BUILDING A FINITE ARITHMETIC STRUCTURE: INTERPRETATION IN TERMS OF ABSTRACTION IN CONTEXT[1]

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Some results of our research on structuring mathematical knowledge, investigated via a student's building of a finite arithmetic structure, will be interpreted in terms of the model of abstraction in context (Hershkowitz, Schwarz and Dreyfus). The contribution has two main goals: to see (a) if and how the model of abstraction in context can be applied to our data, (b) if our existing results are enriched by such an interpretation. The student's construction of the proceptual understanding of inverse reduction was subject to the interpretation in a specific mathematical context, the so called restricted arithmetic.

1. Introduction

In 1998, we started a research project aimed at the processes of building an inner mathematical structure which were investigated via the construction of a new structure as an analogy to the existing structure. The methodology mainly consisted of think-aloud semi-structured interviews with university students – future mathematics teachers. Partial results of the research project have been published in Stehlikova & Jirotkova (2002) and Stehlikova (2002).

Hershkowitz, Schwarz and Dreyfus (Dreyfus, Hershkowitz & Schwarz, 2001a; Hershkowitz, Schwarz & Dreyfus, 2001; Schwarz, Hershkowitz & Dreyfus, 2002) have recently proposed a model of dynamically nested epistemic actions for the processes of abstraction in context which has since been elaborated (e.g. Dreyfus, Hershkowitz & Schwarz, 2001b; Tabach, Hershkowitz & Schwarz, 2001; Tabach & Hershkowitz, 2002; Tsamir & Dreyfus, 2002). The model seems to be in harmony with our view of the acquisition of mathematical knowledge which is based on two assumptions from cognitive science: knowledge is represented internally and internal representations are structured (Hiebert & Carpenter, 1992). We have therefore decided to revisit the data from our research and interpret them differently, this time from the standpoint of abstraction in context.

Our main goal was to see

(a) if and how the model of abstraction in context can be applied to our data,

(b) if our existing results are enriched by such an interpretation.

In the text below, first abstraction in context in which our considerations will be embedded will be briefly described, then our research on structuring mathematical knowledge will be introduced, and finally we will interpret one kind of data from our research in terms of abstraction in context.

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2. Framework – Abstraction in Context

The model of abstraction in context has been presented in detail in the above articles therefore we will only present its brief overview and the author's interpretation of the model. The proposers of the theory characterise abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure". By reorganising into a new structure, the proposers mean the establishment of mathematical connections (making a new hypothesis, inventing or reinventing a mathematical generalisation, a proof, or a new strategy for solving a problem). On the other hand, neither learning to mechanically perform a mathematical algorithm nor rote learning qualify as abstractions.

The authors of the theory also claim that abstraction strongly depends on context, on the history of the learner and on artefacts available to them and in this sense the structure is internal, "personalised" (Schwarz, Herhskowitz & Dreyfus, 2002). Thus hereinafter by a structure we will mean a mental image which a person holds in his/her mind about a mathematical structure.

Note: The authors of the theory call mathematical methods, strategies, concepts, etc. structures. We would have preferred to reserve the word 'structure' for more complex knowledge and simply call what is being built 'mathematical knowledge'. Similarly, the term abstraction has a more specific meaning in mathematics for us, thus instead of 'processes of abstraction', the term 'processes of construction of knowledge' seems to us to be more appropriate. However, in this contribution we will follow the terms the authors use.

The genesis of abstraction is seen as consisting of three stages (Hershkowitz, Schwarz & Dreyfus, 2001):

1. A need for a new structure,
2. The constructing of a new abstract entity in which recognizing and building-with already existing structures are nested dialectically, and
3. The Consolidation of the abstract entity facilitating one's recognizing it with increased ease and building-with it in further activities.

Three epistemic actions which are constituent of abstraction are (Schwarz, Herhskowitz & Dreyfus, 2002):

> *Constructing* is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. *Recognizing* a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement.

3. Study

The **tool** of our investigation of a student's construction of an internal mathematical structure is an arithmetic structure $A_2 = (\underline{A}_2,\oplus,\otimes)$ which we call *restricted arithmetic* and which was elaborated by Milan Hejny. The gate to the restricted arithmetic is the mapping r: $\underline{N} \rightarrow \underline{N}$, $n \rightarrow n - 99 \cdot [n/99]$, which we call *reducing mapping* (here $[y]$ is the integer part of $y \in \underline{R}$). For instance, $r(7\ 305) = 78$, $r(135\ 728) = 98$, ...

Reduction was introduced to students as an instruction illustrated by several concrete examples[2]:

Perform a 'double-digit sum' operation on a natural number until you get a one or two digit number. A double-digit sum operation is similar to a digit sum operation but instead of adding digits, we add two digits at a time. (1)

For example, $r(682) = 82 + 6 = 88$, $r(7\ 945) = r(45 + 79) = r(124) = 24 + 1 = 25$.

Next, let us have the set $\underline{A}_2 = \{1,2,3,...,99\}$ of z-numbers. The reducing mapping r is used to introduce binary operations of z-addition $\oplus$ and z-multiplication $\otimes$ in $\underline{A}_2$ as follows: $\forall\ x, y \in \underline{A}_2,\ x \oplus y = r(x + y)$ and $x \otimes y = r(x \cdot y)$. For instance $72 \oplus 95 = 68$, $72 \otimes 95 = r(6\ 840) = 9$.

In the context of restricted arithmetic, many different problems can be posed and solved (for some examples see Stehlikova, 2000).

In 1995-2000, a series of clinical **interviews** was conducted with several students – volunteers, all of whom were future mathematics teachers. The interviews were semi-structures in that only introductory problems had been prepared beforehand with some possible continuations and the course of the interview depended on the direction the student's investigations took. The first session was the same for everyone – the introduction of restricted arithmetic, some problems on reduction so that the students grasped its meaning and some additive and multiplicative linear equations. The next sessions differed and their content depended on (a) whether and in which order students discovered key objects of A_2 (e.g. neutral elements and inverse elements), (b) what strategy they used for solving linear equations (e.g. if they wanted to use z-subtraction, they had to define this operation first), (c) whether they posed their own questions and problems, etc.

When analysing the transcripts of interviews via the grounded theory approach (Strauss & Corbin, 1998), some categories were identified. One of them is a mathematical category which, among others, includes the student's understanding of inverse reduction.

4. Development of the Student's Understanding of Inverse Reduction

The introduction of reduction (see (1) in section 3) is a procedural one. However, the symbol $r(n)$ points both to the *process* ('carry out the instruction on the number so that you get a z-number') and to the result of this process, the *concept*, i.e. the z-number. Thus the symbol $r(n)$ can be understood in a proceptual way (Gray & Tall,

[2] The isolated models of reduction, e.g. the numerical examples, are introduced to the student at the same time as its verbalised universal model (for the theory of isolated and universal model see Hejny, in press).

2002) as $r(n) = n - k \cdot 99$, $k \in \mathbf{N}_0$, or as $r(n)= n - 99\cdot [n/99]$ (the first symbol is 'closer' to the process than to the concept).

The structure of restricted arithmetic, which is first rooted firmly in ordinary arithmetic, gradually gains independence. The key part of this long process is the understanding of the inverse process to that of reduction.

By **inverse reduction**[3], we mean a mapping which maps any $a \in A_2$ onto the set $r^{-1}(a)$ of all numbers $n \in \mathbf{N}$ for which $r(n) = a$. Again, the symbol $r^{-1}(a)$ can be understood both as a process of finding out the inverse reductions of a number and as a concept, the result of this process. The proceptual stage is reached when a student is able to express in words or in symbols the following: $r^{-1}(a) = \{a + k \cdot 99, k \in \mathbf{N}_0\}$.

Below, we will first describe in detail the growth of understanding of inverse reduction by one student and then our model of the whole process will be given.

4.1 Molly's Understanding of Inverse Reduction

Molly was a 19-year-old student, a future mathematics teacher. She took part in eleven interviews (sessions) over two years. She was very enthusiastic about the topic and continued with her study of restricted arithmetic in her diploma work (for the description of her whole work see Stehlikova, 2002). Here we will focus on her gradual understanding of inverse reduction during the first four sessions. It is important to note that the student was not asked "What is an inverse reduction?". Inverse reduction played a key role in solving different problems in restricted arithmetic. In the following text we will describe parts of Molly's work in which her understanding of inverse reduction appears. We will illustrate our considerations by excerpts from the interviews. E will stand for the experimenter, M for Molly.

At the beginning of Session 1 (16^{th} April 1998) the experimenter introduced Molly to restricted arithmetic (RA) in the way given above (reduction as an instruction, set A_2, *z*-addition, *z*-multiplication). She was asked to add, multiply and reduce some natural numbers for practise. Then she was given some linear equations. When solving the equation $61 \oplus x = 4$, she immediately realised that she had to find a number on the right whose reduction would yield 4 and she first suggested 202 and then 103. Similarly, in the next linear equation she suggested 409 as the inverse reduction of 13. The reason in both cases might be that she reflected on the way reduction is done and decomposed 4 as 2 + 2, 1 + 3 and similarly, number 13 as 4 + 9.

In Session 2 (23^{rd} April 1998) when solving other linear equations the sequence of inverse reductions appeared for the first time: 92 = r(290) = r(389) = r(488). Note that

[3] The words 'reduction' and 'inverse reduction' will be used both for the process ('reducing', 'carrying out an inverse reduction') and for the result of this process ('the reduction of 155 is 56', 'inverse reductions of number 56 are numbers 155, 254, 353, etc.').

the first inverse reduction 191 is missing. She was able to produce the sequence very quickly by adding one to hundreds and subtracting one from units. This procedure (let us call it S1) seems to be connected in her mind with three-digit numbers only because when she was producing inverse reductions of 45 (when solving the equation $13 \otimes x = 45$) she found inverse reductions up to number 936[4]:

E22: *Can't we go on?*

M24: *We would get 45 again.* (pause) *No, we can. We can 10, 35; 1 035.*

When solving the same problem, the question arose what the biggest number is whose reduction is 45.

M31: *It can be 4 500.*

E30: *The x?* (pause)

M32: *No.* (She laughs.) *The result, the product.*

E31: *I see, and why?*

M33: *Because if there were bigger numbers on the first two places, then* (pause) *it would be bigger than 45. For instance, if there were 6300, then it would be 63 and it is bigger than 45.*

E33: *Well, so what if we have 6398.* (pause)

M36: *It is 62. It is logical that it will be one less.*

Again, Molly's belief that the sequence of inverse reductions will be bounded came to the fore. The experimenter's prompting led her to see that it is not the case and in her further work it never appeared again.

In Session 3 (30th April 1998), Molly first presented her work she did at home between Session 2 and 3 to the experimenter and it was obvious that she used the procedure S1 with ease.

When solving the equation $93 \otimes x = 6$, she produced this sequence of inverse reductions: 006, 105, 204, 303, 402, 501, 600, 798, 897, 996. First, her belief that S1 is confined to three digit numbers might have played a role again. Second, S1 does not allow one to find the number between 600 and 798. Molly must have used S1 because it was apparent from her written work that she first wrote 600, 799 (quite logical if you use S1 on number 600), but then she realised that $r(799)$ is not 6 and rewrote it to 798. She did not realise that there was one number missing in the sequence. From her further work, it is clear that she went on using the same procedure S1. She made the same mistake in the sequence of inverse reductions for number 3: 102, 201, 300, 498,...

In Session 3, the experimenter pointed out the missing number:

[4] It is important to stress that when introduced to reduction, Molly also met examples of the reduction of four and five-digit numbers.

E7: *Can I make you aware that you forgot one number between 600 and 798?* (long pause)

M8: *690.* (pause)

E8: *Think it over, I will look at something in the meantime.* (long pause)

M9: *699.*

E9: *Yes.* (pause)

M10: *It cannot be.* (long pause) *I always add 99, it is clear then, it is the zero.*

Molly first suggested 690, probably without much thinking. It was a number fitting between 600 and 798. Then she said 699, we cannot really say why. But what is important is that after a long pause she observed that the difference between inverse reductions was 99 and that 99 was 'zero". (In Session 1 she said that 99 behaved as a zero.)

We expected that Molly would use this important observation for determining inverse reductions in her further work, however, it was not the case. She returned back to her old procedure S1. In one part of her home work between Session 3 and 4, she missed number 399 in the sequence of inverse reductions for number 3 again. In another part of the same work, she determined the correct sequence of inverse reductions of number 8: ... 701, 800, 899,... From the way she reported on her work during Session 4 (7th May 1998), it was clear that she still used the old procedure but became aware of the fact that it failed sometimes and supplied the missing number by adding 99.

After about a year following the above Sessions, Molly was asked to describe in writing everything she had so far discovered about restricted arithmetic. We have 9 subsequent versions of her mathematical description. In version 2 (5th August 1999) she described the process of solving linear equations like this: *We already know that we can write each number from A_2 in different digits and the value of the number stays the same (using the inverse operation to the operation of reduction or by adding number 99).* She mentioned both S1 and adding 99. She did not mention the need to be careful with numbers of the kind *A*00 as given above. It confirms our hypothesis that the fact that S1 fails in certain cases did not lead to the complete reconstruction of S1 but it its enrichment.

In version 3 of her mathematical writing (12th November 1999) she introduced inverse reduction as: *In the operation of inverse reduction we transform a number from $\underline{A}_2$ into the number which does not belong to $\underline{A}_2$. There can be an infinite number of such numbers. For example, 1 = r(100) = r(10000) = r(1000000) = ...* In the same version, in the section on solving linear equations, she tries to prove this statement (to illustrate how the inverse reductions can be made):

$\exists\, k \in \mathrm{N}$, if $a, b, x \in \mathrm{N}$, $a, b, x \leq 99$: $a \otimes x = b + k\,99$.

Regardless of the mathematical validity of the statement, it is clear that Molly was aware of the difference between inverse reductios and she was able to write it symbolically. In her example above she did not use the fact. Why? We can only guess that maybe she considered the statement to be more convincing with respect to the infiniteness of the sequence.

Further development of her understanding of inverse reduction was towards symbolisation. When she was asked to describe reduction and inverse reduction in general rather than on examples, in version 6 of her mathematical writing (30th September 2001[5]) she wrote:

The operation of reduction is defined in general: For all $x \in \mathrm{N}$, $\exists\, k \in \mathrm{N}$,

$r(x) = x - 9k$, *where* $r(x) \in \underline{A}_2$. ...

The operation of inverse reduction is defined in general: For all $x \in \underline{A}_2$,

$x = r(x + 99k)$, *where* $k \in \mathrm{N}$.

She uses the symbolic description of inverse reductions in some proofs, for instance of associativity of *z*-addition.

4.2 General Model of the Understanding of Inverse Reduction

Similar to Molly, the analyses of the work of three other students have been made from the point of view of their understanding of inverse reduction. We thus got four separate models which we wanted to represent by one universal model. We decided to do so in a model which would include not only what the four students had in common but also in what they differed (i.e. the order of steps, the length of time and number of problems they solved before they moved on, etc.). Our model of the student's construction of proceptual understanding of inverse reduction in $\underline{A}_2$ consists of several steps. Each will be first described by examples and our original accounts and then interpreted in terms of abstraction in context.

- The student uses the introduction of reduction to reduce a natural number into a *z*-number carrying out the 'double-digit sum' and gains experience with the concept.

  No *construction* present yet.

- The student solves the equation of the type $x \oplus 98 = 92$ and faces the question how to find a number whose reduction yields 92. The mental reconstruction of reduction leads him/her to the decomposition into 'a double-digit number plus a digit', e.g. 90 + 2, i.e. 290. He/she may immediately realise that there is more than one such number, e.g. 89 + 3, i.e. 389. The need to find a concise way of determining inverse reductions motivates him/her for looking for a pattern. Gradually he/she discovers the procedure 'adding one to hundreds and subtracting one from units' and gets $92 = r(191) = r(290) = r(389) = r(488) = r(587) = ...$

  The *need for the construction* of the procedure arises from the task to solve linear equations of a more 'difficult' kind, i.e. $x \oplus a = b$, where $a > b$. The student

[5] Molly worked on different parts of her diploma thesis at once. That is why there is a time gap between versions of her mathematical writing which represents a small part of her diploma thesis.

builds with the previously given and practised notion of reduction and *constructs* the procedure S1 of 'adding one to hundreds and subtracting one from units'[6].

- The student routinises S1. He/she encounters an instance in which the procedure does not produce the correct answer. Consider this example: $5 = r(104) = r(203) = = r(302) = r(401) = r(500)$. The next number in a sequence is 599, however, if the student uses the above procedure, he/she writes 6 in a hundredth place and then by subtracting one on the right, he/she should have 99. However $r(699)$ is not 5. Thus the student meets a cognitive conflict and has to devise a way of dealing with it (see the next three steps).

  The procedure is *consolidated* by using it in other problems with increasing ease. However, the procedure is not universal, it fails in certain cases. In no case did this failure lead to the complete refusal of S1.

- The student reaches the conviction that there is no bigger inverse reduction of the number in question, i.e. the sequence of inverse reductions is bounded. For instance, he/she concludes that there is no number bigger than 500 whose reduction would yield number 5.

  Building with S1, the student *constructs* a hypothesis (which will later prove to be false) that enriches S1.

- The student meets a contradiction to his conviction (or the experimenter gives a hint, for instance: "What about number 50 000?"). He/she then looks for ways of resolving the contradiction.

- The student notices that inverse reductions of a z-number differ by 99. He/she realises that the sequence of inverse reductions can also be made via adding 99 and uses this as an alternative instruction 'add 99 to get a new inverse reduction'.

  The *need for the new structure* is mainly given by the contradiction above and by the need to get a solution to a problem. *Building with* a sequence of inverse reductions of some z-numbers and probably his/her knowledge of arithmetic sequences (noticing the difference between numbers), the student *constructs* a new procedure of adding 99 to get inverse reductions – let us call it S2.

- The student routinises the procedure of finding the inverse reductions, using the first procedure S1 (it is quicker) and the second procedure S2 when the first fails. In terms of the model of separate and universal models (Hejny, in press), inverse reductions of concrete numbers are separate models for the universal models S1 and S2.

  Building with the two procedures, a process of finding inverse reductions is *constructed* and *consolidated* in problems – let us call it S3.

[6] We are not sure if, according to the authors of the theory, we should use 'the construction of the *structure*' rather than *procedure*. If so, word 'structure' seems to be unnecessarily 'strong'.

- The student realises that any number $a \in A_2$ can be written as $a = x - 99 \cdot k$, where $k \in \mathbf{N}_0$, and is able to manipulate it as an object (he/she reached the proceptual stage). The transition to the final step is motivated by tasks in which students are to prove e.g. associativity of z-addition for which this step is essential (as far as our experience confirms). In most cases, this step must be made explicit to students by the experimenter. If the student has already reached the previous step, he/she can use the knowledge of the last step with ease. Otherwise, he/she is confused by the experimenter's suggestion and refuses it.

  The above *construction* of S2 is expressed symbolically. We consider it to be a separate *construction* – let us call it S4, because the former construction can exist independently and this construction is given by the student's *need*, for instance, to prove the associative law in restricted arithmetic.

Note: The model describes the cases in which the symbolic description of inverse reductions was discovered on the basis of S2. In theory, this description could have been done from S1 as well. Consider this example: 77 = r(275) = r(374). Using S1, we can get from 275 to 374 like this: 374 = 275 + 100-1 = 275 + 99. However, this construction seems to be more difficult to make because it did not appear.

The process of construction of the concept 'inverse reduction' is part of the constructions of higher-order structures (for example, strategies for solving linear additive and multiplicative equations, strategies for solving quadratic equations, the structure of squares, of powers, algebraic structures – groups and subgroups in A_2).

5. Conclusions

To conclude, we will discuss the two goals given in the introduction.

As far as we know, the model has been used for (a) an interview with a single student, (b) an interview with a pair of students, (c) a series of interviews with a single student, (d) a series of interviews with a pair of students. Here we applied the theory to our model of the student's construction of proceptual understanding of inverse reduction in restricted arithmetic which was constructed on the basis of the work of four students. Can we still speak about "personalised" structures? Maybe not, however, we claim that abstraction in context can also be used for this kind of data.

Moreover, we described cases in which (1) two different constructions have been made (S1 and S2) to solve one problem, (2) an erroneous hypothesis has been constructed and later refused.

The model of abstraction proposed by the theory of abstraction in context seems to be able to account of that part of data of our research on structuring mathematical knowledge presented above. We believe that if we look at one thing from several points of view, it will always be beneficial. In our opinion, the main contribution of the model of abstraction in context to our research is that it brought organisation into the results. The model prompted us to ask questions of the following sort: What precisely is being constructed? What is the hierarchy of constructions? With what means is it being constructed (i.e. what is being recognised and built-with)? How and when is it further used (when the consolidation appears)? etc.

It remains to be seen how the theory of abstraction in context can be used for other data from our research and for results which have already been found in terms of the grounded theory approach, procept theory or the theory of isolated and universal models. In addition, our study brought to light some problems with terminology which we had when using abstraction in context.

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[7] In this article, references to the Czech, Slovak and Polish research connected to our research are given.