HOW DOES EFFICIENT LEARNING OCCUR – A HYPOTHESIS

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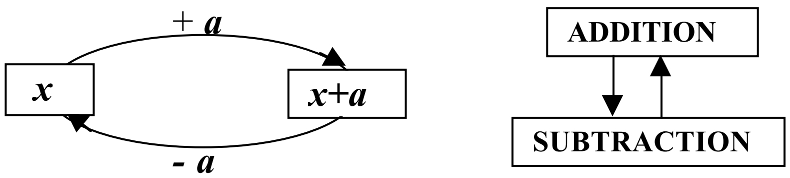
Abstract

The research is developed in the framework of a constructivist approach to learning. I suggest through the example of learning equations how recent developments in cognitive science could be connected to school learning. I propose a methodology of teaching practice consisting in the following three components: systematic training, focused on building specific skills; randomized training, focused on developing variability across skills; structured training, aiming at assimilating the invariants. This methodology has been applied in experimental classes and the results show an increasing interest of children for mathematics and a high level of creativity in problem solving and composing. The methodology reflects a neurophysiological model inferred for explaining the emergence of learning.

Understanding equations: some connections

One of the basic concepts in mathematics, which practically cross the curriculum from grades 1 to 12, is equation. I will use it as a case-concept to exemplify the main ideas of this paper. Understanding the way in which equations work supposes connecting two components: one from the perspective of mathematics and the other from the perspective of building the necessary skills to deal appropriately with the mathematical content.

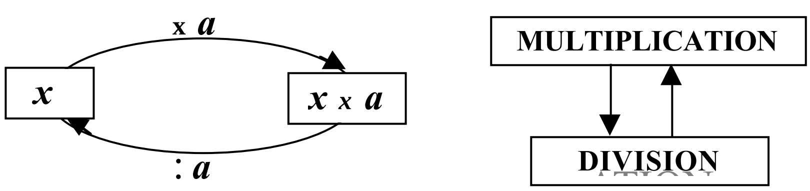
In mathematics, the equations play two specific roles, which connect them to other basic mathematical concepts. The first role played by equations is to connect an operation to its inverse. For example, through functional equations, we are passing from addition to subtraction of real numbers and vice versa:



$$f(x) = x + a$$
$$f^{-1}(x) = f(x) - a$$

Fig. 1: Generating subtraction through addition and vice versa

In an analogous way, functional equations connect multiplication and division:



$$f(x) = x \times a$$
$$f^{-1}(x) = f(x) : a$$

Fig. 2: Generating division through multiplication and vice versa

In general, through functional equations, an operation (addition, multiplication, squaring, etc.) and its inverse (respectively, subtraction, division, squaring root, etc.) are connected in a reversible manner. This connection actually gives the way to solve equations.

The second role played by equations is to extend the sets of numbers. For example, the equation $x + 3 = 1,\ x \in \boldsymbol{N}$ has no solution in $\boldsymbol{N}$. To find and express a solution for this equation is necessary to include the set of natural numbers in a larger set, $\boldsymbol{Z}$. In an analogous way, to express a solution for an equation of the type $3x = 1,\ x \in \boldsymbol{Z}$, it is necessary to build the extension from $\boldsymbol{Z}$ to $\boldsymbol{Q}$; to express a solution for an equation of the type $x^2 = 3,\ x \in \boldsymbol{Q}$, it is necessary to build the extension from $\boldsymbol{Q}$ to $\boldsymbol{R}$; to express a solution for an equation of the type $x^2 = -3$, it is necessary to build the extension from $\boldsymbol{R}$ to $\boldsymbol{C}$.

As we see, metaphorically speaking, equations serve as vehicle between two related operations and between two numerical sets. From this perspective, equations incorporate a dynamic that has to be understood as such; otherwise, their important roles in mathematics are neglected or at least minimalized, with consequences in the lack of understanding the procedure of solving equations. Finding the unknown number in a predicate expressed by a specific equation is just the operatorial algorithmic part. This part is meaningless is the absence of understanding the two roles of equations.

To use efficiently and extensively such a concept-vehicle as equation is, child's mind should internalize a specific dynamic mental structure (Singer, 1995). A mental structure is seen as a scheme of thought and action that have certain stability and could be mobilized in order to solve a specific problem in a given context.

The abstract concept of equality of numbers has an extraordinary advantage, comparing with other abstract math concepts: it has a spatial correspondent in the idea of balance. In addition, an equation has a powerful physical representation as a scale with two plates in equilibrium. Using these two powerful representational "devices", the training could be specifically developed. The characteristics that give to equality of numbers the quality of an equivalence relation (reflexivity, symmetry, transitivity) consistently interfere when solving equations. The same does the invariance of equality at adding or multiplying with the same number in both parts of the equality (with appropriate conditions). All these have to be systematically trained in order to transform them in implicit automatized acquisitions (Singer, 2002).

The concept of equation is not only a concept-vehicle, it is also a transversal concept: it is accompanying the development of mathematical ideas from the beginning of mathematical accumulation (in kindergarten or earlier) to the graduation level. From this perspective, the concept of equation is a developmental one and it has to be correlated both with the formal mathematical content and with ways of internalizing things at different ages. I will describe in the following a methodology to build a dynamic mental structure that could be activated for the concept of equation. I will use the framework of the dynamic skills theory (Fischer, 1980; Fischer & Bidell, 1998) to describe and explain variability in the organization and growth of abilities. The dynamic skills theory started from a neo-Piagetian structuralist perspective and developed as a dynamic constructivism. It is focused on analyzing developmental changes in children and adults in real contexts. Based on this type of analysis, the theory makes predictions on patterns of stability and variation in developing skills. According to the dynamic skills theory, the general pattern in development follows four cycles, called tiers: reflexes, actions, representations and abstractions. I will suggest below some significant tasks and representations that are addressing the properties of equality and follow the recurring cycles of development, in the framework of the dynamic skills theory.

Reflexes. There is a growing body of literature (e.g. Schilling and Clifton, 1998) showing that young children develop a perception for differentiating light and heavy objects, (even if they make a wrong one-to-one correspondence with the volume) very early in life; as the idea of differentiating mass exists, we can build on that the following cycle.

Single sensorimotor actions. The children compare the weights of various objects. For doing this comparison, they could use, for instance, a common hanger and their school bags.

Sensorimotor mappings. The perception of light and heavy objects is coordinated with the intuition of equilibrium as a practical way of comparing real objects. It is the moment to learn weighting: i.e. finding a numerical equivalence that assures the equilibrium of the scale.

Sensorimotor systems. Practical experiments are meant to build on the properties of equality, more precisely, to experience answers to questions like: How a scale will behave when adding identical quantities on both plates? What about when taking away equal quantities? What about when equally double or triple the quantities on both plates?

Single representations. Various models could be used to strengthen the idea of equilibrium. These models are coordinating the skills developed in the sensorimotor cycle in order to “bridge” for a new understanding in the representational phase. For example: How heavy is one box? (All boxes have the same weight).

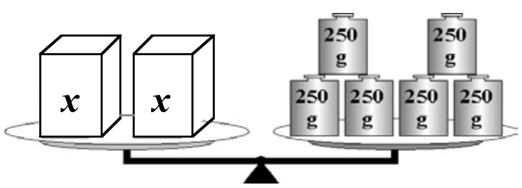


Fig. 3: Using representations to suggest the mathematical operation

Representational mappings. This stage is mirroring the sensorimotor mappings stage at the representational level: finding the numerical equivalence in representational drawings with various non-standard units.

Representational systems. The properties of equality could be suggestively presented through representational designs. For example:

If $a = b$, then $a + c = b + c$

If $a + c = b + c$, then $a = b$

If $a = b$, then $a \times c = b \times c$

If $ac = bc$, then $a=b$, where $a,b,c \neq 0$

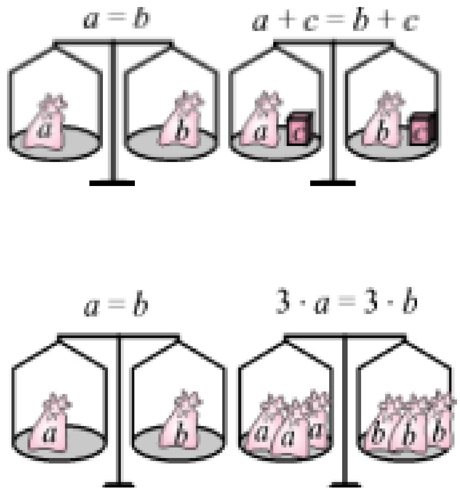


Fig. 4: Using representations for reasoning

The idea of equilibrium involved in weighting with a scale is very suggestive, but it is not the only one. It is important to use also other models to avoid the so-called dependency from a given unique model that risks to generate fixations. To avoid a model-addiction, other systems of representations should be also developed. For example, we can use, instead of weight, volume, area or length:

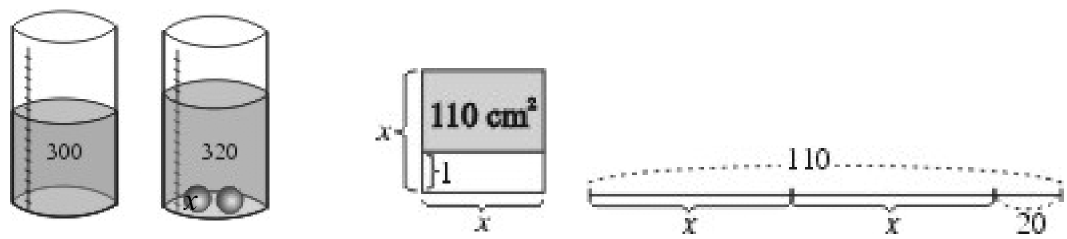


Fig. 5: Using various representations of measures to determine unknown amounts

Single abstractions. The child internalizes and uses the technique to solve equations that suppose a single operation, for example, to solve: $x + a = b$, $ax = b$. In the current teaching, the training practically starts at this level and it is concentrated on memorizing the algorithm for finding the solution. The lack of training the other cycles is manifested in the lack of stability for the skill of solving this type of equations. This is the source of a high rate of errors in applying the algorithm.

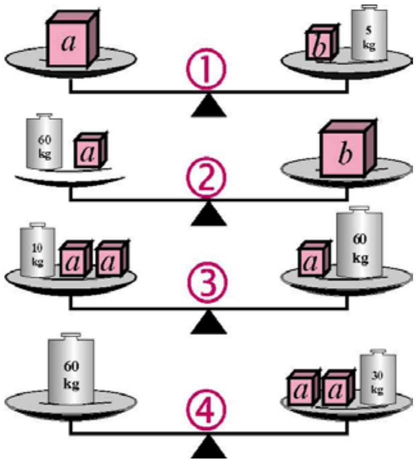
Abstract mappings. The child internalizes and uses the technique to solve equations that suppose two or more operations, for example equations of the type $ax + b = c$, or $ax + b = cx + d$. To give more stability to this skill, it has to be linked to the acquisitions made at the previous level. Below, it is an example for such kind of connection-task: "The next scales are balanced. Match the appropriate equation, scale and statement."

$2a + 10 = a + 60$

$5 + b = a$

$30 + 2a = 60$

$a + 60 = b$



• The double of a is 30 less than 60.

• b is 60 bigger than a.

• The double of a is 10 less than the sum of a and 60.

• b is 5 less than a.

Fig. 6: Connecting various representations

Training for understanding: a methodology

The structural cognitive learning process (Singer, 2001) supposes constructing the content that is to be learned and structuring the training in order to develop in children abilities analogous to the ones of the expert in a specific domain. To achieve these abilities, *mental operations* applied to *information* on different *levels of abstracting*, simultaneously with *transfers* and *crossings* in between, are specifically trained. The methodology developed to organize and structure the dynamic variation of skills covers three components: a systematic training, focused on building specific skills; a randomized training, focused on developing variability across skills; and a structured training, aiming at assimilating the invariants. These training components interact to built dynamic structures of thinking.

The **systematic training** regards, in the main, the development of the following skills connected with the concept of equation: analyzing sequences in order to identify patterns; building sequences using various representations; mental solving; comparing to assess similarities and differences; composing, solving, analyzing and transforming word problems based on equations; estimating and guessing solutions; arising awareness of errors' outcome. The tasks in this training systematically follow a graduation from concrete to abstract, in other words, from manipulating objects through actions, to representations and abstractions.

The **randomized training** is realized by activating the skills starting from isolated information. The tasks, organized as games, may be:

With starting point in isolated numbers. A child proposes two numbers: 4 and 20, for example, and asks to device equations involving the two numbers. The proposed equations are analyzed from the point of view of the existence of the solution. Another game requires the construction of a equation in which the number 4 is the solution and, respectively, it is not. Problems are composed starting from the equations that had solutions. The same procedures are carried out starting from other numbers. These are practiced orally (verbal), silently (in mind), in writing (without or with minimum verbalizing, and the result is required for checking).

With starting point in other equations. The teacher chooses a simple equation and asks for the development of similar equations. The same equation is transformed into a problem. The initial equation is compared with another one (number of terms, operations) in order to determine the similarities and dissimilarities. The number of terms and number of operations are increased. These are practiced orally, silently, in written form, maintaining the interest in exercising as many skills as possible.

With starting point in a simple problem. A simple problem that could be solved through addition is proposed. The problem is reformulated keeping the same numbers (changing the question position, for example). The problem is extended as far as to contain three or four additions, then subtractions or mixed operations. The initial problem is compared with other ones in order to determine the similarities and differences.

The **structured training** plays an important role in consolidating the mental structures acquired by the child. It supposes constantly resorting to models and diagrams in practicing the skills.

The internalization of a concept can be accomplished in very different ways: mainly by memory and rigidly, or only at the sensory-motor level, or at the level of representation (with different degrees of generalization). Initially, the assimilation is superficial: the child is able to solve simple equations but shows an inconsistency when the variability of the tasks is greater. Two targets left to be reached: *the raising of the internalized structure to the level of the formal way of writing and solving equations* and *the transformation of the internalized structure into a dynamic one*. The first target is mostly accomplished through the tasks in the systematic and structured training. The exercises become gradually complicated, thus requiring “the movement of thinking”, in the most various ways: with direct support from objects or concrete schemes, or without this support, inclusively by operating in an internal language, and later by operating with the literal symbols of the numbers. However, to obtain a dynamic thinking structure is much more complicated and this is developed by “shifting” each element (nucleus) of the structure, by confronting it with different operations of thinking. For example, a specific type of equation has to be related to the others on various levels of representation, as the diagram in Fig.7 suggests.

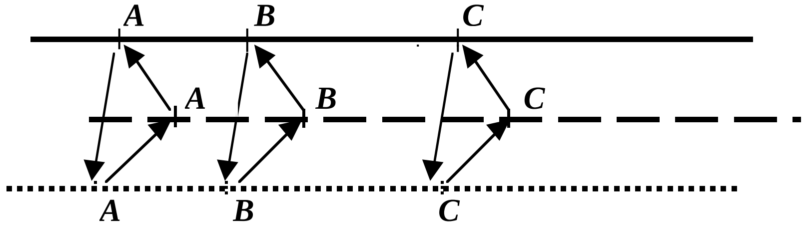


Fig. 7: Transfers on various levels of representation

In this scheme, ***A, B, C*** represents different types of equations that could be studied at a given moment. ***BAC*** represents the formal level of writing and solving an equation, ***B"A"C"*** is the concrete level of operating (with objects) and ***B'A'C'*** is an unconventional representation level (based on using models to represent equality of numbers and its properties). The arrows represent the transfer operations involved in transition between different types of representations. Reversing the arrows, other ways of practicing the transfer from concrete to various levels of abstractness are underlined.

An optimistic perspective: Some results of the experiments

The methodology synthetically describe above has been used in eight classes, surveying each cohort for four school years. The sources for analyzing the results were: observing the students behavior in the experimental and control classes; systematically testing the pupils participating in the experiment; testing, at intervals, the control classes; collecting the teachers' opinions involved in the experiments. The classroom observation was considered as being the essential element for drawing the conclusions, as this permits, in addition, to ascertain the children' individual and group reaction, and to evaluate the motivation level, the interest, the spontaneity, the atmosphere in the class, etc. On short, the results in performing the basic algorithms were relatively similar in both experimental and control classes, with a greater dispersion in the control ones. However, big differences have been recorded in *problem solving* activities. Most of the students in the control classes encountered great difficulties in constructing problems: the differences between the data and the topic were not correctly discerned, the problem's question was not individualized enough, the useless data were frequently taken into consideration, the missing ones were not noticed, while the students in experimental classes showed an awareness of the text meaning. The students in the control classes manifested a blockage on building or reformulating a problem, while the ones in experimental classes showed facility and desire to create problems. Concerning the *language*, the students in control classes manifested a tendency to reduce the language to mathematics operations, while the ones in experimental classes showed a desire to argue his/her approach and a tendency to explain through concrete objects or other models. Language errors were significant (meaningful) in the control classes, while in the experimental ones language errors were less significant. In the control classes, the process of reasoning seamed to be carried out on "atoms of problems" and a certain "structural disorder" was obvious, while in the experimental ones there was an emphasis on a goal-oriented activity.

How does learning occur: Some hypotheses

In spite of a huge body of research concentrated in the last decade on the brain and mind connection, very few data are uncontroversial. A convergent view assumes that learning shapes the brain in an interactive process with the environment (e.g. Quartz and Sejnowsky, 1997). Therefore, again, from the perspective of neuroscience, we know that there is a dual deep interrelation between development and learning.

What is happening in the brain during the "assimilation" of a piece of information? Without being able to give peremptory answers, the neurophysiological model below proved itself effective with regard to the structural cognitive learning. In addition, it is explaining some observable behaviors. The model has been developed based on neuroscience research focused on visual cortex development (e.g.: Kosslyn, 1994; Siegel, Körding and König, 2000). How does learning emerge? Imagine a child staying in front of a balance, trying to solve a weighting problem. When perceiving chunks of information (the balance, the weights, the problem, etc.), groups of cells specialized for form, color, motion, smell, etc. send nervous impulses. Millions of neurons filter and decompose the sensorial information into many waves with different wavelengths and amplitudes. Each cell responds only to gratings (3D waves) with specific characteristics, like proper orientation, spatial frequency (wavelength), and temporal frequency.For a very short time - fractions of a second - these oscillations have the same frequency. If the excitation continues in this universe of vibrating waves, the impulses received by the excited cortex zones modify the wavelength or other components of the oscillation that existed before the impulse. The excited zones start to synchronically oscillate. The pattern of these oscillations creates in the brain the mental configuration of the observed objects.

Let us further refine this model. Uncontested experiments showed that only about 10-20% of the cortex area is used (Hilgard & Bower, 1975). Recent neuroscience findings confirm that much of the cerebral cortex consists of "silent areas", which are not activated by sensory or motor activity, but which mediate higher cognitive functions. This means the central nervous system works in an extremely structured and economical way. The human brain efficiently stores and retrieves an enormous amount of information (It is sufficient to think of all possible aspects of the daily life objects, which are recognized and classified by the brain. Is there a specialized area for each piece of information? Probably, not.). The economy of the cortex use leads to the idea of the existence of a complex *hierarchy* at the level of both the neurons and their elements involved in making the connections: axonal and dendritic arborisations and synapses. Let us go on with our example: the student is analyzing the relationships between the objects implicated into the task. Groups of specialized cells placed in different areas of the brain receive impulses that modify the wavelength and other components of the oscillations in that area. The cells' status of "alertness" "dissolves" into the resonance phenomenon - that is completely separated areas in the brain get synchronized by simultaneously answering to the stimulus represented by the task. According to the complexity perceived for this task, the phenomenon of synchrony can occur on one or several levels. While solving the problem, the oscillations of the groups of cells stimulated by the sense organs (for example) create a primary zone of resonance, which lasts fractions of a second. The oscillation is transmitted to higher groups of cells, which create a second resonance zone, a more specialized one, and so on. As the oscillation propagate "upwards", the lower zones get out of synchronization, nevertheless preserving the damped down "trace" of the previous oscillation. The wave's "way back" ("top-down") will be much shorter this time and will have as effect the synchronous vibration of the sensitive zones. This is the moment of awareness of a possible solution to the problem. The decision-making action takes place within a complex process of synchronization and de-synchronization. In the end, the synchronous oscillation of certain areas selected in the brain sends the answer to the loco-motor system (removing the weights, writing the equivalent equations, etc.). The synchronous oscillation is a phenomenon that takes place according to probabilistic laws. Depending on the individuals, some zones are more permissive or easier to stimulate, but others are less permissive and this fact ads new variables to the learning process.

Learning behaviors: Some explanations

Some conclusions regarding the learning process, experimentally obtained, are easily explainable by the phenomenon of the synchronous oscillations, even though the bio-physical-chemical nature of the process is not clearly defined.

Learning rich in correlated stimuli is more effective. An explanation could be that numerous stimuli and their intercoordination increase the probability of oscillations having similar features, able to generate adequate resonance areas.

It was ascertained that *total interruption of learning for a compact period weakens very much the result*. This happens because, after the cessation of the excitation, the oscillations tend to damp out. The resonance zones separate gradually; they begin to vibrate with oscillations having different physical-chemical features. In this case, in order to accomplish the learning, the process of creating synchronous zones must be resumed, as if those had not existed before.

Forgetting the intention. Permanently, the cells out of synchronized areas are ready to take another task. Therefore, if the phenomenon of "coming back" ("top-down" interactions) of the wave does not occur fast enough, the synchronous vibration of the sensitive zones might not take place.

Persistence in error. The student takes a wrong course when solving a problem, though he/she has good knowledge in the field the problem belongs to. The phenomenon of self-control, or self-correction, does not take place; the resonance phenomenon occurs, but by activating zones in the "neighborhood" of the desirable ones. Classical learning - the inflexible type - presupposes a restricted area for the resonance emerging. Within classical learning, the brain gets the most difficult task (as it is not the subject of training): selecting the areas that can get into resonance.

Summarizing, a question transmitted at the level of the brain modifies the amplitude, the frequency and other possible features of the nervous oscillations, at first chaotically, then, gradually, more and more cells get into resonance as they "adjust" their wavelength. If the mental training accompanying the learning is varied, with emphasis on the stable elements, ***with successive passages, with recurrences and accents from various points of view***, then is more likely that areas of the cortex activated by these stimuli to generate synchronic oscillations. A learning of this type will ensure a good discrimination of the erroneous associations, which are extremely frequent in mathematics. This also is able to stimulate the creative associations. According to this model, the quality of learning could be expressed at the brain level by the quality of the resonance phenomenon. Efficient learning can be interpreted as the *dynamic changing* of the frequencies and of other features of the nervous oscillations so that *the resonance phenomenon might optimally take place.*

Conclusions and future work

Any type of learning is generating mental structures. At the first contacts with a new field, the stress is laid on primary development of the mental structures and on practicing them in simple situations. Later, the internalization continues, as new information is assimilated, but the stress lies on integrating them within the existing mental structures. The aim is that these structures might become extremely flexible in three directions: able to *multiply* (reproduce) on higher levels of abstraction, able to *integrate* in new structures having the same nature or different natures, and able to *mobilize* with great precision when there is a need of them in solving some practical tasks (by their appropriate reduction to the already developed intellectual models, or by their temporary changing for the purpose of solving totally new tasks). In the process of practical training, the mental models confront the external requirements, which are identified as being known or unknown; in the second case, finding a solution implies an effort of creation.

Looking back over the classroom experiments, there is not a huge difference in the type of training proposed by the structural cognitive learning comparing to the classical one; there are only some accents changed, and what was before just accidentally touched, here is much more systematically emphasized. What it is important is that this practice already exists, and teacher-training programs are expected to be able to do these changes, if such a perspective on learning is formally accepted. Where is the difficulty from the point of view of mind–brain research connected to education? We have to identify those conceptual structures, which could be trained in each school grade in order to generate in children the mathematical thinking (the specific intellect) and to avoid the labyrinth of insignificant information (the so called "noise" in complex systems). Future work should identify concrete specific ways to build and develop *dynamic thinking structures* for each domain of mathematics learning.

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