INSIGHTS INTO PUPIL'S STRUCTURES OF MATHEMATICAL THINKING THROUGH ORAL COMMUNICATION

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Abstract: *In this paper we consider the communication which took place between and with two nine-year old pupils whilst they undertook non-standard geometrical tasks. The findings from the analysis of the cognitive and social aspects of the communication showed that there were considerable differences in their ability to articulate their knowledge and in their structures of mathematical knowledge. These differences were not apparent in the teacher's evaluations of their mathematical ability using standard assessments.*

1. INTRODUCTION

Teacher-pupil communication can be the intermediary between the pupil's knowledge structure and the teacher who is trying to determine the extent, depth and complexity of that structure (Pirie, 1998). Communication, in general, can take many forms such as written, spoken, pictorial representation, etc. In our experience most of the communication in mathematics lessons is one-way. First from teacher_pupil when the teacher is explaining the work to be done by the pupils, which is exclusively oral and secondly, pupil_teacher when the teacher assesses the pupils and this is mostly in written form. This usually is time-restricted assessment of the pupil's ability to reproduce algorithms, memorised facts and formulae. Our experimental work, part of which is described in this paper, has shown us that if one wants to determine the level of the mathematical knowledge of a pupil, then it is beneficial to communicate orally with him/her. An implication of this is the need to develop the pupil's communicative skills otherwise they are disadvantaged even if they have a good mathematical knowledge.

Sfard (2002) argues that 'communication should be viewed not as a mere aid to thinking but as tantamount to thinking itself'. Our own experience would support this view. We consider good communication between pupil-pupil and pupil-teacher as a valuable tool in constructive approaches to teaching. It helps linkages to be made between the isolated pieces of knowledge in a pupil's mind and a more generalised and connected view of what were disconnected pieces of information becomes possible (Hejny, 2000). The more precise and unambiguous the mathematical language used, the more profitable it is for the building and the development of mathematical knowledge.

The experiment described in this paper is part of longitudinal research into pupil's understanding of geometrical concepts which was started in 1993 in Prague. The original research was carried out by Jirotková (2001), guided by Milan Hejny. Since 2001, further work has been continued by the authors (Jirotková, Littler, 2002). The research focussed on pupil-pupil communication and its links with learning processes. This paper considers two roles of communication, first that of learning about a structure in a pupil's mind and secondly, its role in the structure building process. That is, we focussed on how oral communication gave us an insight into that part of the pupil's structure of geometrical knowledge related to the tasks and how the necessity to communicate about these tasks by the pupils led them to a more analytical perspective of the solids which resulted in the development of the structure of the knowledge related to the geometrical shapes with which they were dealing.

The main part of this paper undertakes to show how the diagnosed communicative phenomena opened a window to the mathematical structures within the minds of the pupils involved. We base our work on the experience of carrying out the tasks with 14 Czech pupils and undertaking deep analysis of the resulting responses (Jirotková, 2001). The opportunity arose to work with pupils in the UK and it was felt that by working in two different mathematical cultures we could identify interesting differences in students' responses. In addition doing the study using the English language in UK schools would avoid the difficulty of translating idiomatic Czech statements into English.

The original objective of the research was to see how UK children responded to the tasks concerning tactile and visual perception of solids. However during the fourth task (see below), the analysis of the communication between two UK children gave us the clearest evidence of two very different levels of communicative skills and this led us to investigate the underlying structures of geometrical knowledge.

2. METHODOLOGY

2.1 The Sample .

From 15 protocols of pupil-pupil communication we chose the one related to the experiments with two nine-year old pupils from the fifth year of a primary school in the United Kingdom in September 2001. The pupils were chosen by their class teacher because they were considered good at mathematics and the school assessment showed that their mathematical ability was approximately the same. The boy, Ben was very confident about his ability in mathematics when he spoke to us during a warming up discussion. The girl, Gina was more circumspect about her ability and in our general discussion before starting our research, she showed that she had a wide range of mathematical terminology. For the research, an experimenter was present to introduce the tasks. Also present was an observer who made notes about non-verbal expressions, prepared the materials for the tasks, tape-recorded the discussions and took photographic evidence. The tasks were undertaken in the head-teacher's office during the normal school timetable and the experiments proceeded without any disturbance.

2.2 The Research Tools

As stated earlier, the two pupils were involved in a trans-national research project (Jirotková, Littler 2002) and for that research we used four tasks which were devised to test pupil's cognitive processes through tactile and visual perceptions of geometrical solids (Jirotková 2001). The first three tasks were undertaken individually. For the fourth task, the two pupils worked together and it is this task on which we will concentrate. Below we briefly indicate the three tasks which each pupil undertook prior to working together on task 4 and which showed the communication which took place between pupil and experimenter as the pupil described their images based on the tactile perception of the solids.

Task 1: The pupil was asked to dip a hand into an opaque bag containing one solid (truncated pyramid) and perceive it tactilely. Then using the same hand dip it into another opaque bag to try to find the same solid amongst 11 other solids. Before checking his/her correctness visually, the pupil had to say why they thought the chosen solid was the same as the initial one they had felt.

Task 2: The pupil had to work with his/her hands in an opaque bag containing 8 solids. The pupil was asked to choose one, which was different from the other solids and to give reason(s) for his/her choice before s/he took it out of the bag.

Task 3: The pupil was asked to divide 13 solids hidden in an opaque bag into two groups, so that all the members of one group had a common property of the pupil's choosing. The pupil had to describe what that common property was before the solids were taken out of the bag.

Task 4: The two pupils played the game 'Owl'. Fourteen solids were placed on a table in front of them. One pupil was asked to choose a solid in his/her mind. The other pupil had to determine the chosen solid by putting questions to the first pupil who was only allowed to respond with the words 'Yes' or 'No'. The roles of the pupils were then reversed and the game played again. This sequence was repeated once again at the pupils' request.

All the experiments were tape-recorded. These were then transcribed into a form of protocol and analysed with the help of the photographic evidence and the notes taken by the observer. Qualitative methods of analysis were employed (Cohen, 2000; Stehlíková, 1999). We looked for and identified the communicative phenomena which occurred, we described them, assessed them from social and cognitive aspects and verified them by applying them in other experiments.

3. STRUCTURES OF MATHEMATICAL KNOWLEDGE

3.1 Social aspects

In our experiment we observed the communication between and with the two pupils, Gina and Ben. The situation into which we put them was non-standard and this enabled us to show that there was a big gap between their level of mathematical knowledge and mathematical culture as well as their communicative skills. In other words the pupils exhibited two very different cognitive structures, contrary to the results of their arithmetical tests. We will pay attention to these differences and analyse them. The social interaction which had to take place so that the differences could be overcome and not lead to communication collapse will be considered only marginally. However it is worthwhile presenting some facts about the communication bewteen the two pupils, especially those related to misunderstanding.

In the interaction between two subjects we found three basic phenomena:

1. a stronger individual, 2. a weaker individual, 3. a pair.

These phenomena showed the underlying cognitive structures as well as the personalities of the pupils. We list below some of the communicative phenomena which our analysis of the three basic phenomena showed. This influenced the discussions which in turn indicated the cognitive processes in their mathematical knowledge structure.

1. The stronger individual

- was aware that she was on a higher mathematical and communicative skill level,
- tried not to show to her partner that she was on a higher level,
- took responsibility for the ongoing discussion,
- made an effort to reflect her partner‘s cognitive structure (created a model)
- checked if her model was correct, and constantly modified the model according to the situation,
- looked for her own approach to this model,
- tried to avoid complicating the situation and/or misunderstandings,
- preferred to find her own truth rather than get ready knowledge (the value of her truth was placed higher than the value of authority).

2. The weaker individual

- realised that he was on a lower level,
- was ready to accept the role of weaker partner,
- did not retreat into himself initially but kept himself open to his partner,
- tried to re-establish his position of equality by all possible means,

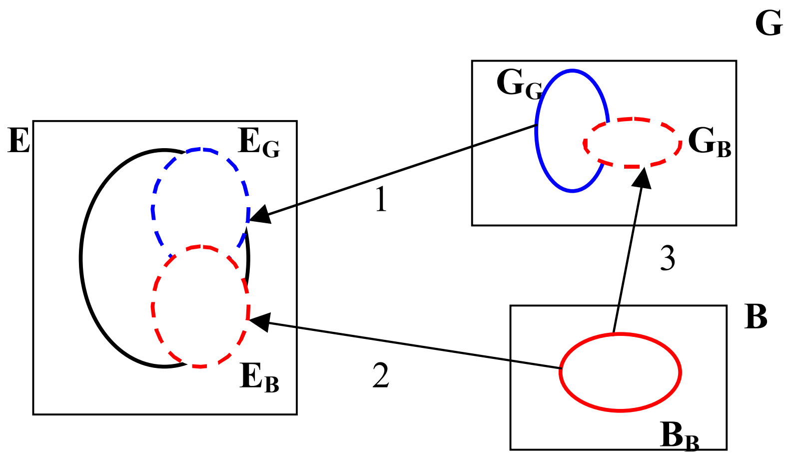
- preferred the value of authority to his own ability.

3. The pair exhibited the following characteristics
   - climate: both pupils made considerable effort to understand each other,
   - values: the girl perceived mistakes as a natural way of learning, whilst the boy considered them as a something wrong and so was afraid to make a mistake.

3.2 Cognitive aspects

The communication between the three individuals - experimenter (*E*), Ben (*B*) and Gina (*G*) - involved in this experiment provided the basis for the construction of pictures of a small part of their structures of mathematical knowledge which they used in solving the task. We analysed the dialogues between $B - G$, $E - G$, $E - B$ focussing on those structures of knowledge which were active during the task. We then constructed and described these structures.

The relationships, which our analysis showed existed between them, are expressed with the help of the following diagram:



E, G, and **B** represent the cognitive structures of the experimenter, Gina and Ben respectively which we assume and of which we only have evidence of a small part.

$\mathbf{B_B}$ **-** Ben's mathematical knowledge used in communicating in Task 4.

$\mathbf{G_G}$ **-** Gina's mathematical knowledge which she used in communicating in Task 4.

$\mathbf{G_B}$ **-** a sub-structure of **G** which is Gina's model of $\mathbf{B_B}$ in her mind.

$\mathbf{E_E}$ **-** the experimenter's mathematical knowledge used in communicating in Task 4.

$\mathbf{E_B}$ and $\mathbf{E_G}$ **-** two sub-structures of **E** and are the experimenter's models of $\mathbf{B_B}$ and $\mathbf{G_G}$ respectively in his mind.

Line 1 represents the projection of $\mathbf{G_G}$ into experimenter's mind to form model $\mathbf{E_G}$, line 2, the projection of $\mathbf{B_B}$ into experimenter's mind to form model $\mathbf{E_B}$ and line 3, the projection of $\mathbf{B_B}$ into girl's mind to form model $\mathbf{G_B}$.

When we analysed the protocol we did not find any evidence of an attempt by Ben to make a model of Gina's structure. In other words he could not fully interpret all of Gina's communications.

3.3 Illustrations

We illustrate some of the mental structures and processes of creating a model of a structure as shown in the diagram, as well as some social phenomena, by giving a concrete example. For this we have chosen the concept of a quadrilateral, which itself is strongly structured and we describe our construct of the structures $\mathbf{B_B}$, $\mathbf{G_G}$ and $\mathbf{G_B}$ of this concept in the minds of the two pupils.

In the following section, F1 indicates the first fragment of dialogue from the protocol etc. All direct speech is written in italics.

$\mathbf{B_B}$**:** Our analysis showed that Ben's understanding of the concept of quadrilateral, according to Hejny (2003), was only at the second stage of the process of construction of a piece of mathematical knowledge, which Hejny calls the stage of 'isolated mental models'. This means that the pupil does not make any linkages between his various isolated experiences of concrete models of the concept of quadrilaterals. In our initial discussion he accurately drew a rectangle and a square on a plane (2D) and described their visual attributes. However during the tasks when the rectangle was a face of a three-dimensional (3D) solid he was unable to differentiate whether the shape was a rectangle or a square. This was obviously a new experience for him, which he had to store as an additional isolated model and through communication, it is hoped that linkages with the existing isolated models will begin. We suggest that until this time he did not have any experience of communicating about rectangles other than those drawn in 2D. In Task 1, he could not recognise the base and upper face of a truncated pyramid as squares or quadrilaterals when perceiving them tactilely. Again the verbalisation of his tactile perception and the recognition of the faces of 3D solids as 2D shapes were missing from his experiences. In the following fragment F1, from the protocol of the experiment, he used the word 'rectangle' but this was a copy of the word used by Gina. It can also been seen in F1 that he was not able to find the linkages between rectangles and squares in this situation (B5, B6, B7) and he lost his confidence when asked to state clearly if a square was a rectangle. From this analysis we derive that the structure of the quadrilateral for Ben consisted of two isolated models, square and rectangle, both in 2D as a drawing in a plane.

Fragment F1 (from Task 4)

G3 represents Gina's third input into the dialogue.

G3: *Has it got any triangles on any of its faces?*

B3: *No.*

G4: *Has it got rectangles as any of its faces?*

B4: *Yes.*

G5: *You do not reckon this as a rectangle.* (She points to a square face?)

B5: *No* (said very quietly).

G6: *So it's not a rectangle?*

E1: *You are looking at me B, it is you who must decide.*

B6: *Did you ask if that was a rectangle?*

G7: *Do you think this is a rectangle?*

B7: *No.*

G8: *Has it got square ends…faces?*

G_G: The analysis of our observations of Gina would indicate that her understanding of the concept of quadrilateral is at the fourth stage of the process of construction of a piece of mathematical knowledge, that of the universal (mental) model, if not further (Hejny, 2003). She showed that she had already passed through the process of generalisation (third stage) and was seeing a general overview of already existing isolated models and getting an insight into the community of models. Her clear understanding of the meaning of 'square' was shown by her use of it when she created a new meaningful name for a solid – *a squared-based pyramid.* She knew the terms rectangle and square and was aware that the square could be considered as a special case of a rectangle. She realised that there should be an agreement about it between the parties involved (see G5 in F1). Gina strictly differentiated between the terminology for three and two-dimensional objects, a situation we have not found in our work with even older pupils. She expressed that she was aware that a quadrilateral is a two-dimensional figure even if it is perceived as a face of a solid. Evidence of this is given in F2.

<u>Fragment F2 (from Task 3)</u>

G4: …..*they* (a group of solids) *have got <u>at least</u> one face which is <u>a quadrilateral in two-D.</u>*

Gina could define quadrilaterals using their attributes like sides and corners. She articulated a structure that showed she had already built linkages between the concepts of square and rectangle in 2D and as faces of 3D solids.

Gina had shown that her mathematical culture was exceptional for a nine-year old. She showed that she understood perfectly the meaning of the used quantifier *at least* (see F2 above). The experimenter was not sure that she was using the expression with a correct logical meaning and repeated her sentence back to her without the quantifier within it. Gina immediately corrected him pronouncing the quantifier with stress.

$\mathbf{G_B}$: During the communication with Ben, Gina modelled Ben's structure of knowledge in her mind. When she started the dialogue between them, she assumed that their level of knowledge was the same. After the initial response from Ben, she began to realise that her assumption was not correct. During their communication she constantly checked and up-dated model $\mathbf{G_B}$.

In Gina's third input to the dialogue (G3 in F1 above), she started to use language, which she thought matched Ben's level and which was a more imprecise mathematical language compared with that she had used during a previous discussion with the experimenter. She used the phrase *on any of its faces*, a rather loose phrase implying could Ben find a triangle anywhere on the face? When Ben reacted quickly to her question G3 (in F1), Gina returned to a more precise use of language in G4, now using the phrase *as any of its faces,* which is much more specific and indicates that it is the face, which is a rectangle. In G5 she checked if her model of Ben's knowledge was correct. The quiet response from Ben indicated to Gina the uncertainty of his knowledge of the relationship between square and rectangle and she politely asked him to confirm his answer (G6). This question caused Ben to be confused and lose confidence, so he sought the confirmation of a 'higher authority', the experimenter, to give him the 'correct' answer. In B6, Ben repeated the question to gain more time to consolidate his thoughts. He finally decided, in the stress of the situation, that a square was not a rectangle in B7 but as shown later, he had not achieved a better understanding. When Ben gave his answer B7, Gina modified her model of $\mathbf{G_B}$ to take account of Ben differentiating between square and rectangle and then formulated G8: *Has it got squares ends....faces?*

Gina showed again in G8 that she was not happy when she felt she had to use imprecise language, *end,* because she followed this by using the correct geometrical one, *face*. This is one of several good examples showing that Gina felt she was the stronger partner in the dialogue, so she took the responsibility for avoiding ambiguities, which could lead to misunderstanding or to the collapse of the game, and for improving Ben's knowledge by first using words at his language level and then giving the correct mathematical language. When he realised that Gina was mathematically stronger Ben retreated into himself for a while until he accepted the role of the weaker partner. Following this acceptance he again entered fully into the dialogue.

We do not describe the structure $\mathbf{E_E}$ because the experimenter was there simply to direct the discussion. We provide two examples of the process of creating and modifying the model $\mathbf{E_G}$. The first example has already been presented in F1. The experimenter checked if his model of Gina's structure was accurate by finding whether Gina had used the expression *at least* in the meaning of everyday or mathematical language. The second example (F3) shows that when the model of the structure does not coincide with the structure itself, it could lead to misunderstanding.

Fragment F3 (from Task 2)

E1: *From the shapes in bag, I would like you to choose the one that you think is different.*

G1: *Different?*

E2: *You decide how it is different, not me.*

G2: *So it has something different from all the others?*

E3: *It could be, but it is for you to decide what you think might be different. You are looking puzzled. ...*

The experimenter did not realise that Gina was not asking what types of differences there could be, but that she was checking if she had interpreted the word 'different' properly. She was aware that the experimenter had not used the word in a correct syntax. In E2 the experimenter insisted that it was the pupil's place to define the difference. In G2 Gina formulated the question in a syntactically correct way recognizing that her question G1 had not been understood. In E3 the experimenter was still insisting that 'different' should be defined by the pupil. Although Gina was aware of this misunderstanding she did not feel the necessity to clarify it further because it did not influence the following discussion. The experimenter did not modify his model of the girl's structure $\mathbf{E_G}$.

It is worthwhile mentioning the phenomenon which related to both cognitive and social aspects and occurred during the game 'Owl', which was played four times, Gina chose the solids cube and sphere, which might appear to be the most common shapes. In the first game she chose the sphere because she considered its attributes to be so different from the other shapes that it would be easily differentiated from them. She knew the attributes and the language involved. Ben on the other hand chose the concave pentagonal prism and pyramid with concave pentagonal base. We consider this as evidence of Ben's attempts to choose the most complicated shapes for Gina hoping that she would fail in the game. He probably did not realise that he was setting a trap for himself, since answering the questions relating to these shapes would also require him to have a good geometrical knowledge of them. He was pleased when Gina failed in one game, even though the failure was due to him giving an incorrect answer to one of Gina's questions. This is an example which shows how difficult it was to separate the cognitive and social aspects.

4. CONCLUSION

Three facts emerged from this analysis.

The first was that using non-standard mathematical tools to assess students gave a clearer indication of their thinking processes and mental structures than did assessment based on traditional forms of algorithmic tests.

Secondly, the use of material, which allows pupils to perceive shapes tactilely and then communicate those perceptions, is a valuable tool which helps them understand the attributes of shapes more deeply. It is only when the pupils have to communicate about the shapes that they come to a better understanding of them and their properties.

Thirdly, a short time spent listening to students and then analysing their responses gives a more complete picture of their abilities, both cognitive and social (Cooney, Krainer, 1996). It also helps to develop the teacher's sensitivity to student's responses (Ollerton, Watson, 2002). The implication of this statement means that teachers should prepare work which requires their students to discuss the work they do and their understanding of it as often as possible and should consider each well prepared and guided discussion as a fruitful learning situation.

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