UNDERSTANDING AND STRUCTURE

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Authors understanding of the process of creating a new piece of knowledge is described and illustrated. The process is broken down into six stages: motivation, isolated models (gaining experiences), generalisation, universal model, abstraction, abstract knowledge and its situation in the cognitive structure. The 'universal model' stage plays a decisive role in the understanding and structuring of a new piece of knowledge. It unifies and organises hitherto isolated experiences of the on going creation.

1. The Aim of the Paper

The problem of how mathematical knowledge is acquired and understood and how the mathematical structure is built have been described and studied from several point of views. Theories of reification (Sfard, 1989), procept (Gray-Tall, 1994), understanding (Sierpinska, 1994), and recently abstraction in context (Dreyfus, Hershkowitz, Schwarz, 2001 and Tsamir, Dreyfus, in press) illustrates this effort.

Our approach to this theoretical problem started in 1975 with following three questions: *Why do so many students not understand mathematics and even do not try to do so? Is it possible to change this state? If yes, how?*

Very soon we found out that the key point in this problem is knowledge without understanding, the rot knowledge. Therefore our research was focused to the question why and how such a piece of knowledge appears in a student's mind. During the long period of the research the author has been influenced by many ideas and people. The most profound has been the influence of Vít Hejny, the author's father, who at the very beginning of the research set its methodological (long-term researcher's experience with teaching at primary and lower secondary classes) and philosophical (constructivistic approach to education – in the conteporary language) frame. Probably the most valuable results of the research was a model of the process of construction of a piece of mathematical knowledge in an individual's mind; see Hejny (1989). This model has since been applied and elaborated by several researchers (Domoradzki, Kopácková, Kratochvílová, Jirotková, Kurina, Littler, Swoboda, – see references). The today state of the model is described in detail in Hejny – Kurina (2001, pp. 98 – 118).

The aim of this paper is to describe the current state of the model from the point of view of a student's mathematical structure and to show key obstacles, which cause the students to develop gain rote knowledge only.

As we said, the methodology of the research profoundly depends on the author's long term experimental teaching. Records of class discussions, students' written

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materials, notes on interesting class events, the author's pedagogical diary and a deep cognitive analysis of students misconceptions and errors in solving non – standard problems were the main source for creating and elaborating the model. The first version of the model was the result of the comparative analysis of three particular models created for understanding rot knowledge of the concept of ratio, one combinatorial problem, and formula for area of a triangle. Later on this model was applied to other parts of school mathematics and several times reconsidered.

2. Brief Description of the Model

The cognitive process which yields a new piece of knowledge starts with the student's interest or even inquisitiveness, continues by acquiring experiences, and terminates in the birth of the new piece of abstract knowledge. This process is a mechanism which consists of six stages.

1. **Motivation**. By motivation we mean a tension, which appears in a student's mind as a consequence of the contradiction between *I do not know* and *I would like to know.* This tension orientates the student's interest towards a particular mathematical problem, situation, idea, concept, fact, scheme,...

2. **Stage of isolated (mental) models**[2]. The acquisition of an initial set of experiences. At first, these experiences are stored as isolated events, or images. (e.g. a child adding 2 apples + 3 apples and later on 2 dolls + 3 dolls does not see the connection between these two cases) Later on, it might be expected that some linkage between them occur.

3. **Stage of generalisation**. The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalising all these models is discovered. The process of generalisation does not change the level of the abstraction of thinking.

4. **Stage of universal (mental) model(s). A** general overview of the already existing isolated models develops. It gives the first insight into the community of models. At the same time, it is a tool for dealing with new, more demanding isolated models. If stage 2 is the collecting of new experiences, stages 3 and 4 mean organising this set into a structure. The role of such a generalising scheme is frequently played by one of the isolated models (e.g. fingers serves as universal model for a simple counting).

5. **Stage of abstraction**. The construction of a new, deeper and more abstract concept, process or scheme which brings a new insight into the piece of knowledge. A frequent consequence of this invention is a strong emotion and an over-valuing of the new piece of knowledge. For example we know that in the pythagorean school the idea of a number was strongly over-valued. The possible explanation of this fact

[2] We feel that the term '(mental) model' is not appropriate but we are not able to find a better one. From now on, the term 'model' means a model of a creating piece of knowledge.

is fundamental discovery of a general proof (e.g. 'odd plus odd is even' holds for all numbers) which changes the abstraction level of the thinking.

6. **Stage of abstract knowledge**. The new piece of knowledge is housed in the already existing cognitive network, thus giving rise to new connections. Sometimes it results in reorganisation of the mathematical structure or its part.

Below we will discuss stages 2 to 5 in more detail. We argue that two mental shifts *generalisation* and *abstraction* play the decisive role in the development of mathematical knowledge. These usually are of a form of AHA-effect which was aptly characterised by B. Russell (1965, p. 53) as "intoxicating delight of sudden understanding". In many cases such a mental shift can be decomposed to A. Sfard's interiorization, condensation and reification and in many cases the product of these shifts is a new procept in sense of Gray –Tall (1994).

The boundary between generalisation and abstraction is blurred.

3. Stage of Isolated Models

Story 1. Five year old Adam added 2 apples and 3 apples. Then he was asked to add 2 sweets and 3 sweets. When he started to put all sweets together his father said: "You already know, two apples and three apples are five apples. Hence two sweets and three sweets must be five sweets, two chairs and three chairs are five chairs and so for everything else. Do you understand?" Adam said: "I do" However he was not very excited about this new piece of knowledge.

Story 2. Two six graders solved the following task.

Problem 1. If you can find the area of triangle *KLM*, do it. If not, say which segment(s) must be measured to be able to find the area of Δ*KLM*. Here are fragments from the discussion of Ben (B) and Cindy (C).

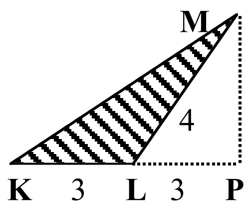

Fig 1

B2: Do you remember the formula?

C2: One half base times altitude (she writes *A* = _ *b.h*)

Ben3: Yeah. But _[3] we have no altitude. __ Do we?

C5: Here is the base (she points to the segment *KL*). __ Where is the altitude?

B6: __ (with excitement) Its here, look (he draws altitude from *L* to *KM*).

C6: It is. _ To this side (she points to *KM*). __ We do not know these.

B8: So we have to measure it. _ Do you agree?

C9: Wait a minute. So what are these three numbers for then?

B10: Well… _ You know, just to confuse us. I am sure…

C10: But the teacher said the area of a triangle can be found in three ways.

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[3] Hereinafter _ means 1-3 seconds pause and __ means more than 4 second pause.

B11: Oh well, she did not speak about such strange triangles (to be continued)

Both stories illustrate the stage of isolated models. Story 1 shows the frequent deformation of the student's cognitive development: the universal model of the new piece of knowledge is not constructed by Adam himself via generalisation but it is directly placed to the child's mind from the outside. Story 2 shows the lack of variety of isolated models of the concept 'altitude of a triangle' in the students' minds. They have no image of altitude outside of a triangle.

The of Adam's piece of knowledge '2 + 3 = 5' is not yet ready for generalisation. The father's 'help' 1. decreases the boy's motivation and self-confidence, 2. prevents this new universal model to become an organic part of the boy's cognitive structure, 3. orients the boy's learning strategy towards receiving rather than creating new knowledge.

Under such teaching styles, the student loses his/her ability to create his/her own knowledge and starts to believe that mathematics depends on memory and skills. The second story serves as an example of such a case. In B2 and C10, memory is considered as the most powerful tool of the solving process.

4. Stage of Generalisation - Stories

Story 2a (continuation). Next day Ben and Cindy solved the following task:

Problem 2. Square *ABCD* is cut into three triangles along segments *AC* and *EC* where *E* is the midpoint of side *AB* of length 6. Find the areas of all three triangles.

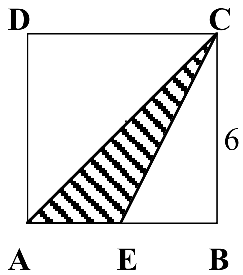

Fig.2

Cindy drew figure 2 and Ben put '6' to side *BC*.

C16: It is the triangle (she points to *AEC*) as yesterday.

B17: You know, the square is thirty six. So here is half, its eighteen (he puts '18' into triangle *ACD*).

C19: Here is the midpoint (she points to *E*) so here (she writes '3' to both segments *AE* and *EB*) _ are threes.

B20: This rectangle (he points to rectangle *EBC*□[4]) is eighteen so, _ let me see, _ here is _ we have nine (he writes '9' into triangle *EBC*).

C20: I know! (hastily) The square is thirty six, cut off eighteen, cut off nine _ you have _ you have nine! (To the experimenter sitting aside) We got it!

After a while Cindy drew figure 1 and said: "We could solve it this way!" Ben drew rectangle *KPM*□ and put '12' into triangle *KM*□ and Cindy put '6' into both small triangles. She said: "Its funny, these are always equal".

[4] By □ we will denote a point which was not named by students.

Story 3. The following problem was given to one class (23 seventh graders, age 13): Problem 3. There is a square *ABCD* on a grid. We know its vertexes *A*(0;0) and a) *B*(1;1), b) *B*(2;1), c) *B*(3;1), d) *B*(7;1), e) *B*(15;1), f) *B*(79;1). Find the co-ordinates of vertexes *C* and *D*. The square is anti-clockwise oriented.

Students started to work and results of cases a), b) and c) soon appeared on the blackboard. Then Dan put the result of the task d): *C*(6;8), *D*(-1;7). He said, that he knows point *D* even in case f); it must be *D*(-1, 79).

Eileen: How do you…_ oh yes, it is… _ (to the teacher) I saw that the first co-ordinate is –1, but I did not notice that the second is that from *B*.

Franc: I know,…co-ordinates of *C* __ well, either they are both even or odd.

Grace: (ran to the board) It's easy! Use the table. (She drew table 1 and it took more than 2 minutes.) You know we did it with those marbles[5]. (Suddenly she interrupted her work and asked Hilda to solve the case f![6]

A		*B*		*C*		*D*	
0	0	1	1	0	2	- 1	1
0	0	2	1	1	3	- 1	2
0	0	3	1	2	4	- 1	3
0	0	7	1	6	8	- 1	7
0	0	79	1				

Table 1

Hilda did it correctly. Grace asked her: "How did you did it?" She answered: "As here" and she pointed to the previous row.

Story 4. Five old Ivy visited her grandmother. Grandma showed Ivy two little plates both covered by napkins and said: "There are three big strawberries on this plate and two on this one. If you tell me how many strawberries are hidden here you will have them all." Ivy hesitated for a moment. Then she put three fingers to the first plate, two fingers to the second one and after a while she said: "Five". Grandma uncovered the plates and asked Ivy to check her result. Ivy did it with a great enthusiasm and asked grandma to continue this game.

5. Stage of Generalisation - Comments

The generalisation of isolated models (experiences and pieces of knowledge) is determined by finding connections between some of isolated models. This web is the most important product of the stage of the isolated models.

Bell (1993), when discussing the psychological principles that underline designing teaching methods, started with *connectedness*: - "A fundamental fact about learned material is that richly connected bodies of knowledge are well retained; isolated elements are quickly lost."

In her analysis of the act of understanding, Sierpinska considers four basic mental operations: identification, discrimination, generalisation and synthesis. "All four

[5] The girl remembered a task solved two days ago by putting all particular results into table.

[6] Grace is Hilda's close friend. Grace gives her a great support in mathematics.

operations are important in any process of understanding. But in understanding mathematics, generalisation has a particular role to play. Isn't mathematics, above all, an art of generalisation? 'l'art de donner le même nom à des choses différentes', as Poincare used to say?" Sierpinska (1994, p. 59).

We agree with this statement provided that 'donner' covers both our terms generalisation and abstraction.

In story 2, Cindy noticed the connection between figures 1 and 2. Hence the solution of Problem 2 served as a universal model for solution of Problem 1.

In story 3 tasks a), b) and c) were isolated models. For Dan, case d) was universal model. The decisive moment of generalisation was finding the pattern in the set of co-ordinates.

In story 4, Ivy's former experiences were isolated models of counting objects: "How many dolls are one doll and three dolls?" "How many balls are two balls and three balls?". At the beginning, the solving process of such tasks concerns dolls or balls rather than numbers 2 and 3. Later on, a child notices that there is something common in these situations, namely the relations between numbers. Ivy, step by step, started to understand that the result of addition does not depend on what objects are added but only on the numbers of objects in both groups which are being put together. All these isolated models help to develop a new idea: it is possible to count strawberries without seeing them. Fingers can be used instead. Fingers start to play the role of a universal model. The idea of using fingers is the mental act which prompted new knowledge.

6. Stage of Universal Model(s)

The universal model, as a result of generalisation, is a starting point for one, two, or even three new mental processes:

1) abstraction
2) further generalisation for which it plays the role of an isolated model
3) restructuring of an existing mathematical structure

Story 2. The universal model of *finding the area of a 'strange' triangle* is based on two isolated models (ΔKLM in Fig 1. and ΔAEC in Fig 2) and previous knowledge *area = half of base times altitude* for all right angled triangles and possibly also for cases where the altitude falls within the triangle. The essence of this discovery is the idea of complement. The further development of this new piece of knowledge can be:

1) understanding the formula $A = \frac{1}{2}b.h$ for all triangles,

2) noticing that the idea of the complement helps in finding the area of a figure,

3) looking for arguments in other formulas for area (or volume)

Story 3. We have to consider two pieces of knowledge:

A) *pattern for finding co-ordinates of vertexes D and C* and

B) *tabulating data is a strategy when looking for the number pattern*.

Universal model A) in Dan's and Grace's mind is a result of his and her own generalisation and it is prepared for the further development:

1) Finding an algebraic notation: A(0;0), B(n;1), C(n+1;n-1), D(-1,n).

2) Solving a more demanding task with A(0;0), B(n;m).

3) Observing linkage between algebra and geometry, e.g. 'segments (0;0)(n;1) and (0;0)(-1;n) are perpendicular'.

In Hilda's mind universal model A) is represented by table 1, particularly by the row with number 7. However, this piece of knowledge is not the consequence of generalisation. It was suggested to Hilda's mind from the outside and therefore it is not prepared for the further development. Grace helped Hilda to improve her confidence, but it did not improve Hilda's knowledge. If Grace would like to help her friend to grasp the idea, she should gave Hilda the hint "put your results into a table" and let her invent the pattern by herself.

For Grace the universal model of piece of knowledge B) is tied to her previous experience with the problem about marbles. This time, she just applies this universal knowledge to the particular situation.

Story 4. The piece of knowledge '2 + 3 = 5' is subsidiary. Ivy's invention is a new piece of knowledge *when counting objects, fingers can be used as representatives of the elements of the couned set*. Fingers are a universal model in all counting situations.

Universal model plays a central role in discovering mathematics - not just in ontogenesis, but also in phylogenesis. On one hand, it unifies and simplifies a variety of isolated models, on the other hand it is the building stone of mathematical structure and frequently also strong tool for solving problems.

7. Stage of Abstraction

<u>Story 5.</u> Following problem was given to one class (25 fifth graders, age 11):

Problem 4. a) Given a hexagon $ABCDEF$ created by squares $AXEF$ and $XBCD$ where $|AB|$ = 11 cm. Find the perimeter of the hexagon if $|AX|$ equals a) 2 cm b) 3 cm, c) 4 cm, d) 5 cm, e) 6 cm, f) 7 cm.

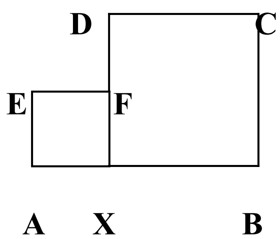

Fig.3

The students started to solve the problem by means of the grid. Soon they found that the perimeter of the hexagon is a) 42 cm, b) 40 cm, c) 38 cm. Jane found the pattern and predicted numbers 36, 34, and 32 as the results of tasks d), e) and f). Her mistake gave rise to searching for a 'trick' which would give the perimeter of the hexagon in figure 3 if 'the big number' (e.g. $|AB|$), 'the left number'(e.g. $|AX|$), and 'the right number' (e.g. $|BX|$) are given. During the four following months long terms were gradually abbreviated by letters '*b*', '*l*', and '*r*' and several formulas for the perimeter had been found. However, neither of them was 'perfect'. Either it was divided to two parts as

perimeter = 3l + 3r + r - l if AX is shorter than XB, otherwise 3l + 3r + l - r,

or it used a description also like

perimeter = 3l + 3r + the length of segment DE.

The final step in looking for the 'perfect' formula was made by Ken. He used Lena's formula *perimeter = 3b + the difference of numbers l and r* and said that his brother showed him how to write the difference of two numbers, say 3 and 5, using 'sticks'. It is like 5 - 3 = 2, but $|3 - 5|$ is 2 as well. Finally he gave Lena's formula following form: *perimeter = 3b +* $|r - l|$.

The new piece of knowledge in this story is the concept of the *absolute value.* It was invented in two steps. First Lena introduced a new idea of *the difference of two numbers.* It is an abstraction of the concept of subtraction, but this new idea does not belong to the world of arithmetic since it is not expressed in its language. The second step was made by Ken by introducing the suitable symbol for this idea.

This story illustrates three main characteristics of the stage of abstraction:

1) The abstraction starts with a solving process that needs a new more abstract idea (with respect to the solver's existing knowledge).
2) Frequently, the invented idea is described by the common language, hence it is not an organic part of the solver's existing mathematical structure.
3) Introducing a suitable symbolic new idea is incorporated to the solver's existing mathematical structure.

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