STUDENTS' ABILITY IN SOLVING PROPORTIONAL PROBLEMS

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ABSTRACT

In this study we proposed and evaluated a model of 6th-grade students to solve proportional problems. The model used the SOLO taxonomy, which was extended by adding sub-levels to original SOLO levels, in order to accommodate the whole range of students' strategies in solving simple proportional problems. The model describes the structure and reasoning of students thinking in answering proportional problems, and thus it can be used by teachers to enhance student learning.

INTRODUCTION

The development of proportional reasoning is one of the most challenging aspects of students' mathematical thinking. Proportional and multiplicative reasoning is basic to many important mathematical concepts and can be regarded as the gateway to success in studying algebra (Confrey & Smith 1995). Although the importance of developing students' proportional reasoning in school mathematics curriculum is recognized internationally, extended research from as early as 1985 until now (Cai & Sun, 2002; Hart 1994; Singh, 2000; Tourniaire & Pulos, 1985) reveals that solving ratio and proportion problems is a very difficult task for most pupils. There is also a wide agreement (Hiebert and Behr, 1988) that if research is to inform instruction, it is important to analyze mathematical structures and children's solution processes in light of the developmental precursors (or, sometimes, prerequisites) to the knowledge needed to function competently in a domain.

Thus, the purpose of this study was to construct an understanding of students' proportional reasoning schemes by developing an assessment model for proportional reasoning in which most of the students' strategies in solving proportional problems as well as misconceptions found in the research are taken into consideration. Specifically, this model was designed to assess pupils' reasoning and development at simple ratio and proportion tasks and to reveal their strategies that need to be addressed during instruction. To this end, we used the Taxonomy of Structure of the Observed Learning Outcome (SOLO) as a means for interpreting students' development in proportional reasoning (Biggs & Collins, 1991).

THEORETICAL BACKGROUND

The purpose of the study is to develop a model for assessing students' abilities to solve proportion problems based on the strategies they use. The framework of the model grounds on the SOLO taxonomy; thus this part refers first on the methods or strategies students apply in solving proportional problems and second on the structure of SOLO.

Strategies for Solving Proportional Problems

Proportional problems include the union procedure of four elements, which according to the way that relate to each other, form two kinds of ratios. The first category consists of the ratios that refer to “within relations”, i.e., relations between quantities of the same kind, and ratios that refer to “between relations”, i.e. relations between equivalents quantities of different kinds (Lamon, 1994). For example, consider the following problem “if 3 balls cost 9 pounds, how much do 12 balls cost?” There are 2 measure spaces in this problem: the first one contains the set of the cardinalities of the two sets of balls and the other contains the cardinalities of the two sets of pounds. The “within relations” compare the number of balls to the number of balls and the amount of money to the amount of money. These two relations form the ratios 3/12 and 9/x. The “between relations” compare the number of balls to the corresponding amount of money and form the ratios 3/9 and 12/x. The distinction between these two types of relations is important because each type demands a different cognitive procedure. The implementation of each procedure determines the kind of strategies that students use to solve a proportional problem. Lamon (1994) names these strategies as “within” and “between” strategies correspondingly. Adult students use more often the “within” strategy (Noelting, 1980). However, the correct implementation of each strategy is directly depended on various factors that refer to the type of the problem and the arithmetic relations between its data (Kaput & West, 1994; Lamon, 1994). Integer relations among elements of the same quantity demand the implementation of a “within” strategy, but, integer relations among different quantities demand a “between” strategy.

The standard school approach to solving missing-value problems is to set-up a proportion equation by identifying and distinguishing the quantities involved (Christou & Filippou, 2002). According to Freudenthal (1983) the missing value and comparison proportion problems can be solved by three distinguishable approaches related to: a) internal ratio (within a magnitude), or ratio between terms within a system, i.e., two lengths, two times; b) external ratio (between two magnitudes), or ratio between terms of different systems, i.e., a length and a time; and c) refraining from computation until the result has been found formally, or set up a relationship that involves all the given data and compute only then. The internal and external ratio methods correspond to the within and between strategies as mentioned above.

In addition, students can handle proportional problems by using informal representations or use self-invented solution strategies, which are primitive, context bound, and based upon counting, adding, and multiplying or dividing (Hart, 1994). These strategies may lead to wrong answers, because they ignore the multiplicative relation between the analogy terms. In the occasion that students realize the multiplicative relations of the problem, it is possible to use strategies that handle multiplication as a repeated addition procedure and division as a repeated subtraction procedure (Nesher & Sukenik, 1991).

The more salient informal strategies are the unit rate method and repeated addition-subtraction procedure. The repeated addition strategy is based on using one given ratio to find the required one following an additive procedure (Nesher & Sukenik, 1991). Nesher and Sukenik (1991) assume that the "unit rate" method is conceptually and computationally more effective because presupposes understanding of the ratio concept. It is a multiplicative strategy that includes conception and understanding of the multiplicative "within" and "between" relations of a proportion problem. However, the exclusive use of the unit rate strategy without a meaningful understanding of multiplicative reasoning becomes a procedurally oriented operation that disembodies from children's initial sense making of proportional reasoning (Singh, 2000).

Besides the relations of a proportion problem, the structure and the type of it have an important role in the strategy choice (Lamon, 1994). Researches (Spinillo & Bryant, 1991; Hart, 1994) assert that students solve more easily proportional problems, which have more familiar content and problems with familiar multiplicative relations. Consequently, the overall frame of the problem constitutes a key factor in the strategy decision-making.

Solo Taxonomy

The SOLO taxonomy, initially proposed by Biggs and Collis (1991), evaluates and categorizes cognitive performance by considering the structure students' answers. A response, the learning outcome to be observed, is prompted by a question, and is indicative of the difficulty of the question and the cognitive ability of the individual. A response varies between 5 levels of complexity, ranging from prestructural to extended abstract.

- Prestructural: Incorrect data or process is used in a simplistic way that leads to an irrelevant conclusion.
- Unistructural: A single process or concept is applied to at least one data item. An invalid conclusion may be drawn because the selected data is not sufficed.
- Multistructural: Processes and concepts are used on one or more data items, but without information synthesis or intermediate conclusions. This may indicate cognitive performance below that required for successful solution of the problem.
- Relational: Response is characterized by the synthesis of information, processes and intermediate results.
- Extended abstract: Responses are structurally similar to relational ones, but data-concepts-processes are drawn from outside the domain of knowledge that is assumed in the question.

THE PRESENT STUDY

The purpose of this study is to develop a cognitive model for the assessment of sixth grade students' ability to solve proportion problems. To this end, we set-up a criterion frame that satisfies the levels of SOLO taxonomy. A response was set to be prestructural when it was characterized by subjectively thinking and was unrelated with the structure and the requirements of a problem. The multistructural response involved students' inability to synthesize all the structured elements of a problem. A response was characterized as relational when students could solve proportional problems conceptually by implementing formal or informal strategies. A relational response also involves students' ability to compare, to explain conceptually the ratios of an analogy and to select the more appropriate solution strategy independently of the relations between the terms of the analogy. In this study, we did not examine the extended abstract level, assuming that sixth grade students do not have the abilities for such responses.

METHODOLOGY

Design and sample

The subjects of this study were 15 sixth grade students at an urban school in Cyprus, 6 males and 9 females. At the time of the interviews the students had not received instruction on solving proportional problems to eliminate the influence of the standard school approaches in solving proportional problems.

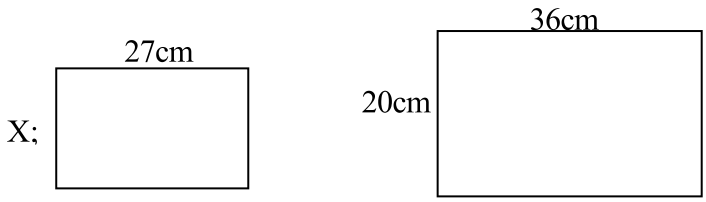
The students were given a 10-item written test (Table 1) to complete in 60 minutes. Students were asked to justify their answers. Based on the implementation of various strategies in students' solution procedures, we selected 4 students for the follow-up interviews. Each of the 4 students was individually interviewed on each of the ten proportional problems. The interviews lasted approximately 75 minutes and were audio-taped. Pupils were provided with pencil and paper to use at their discretion. During the interviews, students were encouraged to think aloud as they worked the problems and they were allowed to change their test-answers if they thought it was appropriate.

TABLE 1: Proportional Problems

1. **Nicholas bought 35 pencils with £7 pounds. How many pencils can he buy with £10?
2. *George worked 4 h and earned £12. How long does it take him to earn £16?
3. *A car of the future will be able to travel 8 km in 2 minutes. How far will it travel in 5 minutes?
4. **Marilena buys 3 drinks every 2 days. How long does it take her to buy 12 drinks?
5. **Marios bought 3 balloons for £2. Helen bought from the same shop 9 balloons. How much did she pay? How much would she pay if she bought 60 balloons?
6. *Costas worked 9 weeks and earned £60. If he earns the same amount of money each week, how long does it take him to earn £20?
7. *A printing machine prints nine books in 4 minutes. How many books can it print in 10 minutes?

8. *Fotini painted 20 chairs with exactly eight cans of paint. How many cans did she use to paint five chairs?

9. *** Mary to needs 15 eggs to bake 6 cakes. Using the same recipe, how many eggs does she need to bake 4 cakes and how many eggs to bake 100 cakes?



10. *** These two rectangles have the same shape. Find X.

(*) The problems denoted with an asterisk are almost the same problems as those used by Kaput and West (1994)

(**) The problems denoted with two asterisks are almost the same problems as those used by Christou and Philippou (2002)

(***) The problems denoted with three asterisks are almost the same problems as those used by Singh (2000)

The problems

The problems used for the test and the interviews are shown in Table 1. The structure and the numbers of the problems were selected in such a way as to represent the within and the between relationships taking into consideration that students perceive the relationships of the problems in the order of the given information in the corresponding problems.

Problems 1, 2 and 3 are easily solved by applying the “between” relations, while problems 4, 5, 6, 8 and 9 are easily solved by using the “within” relations. The relations of terms in problems 4 and 5 imply the multiplication operation, whilst in problems 6, 8 and 9 imply the division operation with an integer quotient. The relations of problem 7 lead students to apply informal strategies. Finally, problem 10 belongs to the “stretchers and shrinkers” category, which is assumed as one of the most difficult proportional problems category (Singh, 2000).

RESULTS

The validation of the model was based on students’ answers to the proportional problems and particularly on their justifications. Table 2 summarizes the characteristics of students' answers in each of the SOLO levels, which, for the purposes of the present study was extended by adding two sublevels in each SOLO level. Following are the characteristic strategies employed by students in solving proportional problems, classified according to SOLO levels.

Prestructural Level

Students' responses at this level were characterized by subjective thinking and were not related to the structure and the data of the problem. These responses referred to wrong numerical answers without justification or contained typical words from the problem without a numerical answer. The following excerpts from students' interviews clarify the kind of responses at prestructural level:

Answer to Problem 4: "Marilena will buy 12 drinks when she becomes 15 years old". Answer to problem 6: "Costas should work 10 years to earn 20 pounds". These two responses are not related to the arithmetic data of the problems, while the semantic structure of the problems seems to be salient in students' cognitive procedures.

Unistructural Level

Responses at this level do not take into consideration the whole range of information provided in the problems or systematically ignore the multiplicative structure of proportion problems. Students at this level also seem to apply additive or one-order multiplicative procedures. To accommodate students' answers at this level, two sub-levels are necessary: the additive and the one-order sublevels.

Additive sub-level: Students at this sub-level lack conceptual understanding and focus on additive relations. Students either use all the arithmetic data to find an additive relation or construct their own additive patterns by using a repeated-addition procedure. For example, Mary's answer to problem 2 is indicative: "George has to work for 8 hours to earn 16 pounds because 16 minus 12 equals 4 and thus 4+4 equals 8, which shows the time George has to work". In this response, Mary noticed that the two amounts of money differ by 4, so the hours must differ by 4.

One-order multiplicative: Students at this sub-level take into consideration only one ratio of the problem and ignore the functional relationships of problems. For example, Nicholas answered problem 6 in the following way: "20 goes three times to 60, so he needs 3 weeks". Obviously, Nicholas extended the "within" ratio of the money domain to the time domain.

Multistructural Level

Students at this level use systematically formal or informal strategies, but fail to synthesize all the structured elements of a problem or their response is procedurally oriented. Based on the cognitive load of each strategy, we divided the multistructural level into two sub-levels: the frailty and the procedural.

Frailty: Responses at this sub-level are characterized by a frailty conception of analogy. Students realize the relations among the terms of the problem but they do not grasp the concept of the analogy as the equivalence of two appropriate ratios. For example, Maria's solution to problem 1 exemplifies her misunderstanding of the meaning of analogy: "If you can buy 35 pencils with £7, then you can buy 50 pencils with £10, because each pencil costs 5 cents. I multiplied 10 by 5 and found 50". Although, the numeric answer is correct, the student's reasoning is totally incorrect.

The Procedural sub-level: Responses at this sub-level are reached by implementing a specific strategy. The most prevalent strategy that students use is the "unit rate" method even if a "within" relation is implied. Although students at this level can solve a problem, they are unable to justify their answer or their justifications are incomplete. Students cannot move from one solution strategy to another one even in the cases where the calculations are very tedious. Kostas's solution to problem 4 is indicative of his persistence on using the "unit-rate method: "I divide 2 by 3 to find how many drinks she buys a day and then I divide 12 by this number {the quotient of 2/3} to find how many days she needs". Obviously, the "unit rate" method distracted him from paying attention to other relationships, which are more appropriate for this problem.

Relational

A response at this level incorporates the conceptual implementation of formal and informal strategies. More specifically, students can use various strategies independently of the familiarity of the type and the structure of the problem or the difficulty of proportional relations. Students also have the flexibility to select the most appropriate strategy solution independently of the relations between the terms of the analogy and have the ability to justify their selection, we divided the relational level into two sub-levels: the semi-selective and the formal-selective.

Table 2: Developmental Model

Levels	**Prestructural**	**Unistructural**	**Multistructural**	**Relational**
Sub-level 1	Subjective thinking answers which do not take into account the data and the structure of a problem. Students lack logical reasoning and use either an isolated wrong numerical solution or isolated words from the data of the problem.	**Additive:** Absence of conceptual understanding of proportion and constant concentration on inventing additive structures. Students use additive patterns in solving problems.	**Frailty:** Students do not understand the equity of the two ratios in a proportion. They also do not synthesize all the proportion's relations.	**Semi-selective:** Students use a variety of formal and informal strategies independently of the kind of the relations in problems. They consider the strategy they apply as the only one appropriate.

Sub-level 2	**One order multiplicative:** Students Understand the one ratio of the proportion and the multiplicative relation between the two quantities. They are not able to extent this relation to the second order relation of the analogy.	**Procedural:** Systematic implementation of a specific strategic regardless of the data of the problem.	**Formal selective:** Students select the most appropriate strategy in solving proportional problems.

The Semi-selective sub-level: Students at this sub-level reach an answer by using a variety of formal and informal strategies according to the type of the relations in the problems. However, students are not able to justify their answers and believe that each strategy is the only one that is appropriate for certain problems.

The Formal-selective sub-level:

Responses at this sub-level are similar to the semi-selective ones with two major differences. The strategy-decision procedure is more flexible and is not influenced by factors such as the numbers or the problems at hand. Particularly, the presence of non-integer ratios does not affect students' reasoning in proportional problems. For example, at problem 7, John answered: "To find how many books it prints in 10 minutes, we can say that 10 minutes is 2,5 times more that 4 minutes, so to find the answer we should multiply 9 books by 2,5". John's strategy involved (a) comparing the number of minutes, (b) determining the factor of change, and (c) multiplying by that factor. The ability to relate all these features illustrates a more formal proportional reasoning and a higher level of thinking (Lamon, 1994).

DISCUSSION

In this study we developed a model, which might be useful in assessing students' reasoning and development in simple ratio and proportion tasks. To validate the model we used the levels of SOLO taxonomy (Biggs & Collins, 1991), and we extended it by adding sub-levels in order to comprehend the whole range of strategies used by students in solving proportional problems. The results of the study show that the proposed proportional reasoning model can be used to develop an effective instructional program in solving proportion problems. Furthermore, the model can encompass the varieties of strategies and ways students find to solve problems, and provides room for instruction to take advantage of the 'built up repertoire' of students and helps teachers to continue extending students' methods into the more common domain of proportional reasoning. Finally, the model can function as a tool that will help the training of future teachers of mathematics in two ways. First, they can be informed on their pedagogical content knowledge about ratio and proportion by trying the teachers' version of such an instrument themselves. Then, they might be able to enhance that knowledge, by delivering the same instrument to pupils and by comparing the actual data with their previous predictions. Consequently, the next stage of the research should be to try and provide robust research findings about the use of the instrument in teacher education and in teaching in general.

Further research is also needed to evaluate the viability of using the model for informing proportional problems instruction in regular classroom situations, that is, to assess the ease with which classroom teachers are able to use the model to enhance student learning. Such research would also provide opportunities for fine-tuning the model and making it more effective for generating instructional programs that build on students' prior knowledge, foster their thinking through problem-focused experiences, monitor and assess their understanding.

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