

# CONSTRUCTING MATHEMATICAL CONCEPTS WITH CALCULATORS OR COMPUTERS

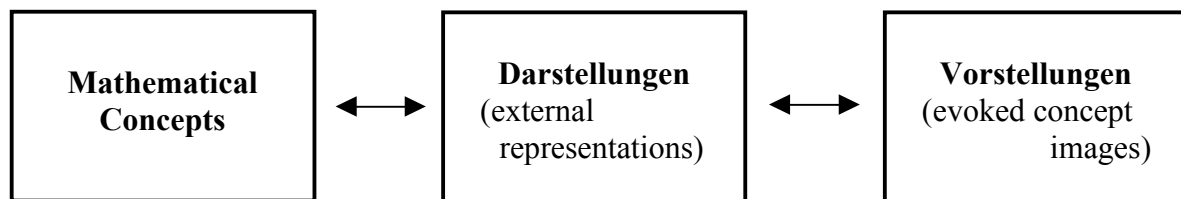
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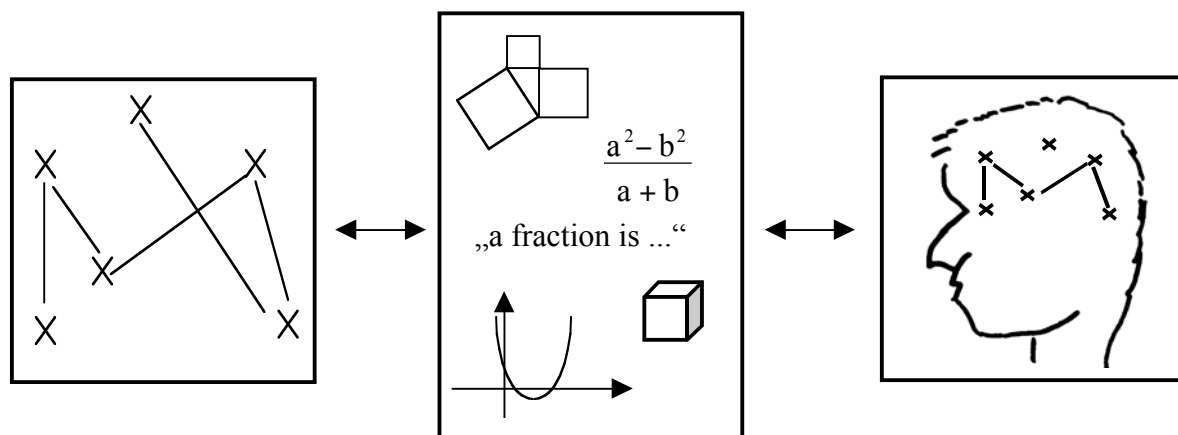
**Abstract:** We will discuss theoretical aspects of learning and understanding mathematics by the use of calculators or computers. We distinguish a syntactical mode of working with these machines and a semantic mode. We discuss a dualism in concept development and reflect, if this also affects the use of calculators and computers. Especially we analyze the role of guess-and-test procedures by the use of calculators and computers (ONE-WAY-PRINCIPLE)

## 1. Constructing Knowledge by the Use of Calculators or Computers

In this paper we will analyze how the use of calculators and computers may influence the learning of mathematics in schools. The process of learning and understanding mathematics may be described<sup>1</sup> by the following Fig. 1:



more "concrete":



We see mathematics as “something” independent from human beings or from human brains like trees, birds, genetic codes, time, space, electricity, gravity, infinity, ... (MEISSNER 2002b). And *Darstellungen* then are (external) representations of mathe-

<sup>1</sup> We will use the German words to emphasize the broad diversity of details.

mathematical ideas, which we can read, or see, or hear, or feel, or manipulate, ... *Darstellungen* can be objects, manipulatives, activities, pictures, graphs, figures, symbols, sequences of key strokes on calculators or computers, tags, words, written or spoken language, gestures, ... In a *Darstellung* the mathematical idea or example or concept or structure is hidden or encoded.

Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each *Darstellung* evokes a personal internal image, a *Vorstellung* (cf. concept image, TALL & VINNER 1981). Thus *Vorstellung* is a personal internal representation. A *Vorstellung* in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same *Darstellung* may be associated with many individual different internal representations, images. Each learner has his/her own *Vorstellung*.

We see two types of *Vorstellungen* which often interfere in mathematics education. According to STRAUSS (1982) young children have a global non-differentiated concept of a certain domain which is appropriate to solve operations or tasks adequately within that domain. The concept is biological in origin and refers to a "common sense knowledge". It is a spontaneous concept in the sense of VYGOTSKY, based on an intuitive thinking (in the sense of Bruner). But then schooling starts and another concept develops - a "cultural" (STRAUSS) or "scientific" (VYGOTSKY) concept - which is reflective and self-conscious, and which is based on analytic thinking<sup>2</sup>.

These two concepts interfere. Abilities relating to the common sense knowledge decrease into a "chaos" while adequate abilities of a cultural knowledge have not yet developed. The global view gets destroyed and the children suddenly cannot solve problems which they could solve before<sup>3</sup>. But step by step the schooling builds up a new and more structured concept. Some of the former abilities "reappear" more powerful than before, now based on a different view. Other abilities are lost for ever.

A similar conflict situation GINSBURG has pointed out: Children sometimes "display a gap between written work, on the one hand, and informal methods on the other". We also observe these contradictions when CARRAHER et al. (1985) report from the mathematical abilities of children who sell fish and can add and multiply correctly with money but never got a schooling.

I.e. there are two different "sides" or views of a problem which do not only exist transitionally as parts of developmental stages. The one side is informal (GINSBURG), intuitive (BRUNER), a common sense knowledge (STRAUSS) or spontaneous (VYGOTSKY). The roots of these concepts are independent in origin from the roots of the written work. DAVIS & MCKNIGHT (1980, p. 42) argue, "that students need to learn to deal with mathematics in both of the two basic modes:

1. as a meaningless set of symbols that are manipulated according to explicit rules;

<sup>2</sup> also the theory of the VAN HIELES can be regarded as a developmental change from intuitive to analytic

<sup>3</sup> examples see STRAUSS (1982) or MEISSNER (1986)

2. as meaningful symbols, where the translation between real-world problems and the abstract mathematical representation of these problems is an essential part of the task."

The same distinction is made by RESNICK (1982) discussing "syntax and semantics in learning to subtract" or by MEISSNER (1978, 1983) in discussing a digital versus an analogues number sense to explain gaps between approximation and estimation or between a semantic use of the calculator by guessing and testing or the syntactic use in just pressing the correct sequence of buttons. This dualism exists in general when we use calculators or computers.

## 2. Procedures, Concepts, and Procepts

Already PIAGET (1985, p. 49) has pointed out that "actions and operations become thematized objects of thought or assimilation". This idea has become very important today to understand the development of concept images (*Vorstellungen*) in mathematics education as a process of *interiorization* or *reification* or *encapsulation*.

According to GRAY & TALL (1991, p. 72ff) there is a "duality between process and concept in mathematics, in particular using the same symbolism to present both a process (such as the addition of two numbers  $3+2$ ) and the product of that process (the sum  $3+2$ ). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema". The successful mathematical thinker uses a mental structure called procept (TALL 1991, p. 251ff), "which is an amalgam of process and concept".

In 1994 GRAY&TALL proposed the following definitions: "An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object. A procept consists of a collection of elementary procepts which have the same object." In TALL et al. (2000) we find examples for symbols as process and concept. For more details on the theory of procepts see GRAY & TALL 1991, SFARD 1987, DUBINSKY 2000, MEISSNER 2002a, and others.

Which now is the role of calculators and computers in the process of developing powerful mathematical concept images? We may quote DAVIS (1984, p.29f): "When a procedure is first being learned, one experiences it almost one step at a time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a *thing*" ... The procedure, formerly only a thing to be done - a verb - has now become an object, ..., it is now, in this sense, a noun".

To use calculators or computers we press sequences of buttons to run algorithms or procedures or programs. According to DAVIS and the theory of procepts we should expect that this continuous use and application of procedures will lead to powerful procepts. But obviously there is no automatism.

Already SKEMP (1978) distinguished between instrumental understanding and relational understanding. Instrumental understanding is characterized by selecting and applying appropriate rules to solve the problem without knowing why ("rules without reasons"). Only a specific *Darstellung*, an external observable behavior, is expected: "Tell me what to do and I will do so". There is not necessarily an adequate mathematical concept image behind.

Thus we have to analyze the activities with calculators and computers more carefully.

Using these machines we distinguish a syntactical use and a semantic use (MEISSNER 1983). The syntactical use is given when we use the machine as an "operator" or a "function machine" or a "computation aid" by pressing the correct sequence of buttons to get a wanted display like a sequence of digits, a graph, a table, etc... For example we compute the four basic operations with a calculator or we press the keys along a given formula, or we press the keys to draw a graph for a given function, or we need certain values in a spread sheet, etc.

Indeed, here it only is necessary to know the correct sequence of buttons to press. An instrumental understanding is sufficient to get the correct solution. From this point of view many manuals for calculators or computers or software packages just provoke the development of an instrumental understanding. The same is true for many manuals for handheld computers like ... (I do not want to blame specific products where big lists of mnemonic codes must be learnt and hierarchies to apply them.)

In the semantic use the meaning of the problem situation is in the foreground and the machine only is an aid to reduce the burden of sophisticated calculations or drawings or it is an aid to visualize relations or properties. There are many nice examples how calculators or computers can be used semantically, starting with the work from DAVID TALL on calculus. Today we can observe plenty of semantic activities: Using dynamic geometry software (*Cabri*, *Euklid*, and others), investigations with computer algebra software, using appropriate software-hardware configurations<sup>4</sup>, ...

But till now there are no general theories how calculators or computers can be used semantically. Thus in the following chapters we will concentrate on the role of guess-and-test procedures to develop mathematical *Vorstellungen*. We assume that guess-and-test may be a bridge to connect the two basic modes of working, to connect the spontaneous common sense knowledge with a "cultural" or "scientific" concept.

### 3. The Role of Guess and Test

The development of understanding is a process of "communication", see Fig. 2. Interaction is necessary and adaptation. In the PIAGETIAN meaning the learner realizes (consciously or unconsciously) a conflict and seeks "equilibrium". Calculators and computers facilitate these interactions. For the syntactical use a specific output only is wanted. The interaction process is simple, a behavioristic style of learning is sufficient. Working semantically however needs an open minded view and flexibility and

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<sup>4</sup> For example graphic calculators, see also the T\_-Project.

creativity and intuition and the knowledge of mathematical relationships. That means we need more than simple stimulus response interactions with a machine.

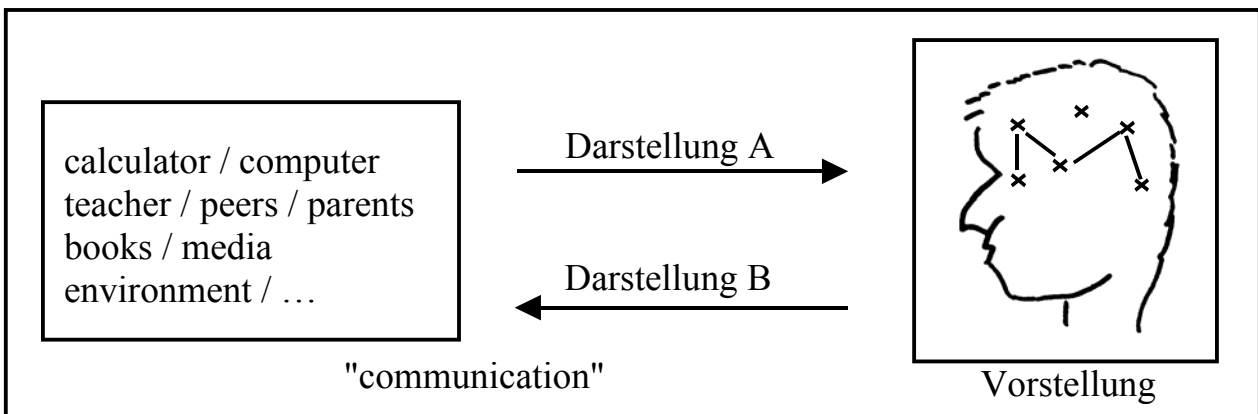


Fig. 2. Developing Understanding

Since more than 25 years we observe children, students, teachers, and colleagues working with calculators or computers. Most of these “experts” have developed not only powerful *Vorstellungen* but also two specific attitudes which were unusual till now in a traditional mathematics class room. These “experts”

- use intensively guess-and-test procedures (often without being aware of it) and
- demonstrate a large and often unconscious *Vorstellung* without an adequate “communicable” understanding<sup>5</sup>.

Our key experiment originates from about 1980. At that time we used a calculator (ABLE from Texas Instruments) where the keys to press had no symbol on it. That means before using the calculator we had to figure out the meaning of each button. Our test persons had to find out the mode of operation of a specific button. There were “experts” with more than 1000 key strokes, there were a lot of “meaningless” repetitions, and each “expert” estimated the amount of key strokes having done three to ten times smaller than it really had been (MEISSNER 1982b).

We also realized that outside from mathematics education a guess-and-test behavior is quite normal to build up a *Vorstellung* of the situation being confronted with and that most of the people even are not aware of their guess-and-test behavior (MEISSNER 1985). There also are systematic vocational training programs based on guess-and-test activities, i.e. for training people at a simulator (pilots, car drivers), using models instead of real situations, etc. Thus we assume that the intensive use of calculators or computers can evoke the development of an intuitive, spontaneous, common sense knowledge which not necessarily corresponds to an adequate “analytic thinking”.

<sup>5</sup> Being asked for rules they are quick in pressing diverse sequences of buttons, but very often they cannot give precise verbal descriptions or explanations.

So we developed a specific approach for the use of calculators and computers in mathematics education, the so called ONE-WAY-PRINCIPLE . It shall help building up intuitive, spontaneous, common sense knowledge by using guess-and-test procedures with calculators and computers.

#### 4. The One-Way-Principle

One of the fundamental ideas of mathematics is the concept of functions. In school mathematics we apply this idea in many domains and we use calculators or computers as powerful tools. Related topics in school mathematics for example are

- the four basic operations,
- square roots,
- percentages,
- growth and decay,
- linear and quadratic functions,
- trigonometric functions, etc.

Each learner must "construct" his/her own individual *Vorstellung* for each of these "functions". Own experiences must be coordinated with external demands. Interaction and adaptation are necessary, see Fig. 2. The "functions" must be experienced, examples and counter examples are essential. And since calculators and computers reduce the burden of computing we can without big efforts study the relations between different variables, we can analyze properties, we can draw graphs, etc.

To do this we recommend a specific method of teaching. To understand the concept of function

$$x \xrightarrow{\textcircled{f}} y$$

(including in particular the four basic operations addition, subtraction, multiplication, division) means to comprehend  $(x,f,y)$  as a unity, i.e. as a procept. We suggest that the three different problems

(a) given  $x$  and  $f$ , find  $y$

(b) given  $y$  and  $f$ , find  $x$

(c) given  $x$  and  $y$ , find  $f$

all are approached using the same algorithm.

The first step for the student is to learn the algorithm for the "direct" problem (a). That means when using calculators or computers the student first must learn the syntactical use of the machine to get the wanted result  $y$ . Finding solutions then for problems of type (b) or (c) we still use the same sequence of syntactical key strokes, but before starting to press the keys we first have to guess a value or key for  $x$  (or  $f$ ).

For example, suppose we have a problem of type (b). An arbitrary choice of  $x$  leads to a "wrong" result for  $y$ . But the magnitude of the "mistake" gives hints for a better

choice of  $x$ . And with that  $x$  we get a "better" result for  $y$ , and new hints for a next guess. Step by step we build up an intuitive model about the relations and rules between  $x$ ,  $f$ , and  $y$ . We develop a concept image of the "procept  $(x, f, y)$ ".

We have called that method the One-Way-Principle (**OWP**), see MEISSNER 1979: There is only *one way* to solve all related problems. Either we just work syntactically (problems of type (a)) or we work semantically by guessing and testing, still using the same syntactical sequence of keystrokes.

We have practiced the OWP in many situations, see below. Our observations show that children are working very concentrated with quite different strategies. Very often they get an unconscious feeling about the new concept before they can explain their discoveries or their good guesses. Important, the students get a (conscious or unconscious) feeling for (mathematical) relations and properties which also is available when they just have to guess or to estimate, especially in situations where a calculator or computer is not available. In this sense the use of machines has developed a certain type of an intuitive "common sense knowledge".

And when guessing and testing gets boring the students themselves start asking for more efficient solution procedures. Then we can introduce reverse functions and algebraic transformations. In this sense the OWP is an intermediate step. It helps developing an intuitive, spontaneous common sense knowledge before an algebraic, scientific concept gets started. The OWP fills an "understanding gap" between function and reverse function, it can reduce the instrumental understanding of algebraic transformations. And when the students get lost in the algebraic approach, in the "scientific" concept, they can "retire" to their guess-and-test procedures in their "informal" concept.

## 5. Empirical Results from Using the One-Way-Principle

We will summarize some empirical findings from using the OWP.

### 5.1. Hit the Target

Hit the Target is a calculator game which trains the understanding of multiplicative structures: An interval  $[A, B]$  is given and a number  $n$ . Find a second number  $s$  so that the product of  $n$  and  $s$  is within the interval  $[A, B]$ . We have more than 1000 guess-and-test protocols from primary school students (age 8 - 10). Our results (MEISSNER 1987) show that the students after a certain training develop excellent estimation skills (guessing the starting number) and a very good proportional feeling (very often less than three guesses to find a correct solution).

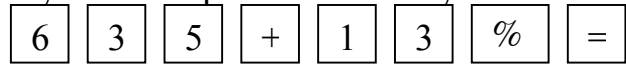
### 5.2. Calculator Games Using the "Constant Facility"

Simple calculators often have a "constant facility", that means "operators" like "-253" or " $\div 47$ " can be stored. Discovering the hidden operator by guess-and-test trains additive or multiplicative structures. In the game "BIG ZERO" we hide a subtraction operator and ask "Which is the input for getting 0 ("big zero") in the display? In the game "BIG ONE" we hide a division operator and ask "Which is the input for getting

1 ("big one") in the display? Discovering these hidden operators by guess-and-test develops an intuitive conceptual understanding of additive respectively multiplicative structures. Playing these games we observe after some training excellent approximation skills. For more details see the dissertation from LANGE 1984.

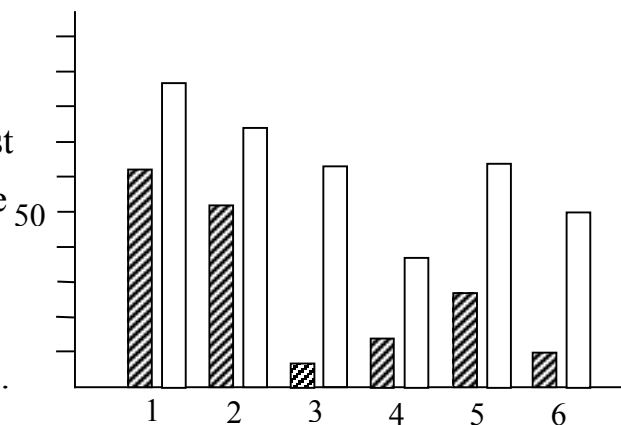
### 5.3 Percentages

There are calculators which work syntactically like we speak in our daily life: "635 + 13 % = ..." needs the key stroke sequence



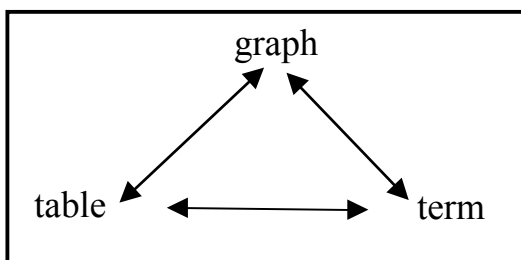
We introduced the topic percentages in about 10 classes (at different times) with calculators using the OWP, results see post test (white bars). We also administered the same post test (6 problems) with about 500 students who got the traditional course of teaching percentages (dark bars).

For more details see MEISSNER 1982a.



### 5.4. Linear and Quadratic Functions

The problem of teaching functions is given by a diagram. The traditional school cur-



riculum has not much success in developing a deeper understanding between the gestalt of a graph and the related algebraic term. In her dissertation MUELLER-PHILIPP (1994) showed that the use of the OWP developed that missing link. For linear and quadratic functions the students very easily could sketch the gestalt for a

given term and determine a term for a given graph.

### 5.5 Guess and Test Protocols

We urge our students to write protocols from their guess-and-test work. Each protocol is an excellent *Darstellung* of the related *Vorstellungen*. By reflecting the protocols also unconscious *Vorstellungen* may become conscious. The protocols often allow effective discussions in the class room, and those discussions can bring the shift from an "informal knowledge" to an "analytic thinking".

### 5.6 Guess and Test Strategies

The guess-and-test protocols allow insight into diverse *Vorstellungen*. We studied more than 1000 protocols (MEISSNER 1987) and found some typical guess-and-test



behavior<sup>6</sup>:

- There are favorite starting numbers, independent from the specific data of the task.
- Many of the strategies used are selected unconsciously.
- Finding a possible strategy the user often does not use all the knowledge he/she has.
- A once chosen strategy is dominant. Arguing logically from the steps already done a change of strategies often happens less and later than it should have been done.
- Most of the guesses follow a “proportional feeling approach”, often interrupted and started again at a different domain when not quick enough.
- Approaching a goal from only one side is more often than approaching alternately from both sides (nesting).
- There are more guesses than necessary (from a logical point of view).<sup>7</sup>

### 5.7 Summary

Teaching the concept of functions the syntactical use of calculators or computers can be expanded systematically to a semantic use by the OWP. The guess-and-test activities seem to allow a shift from an “informal knowledge” to an “analytic thinking”.

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<sup>6</sup> VOLLRATH (1986) describes similar findings. Some of our findings listed above we also realized when colleagues started exploring new computer games (Minesweeper, Freecell, Hearts, Spider Solitar, ...)

<sup>7</sup> Sometimes there are even some more guesses, though the goal already is reached.

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