

EMOTION, INTENTION AND ACTION FOR VITAL MATHEMATICAL EMBODIMENT¹

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Abstract: This paper broadens the notion of the internalisation of mathematical structure, investigated by Nada Stehlíková and Darina Jirotkova at CERME2, to include the roles of emotion and intention in developing a personal internalisation of mathematics. Hence the metaphor of ‘construction’, as used in learning, changes to that of ‘embodiment’; ‘inner mathematical structure’ is related to the emotions, intentions and actions of the mathematics student. The central novel theme is that: emotion and mathematical action are linked through the notion of intention. An application of this theme is applied to students learning with computer algebra systems and, briefly in concluding, to teaching mathematics.

INTRODUCTION

We don't learn without feeling and an aim of this paper is to discuss the role of feeling in mathematical thinking and learning. The ideas explored here follow on from paper presented at CERME2 (Stehlíková & Jirotkova, 2002). They speak of “vitalising” mathematical knowledge through need and creativity (*ibid.*:102) and they acknowledge that “emotional power” directs attention towards potential new mathematical knowledge (*ibid.*:106). These are whole-person orientations to mathematical learning: the body-mind is not separate from its physical or social environments. In this paper, the internalisation of mathematical structure is connected to the actions, emotions and intentions of the learner. Hence the notion of structure is ‘vitalised’, for part of a person’s capacity to structure is determined by their feelings.

In mathematics education there is now awareness that context matters, for example, Boaler recognises “..cognitive structures ...are not abstracted out of their learning environments” (2002:42) when investigating mathematics learning. While the orientation that Boaler and others take is to enlarge the domain of mathematics education from the individualistic-cognitive to the socio-cultural, my orientation here is more towards the holism of the learning organism whose emotions, intentions and actions are, indeed, tutored by the cultures the organism inhabits. The biological metaphor of ‘embodiment’² is used to capture the living process of mathematical learning. This is used instead of the architectural metaphor of ‘construction’. I

¹ Since naming this paper I found a recent book entitled *Reclaiming Cognition: the primacy of action, intention and emotion* a collection of “neo-cognitivist” essays edited by Rafael Núñez and Walter J. Freeman (1999). Their central purpose is to illustrate and explain the myriad of aspects of knowing and particularly to show that the whole body is involved in learning and knowing not ‘just the brain’.

² ‘Embodiment’ has been used by several authors, for example, Lakoff and colleagues, cited below, as well as others (e.g., David Reid, Brent Davis).

believe ‘construction’ communicates a solidity about mathematical knowledge which does not accurately reflect the tentative and revisable nature of learners’ knowledge.

Structure of this paper

Firstly, an illustration, through an example of a mathematical teaching game, is given to show how emotion and intention facilitate the mathematical action that ‘building an Internal Mathematical Structure’ (IMS) (Stehlíková & Jirotkova, 2002) requires. Secondly, the notion of ‘mathematical action’ itself is considered, followed by a discussion of the functions of emotion and intention in mathematics learning. An application to the efficacy of technological aides is then considered. In conclusion, some points about teaching mathematics are raised.

AN EXAMPLE FOR ‘VITAL MATHEMATICAL GROWTH’

Emotion, intention and action in games in maths lessons

One of the positive policy developments that have been brought into English mathematics classrooms in the past few years is the increased use of games as pedagogical devices. A case for the use of games on the cognitive and social levels is given by Gillian Hatch (1996): games are themselves structural and axiomatic, while often having concrete referents like counters; and they engage the whole-person: their social interaction, language and goals. The practice of using mathematical games has evolved from work of the Association of Teachers of Mathematics (in Britain), Gattengo, de Bono, Dienes and others, in which a play dimension to mathematics is incorporated into teaching.

Here is a family of mathematics teaching games used with 13 year olds to introduce the structure of directed numbers through operating with these numbers in a context which seems to have involved the pupils as whole people:

Pupils play in pairs; this reduces anxiety as they are not put in a position where they could be embarrassed, they also usually play with friends which increases fun and pleasure. Each pair has a numberline marked with integers from -10 to +10, with some space so they can extend the range. They also have a set of 10 (unsigned) digit cards (0 to 9) and 10 sign cards (five + and five -). They use their own pencil as a place marker.

The basic game starts as a game of chance and is then unleashed into a game of strategy. To start with, each player starts with his/her³ place marker at 0 on the numberline and a set of five of each type of card. In turn, they turn over a sign card with a digit card, move their own place marker and play out their set of cards (e.g. player A might have this sequence of cards: +5-0+3+4-7 ending up at +5; hence B will have {-,-,-,+,+} and {1,2,6,8,9} in some combination). The winner is the one with the maximum final position. This game soon becomes boring! This is because the action (of turning up cards) is not linked to the intention (to win) and so feelings

³ Henceforth I’ll use ‘he’ for a generic pupil and ‘she’ for a generic teacher in this paper, for reading ease.

do not control or power the purpose of the game. Intentional beings impose themselves on their environment, in the case of playing a game, they want to act to win, so they adapt the game, hence engaging with the structure of directed numbers. For example, they have decided to look at the dealt cards and choose the + or – cards to go with the digits. When that has run its course of interest, they have marked signs on their digit cards to make them directed numbers and play with one player placing down a sign and the other choosing a directed number. They are encouraged to invent their own rules and adapt the basic game further thus increasing their play-exploration with the structure of these numbers.

Emotion and intentionality feed mathematical embodiment which is played out in the action of the game: rally within play, emotionally charged, feeds self-worth and empowers the sense of self as agent; the win-focus, both cultural and instinctive, directs intention. As this is all done in a context subject to the mathematical structure of directed numbers, the desired learning outcome of internalising the mathematical structure of directed numbers has a better chance of success than if a cognitive framework alone was presented to the children.

The next section explains why having a particular IMS has similarities with having a capacity for a particular type of mathematical action.

MATHEMATICAL ACTION⁴

Of a ‘capable pupil’: “he proved the remaining algebraic theorems freely, without reflection”. The pupil, who “To the experimenter’s question ‘How did you solve it?’ replied: ‘Here there is nothing to think about - just look at the example and write’.” (Krutetskii, 1976:246;266).

The notion of ‘mathematical action’ used here is that of working in a fluent, non-rote, curtailed manner. It can be distinguished from rote working where there is no capacity for unravelling the process, of flexible adaptation of the procedure or explaining the moves. Pupils’ facility to play and adapt a directed numbers game (illustrated above) is not like a rote stimulus-response question answering, or, in Skemp’s (1976) sense, a *purely* instrumental response. While non-reflectivity is a characteristic of this facility in action, the facility holds ‘relational potential’, (adapting Skemp’s term). Some of the current debates on the British educational scene, are trying to address this ‘mathematical action’ (for example, LMS, 1995). One of the points about mathematical action is its natural, fluent, almost instinctive, feel, as the quotation from Krutetskii above illustrates. Krutetskii uses the notion of ‘curtailment’ to express a capacity to work without attending overtly to structural

⁴ In my PhD thesis, (Rodd, 1998), I make stronger claims about this capacity being a form of mathematical knowledge, in which such mathematical actions are referred to as ‘automaticities’. I argue that automatically executed actions with mathematical entities can constitute knowledge by virtue of their (warranting) form. This is due to mathematical objects and mathematical forms (like some algorithms) being inextricably linked. Furthermore, individuals require some procedure-embodiment for mathematical knowledge, i.e. learning mathematics involves developing a capability to execute some mathematical procedures automatically.

aspects of the mathematics as these have been internalised, embodied⁵. The ability to operate without reflection, yet be able to fill in the gaps and give the rationale for certain moves on request, distinguishes ‘mathematical action’ or ‘curtailment’ from a ‘théorème en acte’ for which reflection is not available, (Vergnaud, 1981). Procedural efficiency requires a ‘just do it’ attitude to routine mathematical calculations but has a meaning and significance to the student who has internalised the structure of that mathematics⁶.

The notion of ‘embodiment’ I am following was theorised by linguist Mark Johnson (Johnson, 1987) who explores “some of the more important embodied imaginative structures of human understanding that make up our network of meanings and give rise to patterns of inference and reflection at all levels of abstraction.” (p xvi). More recently these ideas about the body and the imagination being inextricably linked have been developed in Lakoff and Johnson (1999). Johnson’s co-author, George Lakoff has, with Rafael Núñez applied a ‘cognitive science’ theoretical stance to the mathematical context in Lakoff and Núñez (2000). An account of meaning is given that incorporates action-cognition. Abstraction starts with bodily action, “our neural motor-control systems may be centrally involved in mathematical thought” (Lakoff and Núñez, 2000:34). The bodily experience is of intrinsic importance in developing meaning⁷. And emotion is central to the bodily experience.

EMOTION AND INTENTION

Emotion is a basic life regulator. Part of our evolutionary adaptation is to have developed non-reflective and action-stimulating responses to danger, pleasure and other survival demands (I have used the philosophically-oriented biologist Damasio, 1999, for a source on the primacy of emotion in embodied reasoning). Even learning an abstract subject like mathematics hinges on emotional reactions. Though many of our emotions are honed by our social environment, they are felt by an individual and more often experienced unconsciously. The ability to ‘reason’ is intimately entwined with emotion, particularly as it pertains to our feelings and identity (Damasio, *op. cit.*, Lakoff & Johnson, *op. cit.*). Emotions stimulate the neurons - hence the organism - towards intention for action. To learn, organisms need to act. How you feel affects your capacity to act.

Intention is mental. It is an antecedent to action. But intentions can be modified by the individual in a way that expressions of intention, orders and predictions cannot (Anscombe, (1957:9). Anscombe’s example of Peter’s denial of Christ demonstrates

⁵ The term ‘compression’ is used by Thurston (1995). Dubinsky’s notion of ‘encapsulation’ is also similar, as is Tall’s ‘procept’ (both in Tall, 1991).

⁶ This point underlies a proposition investigated in my PhD (*op. cit.*): ‘automatically executed actions with mathematical entities constitute knowledge, per se, by virtue of their (warranting) form, and that learning mathematics involves developing a capability to execute some mathematical procedures automatically’.

⁷ An embodied-aware account of the bodily basis of elementary logic can be found in my PhD (*op. cit.*) and refined in Rodd 2000.

the subtlety of the concept of intention as well as gives an insight into why it is inextricably tied to subconscious feeling⁸:

“... St. Peter could do what he intended not to, without changing his mind, and yet do it intentionally.” (Anscombe, 1957:94)

The subconscious character of intention comes from the organism being driven by its feelings. As an illustration, pain is intentional but not an emotion - it can cause emotion and induces action as a consequence of the local dysfunction in living tissue (Damasio, 1999:71). Subconscious aspects of learning are elusive yet central: Lakoff and Johnson (1999) exemplify a dozen essential mental capacities which take place below conscious awareness when engaging in conversation. All of these are intentional: e.g. “Accessing memories relevant to what is being said. ... [and] ... Anticipating where the conversation is going” (*ibid.*:10-11).

The context of conversation prompts consideration of the psychoanalytic perspective on intention and emotion. The psychoanalyst and sociologist Nancy Chodorow gives a feminist-friendly interpretation of the concept of ‘transference’, (initially coined by Freud), through which persons create their own meanings: “In transference, we personally endow, animate, and tint, emotionally and through fantasy, the cultural, linguistic, interpersonal, cognitive and embodied world we experience”. (Chodorow, 1999:14). Chodorow’s thesis is that feelings power development and that “emotions, however they may appear or we may label them, are rational or reaching for rationality” (*ibid.*:23). Going back to Lakoff and Johnson’s conversational capacities, it is well known that trauma can block memories: intentional capacity is constrained by emotional power beneath conscious awareness. The emotional intentional subconscious affects all learning. Even when a student complies with the instruction given and has some meta-cognitive awareness of the ‘things’ he is supposed to learn and how his teachers believe that learning is best achieved, his feelings still power his subconscious and hence his capacity to learn.

Emotions are biological-environmental mechanisms for survival. People have emotional reactions to learning mathematics and their intentions, which are sometimes articulable, direct their activity in learning mathematics. Hence, we come round full circle to doing, to action. Specifically, mathematical action, which can occur when mathematical structure has been internalised, as Stehlíková and Jirotkova (*op. cit.*) documented, is a capacity inextricably linked to the person’s emotion and intention. Stehlíková and Jirotkova (*op. cit.*) employ a methodology which can be used to track the emotional blocks and spurts that learning involves. Their report was on the cognitive structuring dimension of Stehlíková’s learning through problem-solving, but the experience will have had emotional and intentional aspects, many of

⁸ The distinction between ‘goals’ and ‘intentions’ is that the former directs behaviour (consciously or subconsciously) and the latter is mental, it is the “action of understanding” (www.oed.com). This distinction is fuzzy, as I *intend* to show here in the mathematics learning context; a *goal* of writing this paper is to link the learning of structures in mathematics with personal feelings and actions.

which are subconscious. These aspects are so difficult to capture in ourselves, how do we go about capturing it in others? Jeff Evans has been working in this area for some time and he has had success in capturing retrospective emotion (see Evans 2000, pp135-149). Emotion of the moment feeds the intention which directs an organism, our pupils, to learn, and this is much more difficult to capture.

To flesh out the purpose of enlarging the notion of internalisation of mathematical structure to more a amorphous acting-emoting-intending being, let us turn to a classroom application concerning student learning with computer algebra systems (CAS)

Application: Embodiment or anxiety with CAS

Picture a student of maybe 16 years learning about, say, polynomials' factorisation and expansion. The mathematical structure that he is to internalise includes, for example, being able to apply the quadratic formula, knowing there's at least one real root from an odd degree polynomial, being able to interpret results from the remainder theorem, etc. Suppose too that this student has his own advanced calculator which is equipped with a CAS, so factorisation and expansion can be investigated empirically with the machine, as long as the student has (to a certain extent, at least,) internalised the functions of the CAS. Thus, in this scenario, there are two different types of structure which may be internalised: the mathematical structure of the polynomials and the structure of the CAS tool. Ideally, these structures are mutually reinforcing in that knowing how to work the CAS would help develop the concepts of factorisation and expansion.

However, employing ideas developed above of intention, emotion and action leads to an explanation (only outlined in this paper) of why 'helpful' tools (like CAS, or, indeed, at a younger level, regular calculators) may be, for some students, blocks to internalising mathematical structure and internalising the structure of the CAS does not lead to the 'traditional' mathematical structure. For example, let us consider two differing cases: a 'capable' and an 'average' student (in Krutetskii's 1976 sense where 'capability' is associated with having internalised mathematical structure⁹). What might their intentions be in a mathematics learning situation? I suggest that they might both intend to solve their problems on polynomials with the minimum amount of pain; to get the problems done as efficiently as possible; to seek satisfaction that is achieved by completion of a page of answers. Thus their intentions are honed by their emotional desires, what ever their 'capability'. But how might they proceed to achieve their desires, what might these students do, or act, if each had his own, personal, CAS?

⁹ "in the usual method of generalising, the average pupil perceives the generality of features by contrast, the capable pupil infers the features' generality from their essentiality" (Krutetskii, 1976:259). I have made an interpretation that the 'perception' of 'essentiality' of the 'general [structural] features' corresponds to an IMS.

My observations of students suggest (and really one cannot get inside a student's head) that the intentions of the structural thinker guide him to add detail to this structure with the data from the CAS and because not all understanding or power to solve problems rests on the tool, this can be done with low anxiety even with some playfulness. Intention directs action to problem solving and the CAS can, as advocates suggest, liberate the student from mundane calculations, revealing more structure. This is a satisfying loop potentially increasing relational understanding. The other sort of student may be equally focussed on solving problems (factorisations, or solutions, etc.), but because there is no embodied germ of the concepts involved, the approach is 'instrumental' and to solve problems instrumentally surely an instrument like a CAS is a promising method? So his focus is directly on the machine – let it give me answers please! – The student's emotion (desire to complete linking, perhaps, with anxiety) orients intention to action to focus on the machine. Unlike the 'capable' student, there is not the structure to embroider or to add too in a relaxed and playful fashion; what becomes embodied is the facility with the machine (e.g., a sequence of button presses is factorisation). The scenario I have sketched portrays the 'capable' as acting on intending to solve problems anticipating satisfaction, whereas the 'average' acts on the machine intending to produce answers potentially with anxiety.

Kendall and Stacey's 'teacher privileging' project (Kendall and Stacey, 1999) supplies evidence for the claim that the actual cognitive structures of knowledge appropriated by students using CAS in different ways, as illustrated above, is different. In their research, Kendall and Stacey found that the students acquired "different conceptual understandings, a different set of competencies and different abilities to discern whether or not it would be advantageous to use the various features of the calculator [CAS]" (ibid.:244) depending on their teacher's orientation to CAS. While Kendall and Stacey do not use the notions of emotion, intention or action, the notions do implicitly arise within their report. For example, they note that different classroom climates with different levels of teacher and student enthusiasm, different teaching goals and, crucially, different student actions (with respect to CAS use) in problem solving do result in different IMS.

Finally, as a specific instance of in appropriate action for mathematical development due to intention being limited because of negative emotion: we found in a recent research project, (Rodd & Monaghan, 2002), that some students took up to three calculators into an exam as they only knew how to use each calculator for specific functions. Their anxiety limited their action; their intentions were superficial, focussed on detail not structure.

CONCLUSION: ON TEACHING

The holistic view of mathematics learners put forward here is to counter-balance approaches which focus on mathematical detail coupled to cognitive progress. For example, in England currently there is currently a big emphasis on anticipating 'misconceptions' conceived as purely cognitive, and to seek or devise methods of

instruction which can be ‘delivered’ to remedy the children’s deficits. But learning doesn’t happen just like that; the emotions are always involved, the unconscious is constantly fueling the organism’s attention and effort.

However, a cognitively-focussed account can give a starting point for the wider story. The cognitive domination of educational learning theory is unlikely to provide a basis for pedagogy for all children. This recognition has indeed been addressed by a move towards a socio-cultural understanding of classrooms and their participants and a respect for the cultures of the students and the teachers, at least within educational research. Boaler’s recent paper on mathematics classrooms (*op. cit.*) links the social and the individual with the “discipline” of mathematics, which includes the structural thinking elaborated on in Stehlíková and Jirotkova and here.

The example of developing an IMS described in Stehlíková and Jirotkova (*op. cit.*) has at its centre an investigative, enquiry-based approach to an open problem situation, (the number system A_2), which was done over some length of time. The emotions of curiosity, pride or fear-of-failure, euphoria, etc., can be imagined as motivating the continued ‘building’ of the IMS. Less likely, I guess, would be the emotions of anger, panic or satiatedness which are often experienced by pupils.

In England, nowadays teachers’ praise of children’s efforts is considered good practice. This is generally positive – it indicates a respect for the feelings of the students, and a belief that people learn from their mistakes. But, excessive praise-giving misses the point of praise: soft praise is a sop which at best does not damage a pupil for whom the teacher has no particular feelings. Energy to learn comes from love and hate and passion and anger, not politeness and political correctness. A satisfying teacher-pupil relationship is dynamic and emotionally fulfilling. Because of the distilled nature of mathematics – and this is why mathematics is applicable, universal, adaptable – the teacher does need to lead the way in charging the emotion to engage her pupils’ intentions to do mathematics.

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