A CLOSER LOOK INTO COGNITIVE ABSTRACTION PROCESSES IN THE LEARNING OF ABSTRACT MATHEMATICS

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We present findings from our research of cognitive abstraction processes, where we have used an alternative computerized research tool and methodology. This methodology offers a new kind of insight into cognitive processes, which is unique in the kind of conclusions it suggests, and in the wide scope of conclusions it allows one to make. Also, the research tool investigates processes rather than their outcomes, quantifies qualitative situations, and 'externalizes' cognitive processes of an individual. The latter is done by means of a comprehensive individual profile that is composed of about 60 graphs, most of them continuous, and some non-graphic data.

1. Introduction

The wish to understand how students construct mathematical concepts and schemes led us to develop a methodological approach that focuses on processes rather than on their outcomes. Then, since the essence of Mathematics is abstractions and abstract concepts, we used this methodology to study the Cognitive Abstraction Processes that are involved in the learning of a piece of abstract mathematics.

Hence, we present this methodology vis-à-vis our study of the cognitive abstraction processes. We begin our presentation with an overview of our research and then we get into a more detailed description of: a) The subjects; b) What concept we used in the learning experiment and why; c) What processes we observed and how we initiated them; d) How we conveyed the learning text and why; e) The assessment attribute of our tool or, How we monitored the processes; f) The products of our tool - The individuals learning profiles; and, g) The analytical attributes of the method - Our research conclusions. For obvious reasons this paper treats these issues only in part. But first, we give a brief outline of the conceptual frame that leads our work.

2. The Conceptual Frame

Our Conceptual Frame focuses on three aspects of our methodology: The first one is the broad experimental context: the clinical attribute of our study, which was designed with the intent to reduce the effects of some of the more common aspects of learning, such as The "Human" Teacher (by using a computerized learning module); The Teaching (by using learning from examples paradigm - not worked examples); The Text (no elaborations or explanations), and The Context (individual computer lab sessions). The second aspect is the method that we chose to monitor and to observe the processes that we were investigating, which was a computerized device. Finally, the third aspect that we discuss here is the Mode of Representation (MoR) that we chose to present the processes that we have monitored, which was a graphic profile.

When the goal of a study is to understand the basic learning mechanisms themselves, and if "learning takes place inside the learner and only inside the learner" (Simon, 2001, p. 210), then a lab experiment seems to be the best choice for this study. Clearly, one could control those variables that have an effect on the learning more easily in the lab than he/she could in a class environment or in any other social structure. Also, Simon argues that "…Putting learning in books does not desocialize it" (p. 207). By continuing this line of thought, we can assert that laboratories are a 'social make' and therefore they also incorporate the social values and beliefs of the 'social construct' that built the specific lab. Thus, conducting research on learning in a lab context, does not necessarily disagree with the social trends of the educational milieu; rather it offers another level of observation. Though, some constructivist educators could argue that from a pedagogical point of view a "clinical" setting for a learning experiment is not appropriate, as their main interest is in learning as a social activity (usually in a class context). Still, Atkinson et al. (2000, p. 185) contend for the transferability of lab experiments to class context.

One way to partly control the effect of the (human) teacher, and its inevitable partiality is to use a computerized learning module; One way to partly control the sub-text that is hidden in any learning text is to present the text in a very 'facts-only' rigid way without any elaboration or any explanation; and, finally, one way to partly control the effect of the teaching on the learning is to present the concepts to be learned by examples of the concept.

Atkinson at al. (2000) present a comprehensive review of the literature about the Learning-From-Examples (LFE). They refer to learning from 'worked' examples rather than from examples as instances of the concept, when they discuss the LFE paradigm that is used in much of the research on Problem-Solving (PS) in the last couple of decades. Here the LFE serve as the focus or the target of the research, and not as its context. This kind of research aims at gaining insight about the LFE as a preferred mode of teaching or learning. Indeed, as a result of the wide-ranging research that focused on LFE many teachers today use positive and negative examples in their teaching. However, researchers already used LFE in the mid-1950's in Concepts-Formations (CF) experiments (Atkinson at al. 2000, p. 182). Though, the role of LFE in the PS paradigms differs in principle from its role in the CF paradigms. Langely (1987, p. 133) describes a research study of learning concepts through generalization from positive and negative examples (using a computer simulation). Thus, when the LFE serve as a tool to convey the learning text in a learning experiment, it is a part of the context of the experiment rather than its focus.

Until recently research on learning processes was primarily based on initiating the desired processes, observing/monitoring the outcomes of these processes, and then using these outcomes to deduce about the processes themselves. It goes without saying that the conclusions that this kind of research might offer could only be of very general attributes. It could be suggested that under specified conditions one process is better than another (in some respect), but it could not have said much about Why this is the way it is. Correlation analysis might have suggested some possible "reasons". But, then again, too much was veiled to enable one to accept these conclusions without reservations. Obviously it would be unjust to ignore the technological barriers that past generations of researchers had to face, and which lead them to investigate the cognitive processes through their outcomes.

Today, the call to concentrate research on processes rather than on their outcomes, though long overdue, has undoubtedly manifested itself in most of the different camps of the educational milieu, and is nicely stated by Simon (2001, p. 210):

> "The implication of a learner-centered learning theory (it sounds obvious when you put it that way) is that research on learning must aim at understanding the intracranial learning processes and mechanisms."

Thus, the rapid development of new technologies that is reaching new climaxes practically every day, allows one to attend to the task of a more careful study of the learning processes themselves (Atkinson at al. 2001, p. 215):

> " Our ability to do this is enhanced whenever we find new technique for observing behaviors ... tape recorder, verbal protocol that it records, the eye movement camera, the videotape…EEG, PET scans and fMRI are now just beginning to deliver important new information".

Recently, many cognitive researchers began to use computer simulations as models of cognitive processes (Langely, 1987). Though, computer-simulation models do deal with processes, still they are only simulations and not the processes themselves (see also Langely, 1987, p. 134), or, as Dennett (1987) argues forcefully:

> "There is no way an electronic digital computer could be programmed so that it could produce what an organic human brain produce …" (P. 324)

Nevertheless, computers could be very efficient in monitoring the very complex learning processes. Olive (1986) describes such an experiment where he uses the computer to trace a learning process.

The first thing that beginning researchers learn is that the quality of their conclusions strongly depend on the methods they are using to organize and to describe their findings. Researchers always faced the problem of finding an adequate MoR for their findings. Needless to say that quantitative educational research paradigms tend to use mathematical tools as MoR (statistical tables, diagrams, graphs, etc.), for example, the Forgetting curves (number of words/time graphs) that Ebbinghaus already used at the end of the nineteenth century in his research on memory (Driscoll, 1994, p. 16). Later on, the Behavioristic researchers who study behavior change recorded their experiments on graphs (of numbers of occurrences of behavior/time) (Driscoll p. 50-52). In the last few decades, qualitative researchers have tended to use statistical MoRs.

Though the qualitative milieus do describe processes (by means of a written description), yet, from all the MoRs that have been used hitherto, graphs are the most adequate MoR for describing continuous processes.

3. An Overview of the Research

We closely observed two cognitive abstraction processes that we had identified theoretically: *Cognitive Abstraction Process* (**CAP**) and *Reversed Cognitive Abstraction* (**RCAP**). For the purpose of this report we define the two different learning processes by the mode of their initiation. One is initiated by learning from a text which is presented in an *increasing* order of abstractness level (CAP - group U in the experiment – ten ninth-grade subjects), while the other is initiated by learning from a text which is presented in a *decreasing* order of abstractness level (RCAP - Group D in the experiment - ten ninth-grade subjects). Computerized learning modules presented the learning texts, which were completely identical in both modules except for the order in which the chapters appeared in each of them.

The module kept track of all the time periods that the subjects spent at each point (assignment) as well as their answers, during the experiment (about 3 hours). At the same time, it created a personal file for each of the subjects, which was later used to assess the learning process of the individual. Moreover, the module enabled the researcher to assess each of the answers (about 204 "small questions") according to approximately 50 different indexes (a few of them were not used in the final analysis after proving to be redundant or too hard to interpret). Subsequently, the module created a file of numerical vectors for each subject as a product of the evaluation stage, where each of the vectors represents the progression of the learning process according to one of the indexes. These vectors were readable by the computer algebra system, *Mathematica,* that we used for constructing a graphical profile for each individual subject. Eventually, the analysis of the outcomes of our experiment was based primarily on these graphical profiles.

4. The Research

We chose the definition[1] of *Mathematical Relation* (MR) to be the concept to be learned by the subjects in order to study the cognitive abstraction processes, primarily because MR is a highly abstract concept. Also it was guaranteed to be completely new to all our subjects, thus allowing us to relate the subjects' knowledge of a MR to the learning processes we were observing. Moreover, the MR generalizes most of the main mathematical concepts that students learn in school. Hence we were able to investigate the initiation of abstraction capabilities in the participants as they were trying to relate the new MR concept to other concepts that they had previously learned.

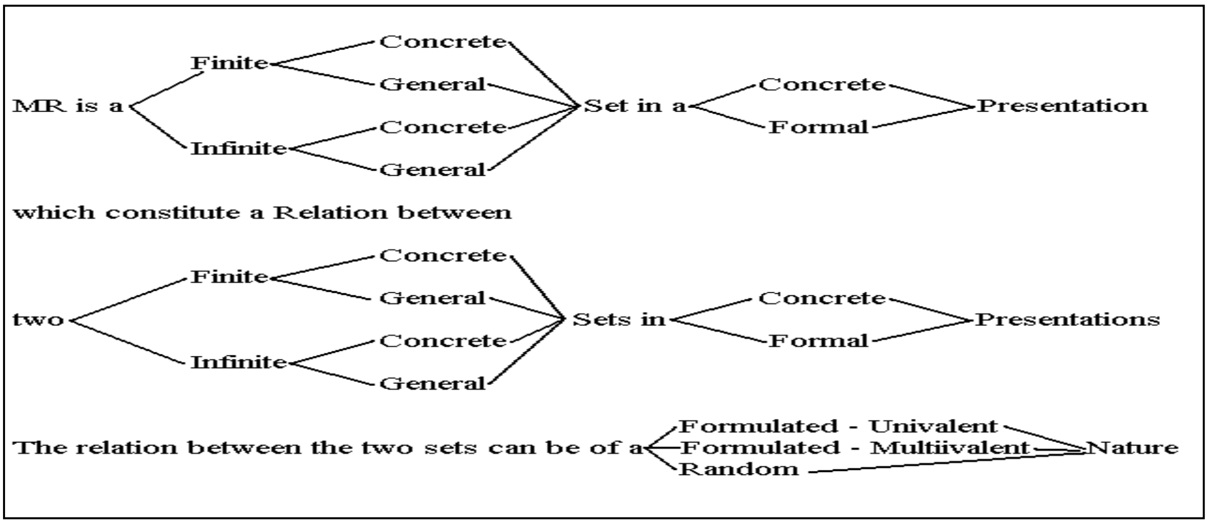
Also, we chose to convey the learning text to the participants via a computerized learning module as a kind of 'non-partial' teacher.

[1] A is any set, B is any set. A Mathematical Relation, from set A to set B, is a set Each pair is in the form (a,b), where the left component is an element of A, and the right component is an element of B.

Furthermore, in order to reduce the effect of the sub-texts we chose to present the learning subject with many exemplars of MR, rather than with explanations. These exemplars were followed by fixed repeated reasoning assignments to force the learners to "think aloud".

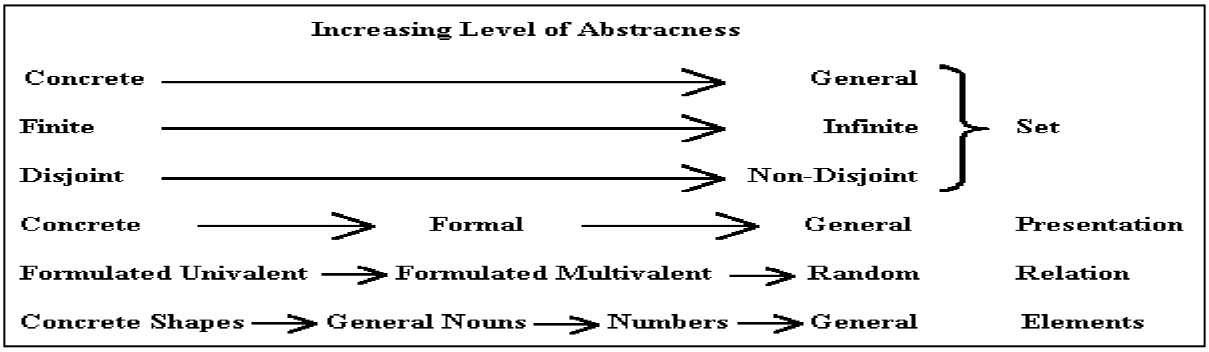
We monitored separately the learning processes of ten different aspects of the MR concept, and also the progression of three general aspects of the learning. The latter are the Time aspect, the Fixation (in degenerate examples) aspect, and the Affective aspect. The different aspects of the MR were determined according to its different components and their characteristics, as they are presented in Figure1.

Figure 1 The MR Components and their Characteristics



Thus, we determined the abstractness level (figure 2) of each of the exemplars of the MR according to the relative location of its components and their different characteristics, in the widening circles of mathematical abstractions (Mitchelmore & White, 1995)

Figure 2 The Abstraction Scale



4.1. The Tool

Each of the fifteen chapters of the learning module contained various examples (positive and negative) of MRs, all of which are of approximately the same level of abstractness. Where a Defining Statement (DF) accompanied each example (to illustrate a DF: *R is/is-not a MR from set A to set B*). In one module the chapters were in an increasing order of abstractness (group U – CAP) and in the other they were in exactly the opposite order (group D – RCAP).

When designing the assignments we aimed towards a maximal exposure of the participants' thoughts. Thus, following each of the examples, the module presented several kinds of assignments: a) Reasoning assignments; b) Construction of examples assignments; c) Counting elements assignments; and d) Identification assignments. At the end of each of the fifteen chapters the learners were asked to: (a) Describe their feelings, (b) Define a MR verbally, and (c) Construct an example of a MR.

Subsequently, the assessment of the learning processes was done manually according to thirty-five different indexes that were designed to evaluate the conception of the different aspects of the MR and the general aspects of the learning. We used these indexes to grade each point (point = assignment, 204 points in total) of the learning process, in an accumulated manner, and the grading was typed into the module. Also, we normalized the indexes to have values between -1 and +1 for comparison reasons.

In addition, positive progression and negative progression (i.e., errors) of the participant's conception of each of the ten different aspects of the concept were monitored separately, as well as the "total" (taking into account positive and negative progression). We evaluated the learning processes separately for the positive, the negative, and the total progression. Furthermore, a few more indexes were designed to assess total (i.e., "sum" of all aspects) positive progression, total negative as well as the total of the "totals" for each subject.

The grading at each point was done in a positive manner; i.e., grades were assigned only to the participant's statements, and there was no negative grading for a missing statement or for an erroneous statement. For instance, if a participant neglected to use "ordered pairs" in his example of a MR, it was not documented in either his negative or positive indexes. On the other hand, if he did have pairs in his example, but with curly parentheses rather than round, it was documented in both the positive and the negative indexes. Here, his positive index was scored for the kind of elements (i.e., pairs) and on the existence of parentheses, but not on the shape of the parentheses. But his negative index was scored for the shape of the parentheses.

The progression of the indexes' values was graphed (using *Mathematica*) on three axes: the Process Progression Axis[2], the Location Axis[3], and the Time Axis.

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[2] Point x on this axis marked the xth assignment that was treated (one of 204).

[3] - Point x on this axis marked the xth assignment in the Bottom-Up module (assignment x in U is the same as 205-x in D).

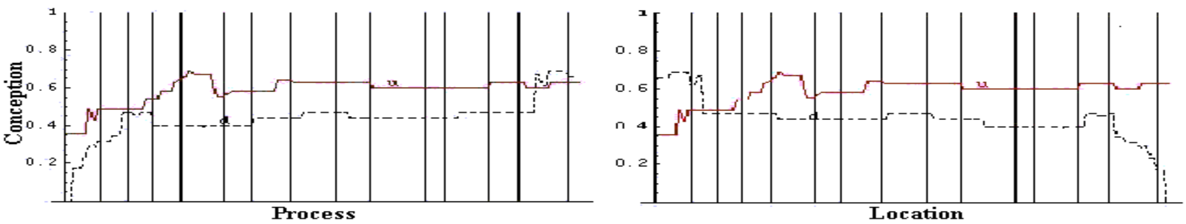
Furthermore, each of the individual's observed cognitive processes was presented as an individual profile consisting of approximately 60 graphs (most of them are continuous) as well as of a non-graphic set of data. Also, the 'average processes' of each of the two groups of learners was presented by such a profile, as well.

Subsequently, the analysis was done on the group level (based on the graphs of the average conception of each of the 2 groups), as well as on the individual level (based on the individual profile of each of the participants in both groups).

4.2. The Findings

In our study, we derived dozens of conclusions about the conception of each of the ten aspects of the MR and about each of the three general aspects of the learning. Likewise, we obtained conclusions about the whole conception processes and about the initiation of abstraction abilities that were developed in the participants as a result of their participation in the learning experiment. For obvious reasons we cannot present all of our findings here. Therefore, we will present only a couple of the graphic products and a few conclusions, just to give the reader some idea of the kind of products and conclusions this tool offers.

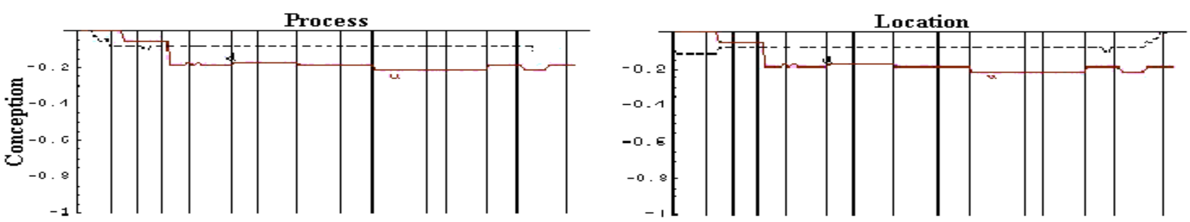
Figure 3 The Progression of the Average Positive Conception of the Set aspect



Figures 3 and 4 present the graphic portrayal of the progression of the Average Conception (positive and negative) of the Set aspect (i.e., understanding that a MR is basically a Set of elements) in the two groups of learners. The students of group U learned via CAP while students of group D learned via RCAP. The progression of the average conception is presented on two axes: the learning process axis and the location axis. The vertical lines represent the different chapters of the module:

Consequently, these graphs, and other data, suggest the following (partial list) conclusions:

Figure 4 The Progression of the Average Negative Conception of the Set aspect



1. The conception processes of the set aspect of the MR are more efficient via CAP than via RCAP (Figure 3, U is leading throughout almost the whole process).

2. The CAP enables the construction of a more stable schema than the RCAP does (based on findings, not presented here, in regard to the number of changes in the conception of the concept "Set").

3. The RCAP is more influential than the CAP (Figure 3. Conception via RCAP surpasses the conception via the CAP by the end of the process even though the situation is exactly the opposite throughout almost the entire process).

4. The RCAP directs the construction of the conception schema towards the positive direction more than the CAP does. (Figure 4. The negative conception: D is above U throughout almost the whole process).

4.3. Discussion

We chose to discuss our research in the context of a methodological framework rather than in the context of a conceptual framework that focuses on research objectives (the cognitive processes). This is mainly because we view our methodology as the main contribution of our scholarly work. As an exploratory research our conclusions suggest intriguing conjectures for continuing investigations rather than well-established new scientific results.

Though, we conducted our research in a lab context, and took different measures in order to attain a higher degree of precision and to obtain more generable outcomes (see Section 2), yet, noticeably one cannot achieve ‘absolute’ precision in such experiments. Furthermore, our study does not escape yet other weaknesses: It focuses on a very short learning episode (about three hours), while meaningful learning could take much longer (even years); It monitors the conception of a very small “piece” of a whole concept (MRs could be the subject of years of studying); It “flattens” the very complex research situation (putting a very rich learning process into a graphical profile, even if it does contain dozens of graphs); It “extrapolates” without sufficient justification (when stretching a line between the conception in two adjacent points to achieve the continuousness of the process); and, among others things, it also attempts to derive general conclusions which are based on a non-statistical sample (small and non-random).

Notwithstanding this un-encouraging image of our study, we believe that its products (the graphical learning profiles), along with the kind of conclusions and the conjectures it suggests, have promising potential for the study of the processes in which we come to know Mathematics: The short learning episode allowed us to concentrate on a very deep and detailed observation and the “flat” graphical portrayal enabled us a precise analysis of the products. While the design of the two different modules could have an effect on the processes that we were observing, they did facilitate a closer monitoring of the learning.

What we consider as most important is the large scope of conclusions and the comprehensive profiling of the process that our research tool offers, which regrettably, because of obvious reasons we cannot present here in full. These attributes allow one to construct a very detailed and comprehensive picture of the research target, which is receptive to analysis in various levels of automatization. As a matter of fact, we derived dozens of “local” conclusions (about each of the thirteen aspects that we have monitored), as well as dozens of conclusions about the whole process. Interesting observations in regard to the way that the conception of the different aspects of the MR progress led us to the conjecture that conception progresses in three parallel channels (The Visual-Morphological, the Set, and the Relational channel). Furthermore, the relations between these three Conception-Channels have intriguing pedagogical derivations.

Finally, to end this partial list we’ll just mention that although our findings do seem to be too “pretentious” for the amount and for the quality of data it is based on, they do have a very appealing potential which we think make them ideal conjectures for future studies.

As a final point we’ll mention that our proposed methodology is highly suitable to utilize many technologies and methodologies simultaneously and could be continually improved as more advanced technologies are developed and more creative scholars utilize it in innumerably different situations.

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