DEFINITION-CONSTRUCTION AND CONCEPT FORMATION

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Abstract: This paper aims to explain the main ideas concerning the activity of constructing a definition (definition-construction) and to question the analysis of the process of definition-construction and mathematical concept formation relationship in terms of the "understanding" of the latter.

INTRODUCTION

In the mathematical activity of a researcher we observe a parallel between the construction of a concept and the construction of its definition. We share Vergnaud's idea that "a concept cannot be limited to its definition, at least when we are dealing with its learning and its teaching".

The existing dialectic between the construction of a definition (definition-construction) and the construction of a concept raises questions about the implementation of defining activities in education and their purpose from two points of views, in two ways: knowledge acquisition about the notion of definition on one hand and the understanding of a mathematical concept on the other. In this paper, we use the word "understanding" as "we understand something if we see how it is related or connected to other things (also from other fields) we know"; in particular, this definition serves as to underline the relations between concepts in mathematics, and we come round to Arsac's view on the importance and the interest of the heuristic approach, experimentation and additional research, different from a prewritten reasoning.

In this way, we aim at proposing activities of definition-construction, and to give elements for the analysis of the evolution of the definition-construction in parallel with the construction of the mathematical concept. That involves a knowledge of students' existing conceptions relating to the concept of definition in mathematics, in order to be able to differentiate what is relevant to conceptions on definition from what concerns the understanding of the mathematical concept involved. Moreover, we are aware of the distance between a sentence (even if it is produced by students) and something concerning the possible knowledge of the concept involved, that is why one key question for us, with regard to learning a "new" concept, is how to study the produced statements as a mark of the concept formation.

We assume that an activity of definition-construction allows an actual work on a concept and we think that it is the necessary way to get in order to have access to a mathematical concept in connection with the possible work on the concept during the defining activity. Vinner (1980; 1991) introduces the notions of concept image and concept definition and thus brings a model which seems to have a certain potential to explain phenomena in the process of concept formation, especially for the analysis of a definition construction process because

> "the cognitive reason for asking somebody to give definitions should be the assumption that definitions help to form concept images" (Vinner-Hershkowitz, 1980)

Considering that the definition construction process is a new way for the analysis of concept formation (let us notice that in the defining activity we propose, unfamiliar concepts are at stake), we need to reread Vinner's notions of concept image and concept definition and we assume that the notion of "common cognitive paths" (Vinner-Hershkowitz, 1980) seems to be more appropriate for our framework. At this stage, we require to define what we mean by "concept" (because to analyse the construction of a concept constitutes a new point of view about 'concept'); we define by "concept" a set of three elements (always in extension during the construction process): definitions, use (it means that the concept is available in specific situations), significant examples and counterexamples. Let us underline that one can define through usage(s) and that a definition discriminates examples and counterexamples hence the interest to study links between definition construction and concept formation. When all is said and done, we will use the notion of "common cognitive paths" with the explanation of the relations between different components of the mathematical concept the students proposed during their definition construction process.

EXISTING CONCEPTIONS ABOUT DEFINITION IN MATHEMATICS

We will quickly remind you of some general conceptions about definitions which will come mostly from teaching practices and handbooks (it depends of course on the teaching level). For example, at university level, we find some discourses on definition like: "we should prove the existence of the defined object", "we can check a definition relating to previous knowledge", "a definition should be minimal, non redundant, non circular …" etc [1]. Some of them originate directly from epistemological and logical representations (in a certain philosophical tradition of definitions) such as the existence problem (Aristotle, Leibniz …), the abbreviative process (Pascal, Popper …), the equivalence between the defining sentence and the defined thing (logical point of view) and so on. We assume that this kind of conceptions is present in students' minds (more or less implicitly), granted, but is still fuzzy because it strongly connects with teaching practices, and thus is variable from one class to the other. Moreover, we underline the fact that, in education, the concept of definition (in mathematics) is not problematised; in general, one definition of a concept is given to students and thus is not questioned from both the mathematical concept and its definition viewpoints. De Villiers (1998) notes down this aspect too:

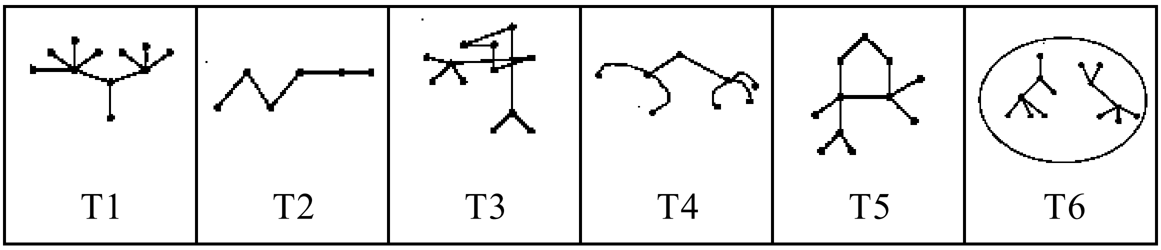
> the construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching.

What are the students' conceptions involved in an activity of definition-construction? Students were immersed in a mathematical culture and that may influence them in such an activity, at two different moments: first, at the end of the activity, in terms of the didactical contract, that is to say students try to answer the teacher's request with some rules of conformity for their produced definition(s) (as above, with a minimal sentence for example). And secondly during the activity of definition-construction, students may call on other sources of knowledge (and thus other conceptions – mathematical or others - may appear). That is the reason why we should explain both students' conceptions on definitions and the process of definition-construction in parallel with the concept-formation.

PROCESS OF DEFINITION-CONSTRUCTION

The activity

The activity concerns the mathematical object 'tree' (called in P1'thingummy').



***P1**: how could you define the mathematical object 'thingummy', knowing that: T1, T2, T3 and T4 are representations of 'thingummy' and T5, T6 are not representations of it ?*

***P2 (exercise-given when students think they have done with the definition)**: let G be a graph (i.e. a collection of dots and lines between two dots) connected (i.e. in only one piece). Prove that G admits a spanning tree (i.e. a tree with same vertex set as G)*

We think that this activity is conceivable for students from grade 10 to university. A more precise presentation (with a whole resolution and an analysis of it) is available in Ouvrier-Buffet (2002). Let us underline that the main characteristic features of this situation of definition-construction concern: starting from given examples and counterexamples, asking for a definition (it is not a classification task, the recognition of the object with regard to previous knowledge is not involve), and using it in an exercise (a proof is required). Whereas here the exercise allows students to return to the definition and to use the definition. Notice that the definition of the chose object (tree) is not available to the students who take part in this experimentation (NB: 'tree' was only a tool of representation in France, often use that way, hence there was no institutionalised definition before University level, before the new curricula for 2002, in which 'tree' appears as a mathematical object in grade 12). We would like to remind you that the meaning of "defining activity" in recent research does not mean all the features explained above. De Villiers (1998) and Borasi (1992) propose defining activities, which are really re-defining activities of known mathematical concepts (known by students), circle or geometrical basic concepts for instance. More specifically, Borasi proposes three instructional heuristics for the design of defining activities:

1. The in-depth analysis of a list of incorrect definitions of a given concept
2. The use of definitions in specific mathematical problems and proofs
3. The exploration of what happens when a familiar definition is interpreted in a different context. (Borasi – 1992 – p.155)

and she underlines the difficulties in building defining activities in which unfamiliar concepts are at stake (and uses the notion of "à la Lakatos").

Process of definition-construction

In this paragraph we propose to outline the main features of the process of definition-construction as it was observed in three groups (students in the first university year) [2]. Different functions attributed to a definition guide students in their activity; for example, at the start of their investigation, the everyday meaning of "definition" (i.e. to define something means to explain it) gives students the possibility to get to grips with the activity. Other functions (to build the mathematical object, to recognize it) bring a goal to the students for this defining activity, and thus enrich it. Moreover, the students quickly go into a process of comparisons between the given examples and counterexamples on the one hand, and analogies with objects from mathematics (e.g. geometric form, closed polygon, circle …) or from other fields (e.g. genealogical trees, organization charts, roots, chains …) on the other hand. Both these processes (of comparison and analogy), even if not only intramathematical, contribute to the definition-construction and thus allow to work on mathematical properties. After that "introduction", all the processes of definition-construction feed itself with the use and the generation of examples and counterexamples and the exchanges between the students. This part can be analysed in the light of Lakatosian elements as zero-definition and proof-generated definitions [3] that is to say to analyse students' statements with the notions of zero-definition, definition, property, proof-generated definition as proposed by Ouvrier-Buffet; this kind of analysis permits to understand the chronology (and the progression) of the students' statements in terms of definition.

Students' conceptions on "definition"

As explained above, there are valid conceptions when a mathematical concept is constructed, in particular when a (new) definition is inscribed in a theory, it must be minimal, non redundant etc. The core students' conceptions, which appear during the defining activity, concern sufficient condition(s) and the independence of the representation (of the mathematical object) used; and at one stage for students (i.e. they think they have done with a particular property), the students' conceptions concern more the form of the definition (for example minimal definition i.e. short sentence) and the nature of the words used in it; indeed the students try to make their sentence look like a definition, without using everyday words (when they think their sentence – as in only one piece, without circuit - is fuzzy, not precise), to formulate "THE" sentence. This brief presentation of the students' main conceptions mobilized in our activity now allows us to focus our discussion on the concept formation.

DEFINITION-CONSTRUCTION AND CONCEPT-FORMATION

We chose to present to you three excerpts (groups number 3, 1 and 2), each allowing a questioning in terms of the parallel between definition-construction and the understanding of the mathematical concept 'tree' involved. For each example we propose a brief chronological story of the evolution of the students' definition-construction and an explanation of the links between student's definitions of 'tree' and mathematical definitions involved.

Example 1: emergence of a new property, production of a counterexample

Definition 1	a tree is a connected graph without cycle.
Definition 2	between any two vertices, there exists a unique path.
Definition 3	a tree is a graph with n vertices, (n-1) edges, without cycle.
Definition 4	a tree is a connected graph with n vertices and (n-1) edges.
Definition 5 (inductive)	a tree is a vertex or a tree T for which we add a new vertex adjacent to only one vertex of T (induction step).

Mathematical definitions of 'tree' at stake for group 3

<u>Story</u>: Yohann and Arnaud's first "perceptive definition" was a set of points and lines, in only one piece and without cycle (def1). Afterwards, they defined 'thingummy' (part 1 of the activity) as a set of points and lines, with a unique path between any two points (def2). The defining activity carried on, because they had other ideas, and we notice that the continuation of the defining activity inscribes itself in the "descriptive defining" (according to Freudenthal) [4], and thus this sequence is propitious to the observation of new knowledge on 'tree. We propose an excerpt with a discussion on the relation between the number of points and the number of lines of a 'tree'; here both definitions 3 and 4 (see above) are at stake.

Yohann: we need one line fewer than the number of points. Perhaps we can define 'thingummy' starting from that … It may replace the idea of the path. It is the same thing when there is only one path between any two points.

Arnaud: it is the same idea.

Yohann: it is the same condition, but it exists in another form; and then we need other things … with n points and (n-1) lines …

Arnaud: here is a counterexample with 3 lines and 4 points.

Yohann: yes, this condition is not sufficient.

At this time, they have difficulty grasping the condition to be added. However, they define 'thingummy' with an inductive relation (def5) and are satisfied to have found two so different definitions (different in terms of their use: for these students, the inductive definition (def5) is much of interest in order to construct a 'thingummy' and the definition 'a unique path' is useful in order to check if an object is a 'thingummy' or not). From the understanding point of view, we note, in this example, a progression in the understanding of the mathematical concept 'tree' because of the presence of new knowledge in terms of: the existence of an another property (n points and (n-1) lines) on one hand, and the insufficiency of it to define 'tree' on the other (let us notice they share the conception that a definition is a sufficient condition).

The objective of the exercise (proof) was reached for this group because these students were able to write the proof (with the inductive definition 5) and came close to definitions 3 and 4 (see above).

Example 2: proof-generated definition as a mark of good understanding?

The mathematical definitions of 'tree' at stake for group1 are definitions 1, 2 and something relating to definitions 3 and 4.

Story: as did the previous group, Fabien and Abdel described 'thingummy' with a sentence close to definition1 and defined it as a set of lines and points, with only one path between any two points (def2) in the first part of the activity. During the second part, they formulated more precisely that a tree is a graph (the knowledge on the concept of tree as a subclass of a graph appeared and they explained that a graph has specific properties and a tree has more specific properties). Moreover the search of the proof enables the students to grasp a new property (which concerns both definitions 3 and 4), hence a proof-generated definition (Lakatos, 1961) was observed. The excerpt below presents the emergence of the proof-generated definition.

Fabien: there is always one line fewer … than the number of vertices.

Abdel: are you kidding ?? (10 seconds, they check this property on the given examples and counterexamples) (…)

Observer: is it an another definition of 'tree' ?

Abdel: no, it is what we need to transform the graph into a tree. During the first exercise, we didn't need this relation, it was not useful, because if someone draw a tree, they needn't know that. (…) We can define 'tree' with that, it is sufficient to define 'tree'. (…)

Fabien: when we have a closed circuit, we have as many vertices as lines. And with one line fewer than the number of vertices, we can't have a circuit. (…)

Abdel: thus, to prove that we have a tree, we count the lines, the points (…) and the points must stay in a group.

Fabien: hence, connected !

We have difficulties to express what went through the students' minds, and we cannot limit ourselves to the analysis of the produced statements. This excerpt attests to an evolution in the understanding of the concept 'tree', and it is an opportunity to relate this production of definition to the proof-generated definition, because produced during the making of the proof. We underline the fact that, in this group, in spite of several produced definitions (def 1, 2 and a part of def3 or def4), the students were not able to write the proof (they 'understood" the idea of the proof, but that is all). An explanation of this phenomenon may be the difficult concept of connectivity, involved in this activity and the problems of this group to connect 'connected' (this word was given in the exercise) and the existence of a path between any two vertices. This extract confirms us in the opinion that the production of different (correct from the mathematical viewpoint) definitions does not attest (even if they are produced by the students) that students have a "good" understanding of the mathematical concept. In such activities, we need a place in order to observe the concept in action: the exercise (a proof is required, the definition appears as necessary) is one, but is it sufficient?

Examples 1 and 2 show a common cognitive path with four different components that partially form a certain concept image of 'tree': in only one piece without cycle _ a unique path _ n points and (n-1) lines. The concept of "connectivity" is absent in this path. We will observe now a different cognitive path, including the "connectivity".

Example 3: a better understanding…

The mathematical definitions of 'tree' at stake for group2 are definitions 1, 2 and 5.

Story: this group (Angelique, Thibault, Vincent) defined 'thingummy' with an inductive definition (def5). But they grasped other properties and they did not agree on the status of the latter, because for Angelique and Vincent, one definition is enough, as testified by this sentence: "that will be included in the first definition (…) I have the impression that we have finished our work". Nevertheless, …

Thibault: and can we speak about path? Perhaps it will help us to define 'thingummy'. For T6, there exists no path between this point and that one. Thus, it is not a thingummy.

Angelique: yes … there is a path between any two points, it is always the case.

Vincent: we need a unique path from a point to another one.

(5 minutes later … they try to understand the relations between the properties)

Thibault: we have points connected with lines, without circular form, in only one piece. (…) One path because we have one piece and the uniqueness because there is no cycle.

This activity allows the observation of two processes (according to Freudenthal) i.e. the descriptive defining (a posteriori) and the constructive defining (a priori) [4]. For this group, we are tempted to analyze their production in terms of a better understanding of the mathematical concept 'tree' (relating to the previous group) because of the connections they are able to make between the properties such as 'connectivity' and 'path': they formulate the relation between 'connectivity' and the 'existence of path between any two points' on one hand and the relation between 'without cycle' and the 'uniqueness of the path' on the other. In spite of that, we think that with this first analysis in terms of understanding (and better understanding) in order to be more objective, we need other elements. For example, we may propose other exercises to make students use the concept of 'tree'. Two kinds of such exercises could be proposed: ordinary exercises (where the mathematical concept 'tree' is explicitly involved) and unordinary exercises (where the concept 'tree' is at stake non explicitly), for instance this one: "construct a labyrinth with the property: between any two points, there is only one path". In such an exercise, the concept of 'tree' is mobilized non explicitly.

CONCLUSION

We assume that we shall use both the Lakatosian notions (zero-definition, proof-generated definition) in such a situation of definition-construction which deals with a heuristic approach, and the students' conceptions on "definition" in order to analyze the students' statements. Furthermore, in terms of the "understanding" or the "better understanding" of the mathematical concept involved in this kind of situation, and in view of the deep discrepancy between the formation of a concept and its verbal definition, we shall underline what kind of relations and characteristic properties students grasp, and how they deal with them. In particular, the notion of "common cognitive path" underlines the main features of the definition process. The present results concern a work in progress and we venture as an hypothesis that the choice of the exercise (in order to observe the use of the concept) is central but may be limited. Indeed the exercise (only one was proposed) is an inappropriate one to make certain definitions of 'tree' worthwhile, for example this one: "a tree is a graph which has no cycle and if we add a new edge then we create a unique cycle". Hence, we must propose other exercises (for which proofs that used definition are required) in order to complete our analysis and to consolidate the results in terms of "better understanding".

NOTES

[1] It is not an exhaustive list !

[2] [3] see Ouvrier-Buffet (2002)

[4] " … the describing definition … outlines a known object by singling out a few characteristic properties" ; "… the algorithmically constructive and creative definition … models new objects out of familiar ones" (Freudenthal - 458)

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