, p. 43-52, 1987.

Wagner, C.Angelika, Ingrid Uttendorfer-Marek, and Renate Weidle. Die Analyse von Unterrichtsstrategien mit der Methode des „Nachträglich Lauten Denkens“ von Lehrern und Schülern zu ihrem unterrichtlichen Handeln. In Unterrichtswissenschaft, No. 3, p. 244-250, 1977.

Zhang, Li-Fang, and Robert Sternberg. Thinking Styles across Cultures: Their Relationships with Students Learning. In Robert Sternberg, and Li-Fang Zhang (Eds.), Perspectives on Thinking, Learning and Cognitive Styles, p.197-226 London: Erlbaum, 2001.