REASONING AND PROOF: METHODOLOGICAL KNOWLEDGE AS A COMPONENT OF PROOF COMPETENCE[1]

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Abstract: *Methodological knowledge is an important component of proof competence. In this paper, we argue that three different aspects of methodological knowledge may be distinguished, which we describe as proof scheme, proof structure, and chain of conclusions. These theoretical aspects guided part of an interview study on secondary students' methodological knowledge, which serves as a qualitative supplement of a large-scale quantitative study. We present some results of this investigation in which the students had to evaluate correct and incorrect proofs.*

1 Introduction

During the last few years, reasoning, proof and argumentation in the mathematics classroom has become an important issue in mathematics education research. There is an increasing number of empirical studies on this subject; their results were partly supported by the outcomes of international comparative studies like TIMSS or PISA and their intensive discussion in the scientific community. Our research is part of this context. Its aim is to identify cognitive and non-cognitive factors which play a role in proof competencies of students. We will suggest a theoretical framework and present empirical results, which demonstrate that the methodological knowledge of students is an important prerequisite of their proof competencies.

2 Theoretical framework and research questions

2.1 The role of proof in mathematics and in the mathematics classroom

Axioms, definitions, theorems and their proofs, and conjectures form the scaffold of mathematics as a scientific discipline. Mathematics is a proving discipline, which represents the main difference between mathematics and any other scientific discipline. Obviously, mathematics is also the result of social processes, but there is a relative high degree of coherence and consensus among mathematicians (Heintz, 2000; Manin, 1977). For evaluating a proof, the mathematical community uses the theoretical construct of a formal proof: starting from a "real" proof one tries to approach a formal proof by adding information (which is part of the general knowledge shared by the scientific community) until mathematicians are convinced that the real proof is correct.

The role of proof in mathematics has consequences for teaching proof in the mathematics classroom. On the one hand, the shared knowledge basis of students differs from that of the mathematics community. Consequently, one has to accept

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other kinds of proofs in the classroom. On the other hand, the students should get an authentic image of mathematics and, in particular, in high schools they need to be prepared for scientific mathematics, which plays an essential role in many domains at the university level. Accordingly, proof in the classroom will aim at the consensus in the mathematics community, but will probably be less rigorous in its nature (cf. the discussion in Knipping, Dreyfus & Krummheuer, 2002, and Heinze & Reiss, 2002). This approach is the basis for our research.

2.2 Methodological knowledge as a component of proof competence

Research results, which are based on analysis of students' proofs, suggest that the knowledge of concepts and rules (or theorems) is not sufficient for performing mathematical proofs (Reiss, Klieme & Heinze, 2001). In addition to this, understanding and knowledge of correct mathematical proof procedures are necessary prerequisites. This *methodological knowledge* consists of three aspects, which are independent of each other. We would describe these three aspects as *proof scheme, proof structure* and *chain of conclusions*.

1. **Proof scheme**: A mathematical proof is a deductive reasoning pattern. This means that for each conclusion in the proof there is a supporting argument with a deductive character. Other kinds of arguments like empirical-inductive arguments, reference to a higher authority, or perceptual arguments are not adequate for a mathematical proof. Notice that in our definition an argument with deductive character must not necessarily be correct.
2. **Proof structure**: A proof starts at given premises and ends at a specific assertion. This assertion is proved if all arguments are valid from a structural point of view. In other words: a proof has to prove what it should prove, and the use of the assertion as an argument is not adequate. Moreover, gaps that form disruptions in the structure of the argumentation are not accepted as part of a mathematical proof.
3. **Chain of conclusions**: Each step of a proof can be concluded from the previous step (possibly supported by additional mathematical information).

As mentioned above, these three aspects are independent from each other. For example, a student who tries to prove a statement but uses a circular argument (see the example in Section 3.1), contradicts the second aspect (proof structure) but not necessarily the first or third aspect (proof scheme, chain of conclusion). A "proof" which is based on empirical-inductive arguments like the example in Section 2.3.4 contradicts the first aspect but not necessarily the second or third aspect. And finally, it is possible to construct a "proof" that contradicts the third aspect, but not the first or second one (e.g., if there is a division by zero as a proof step).

In the following we will describe findings from different empirical investigations concerning students' methodological knowledge. There are several detailed studies, which investigated the three aspects or parts of it. Moreover, there are some studies based on multiple choice items, which include all three aspects. We will show later, that multiple choice items bear the risk to give only surface information and may not allow a deeper analysis of a student's problem solving process.

2.3 Methodological knowledge for performing proofs – empirical findings

2.3.1 Students' proof schemes

One of the most detailed research on students' proof schemes has been published in a study by Harel and Sowder (1998). The authors investigated the proof schemes[2] of 128 college students using classroom observations, tests, and interviews. They could identify 17 different proof schemes, which can be assigned to three classes: (1) the external conviction (e.g. reference to a higher authority), (2) the empirical proof scheme (e.g. inductive or perceptual arguments), and (3) the analytical proof scheme (e.g. the deductive-axiomatic proof scheme).

An investigation by Martin & Harel (1989) with 101 elementary school preservice teachers gives some quantitative data. In this study, the students were asked to evaluate different proofs of number theoretical statements on a scale from 1 (= no mathematical proof) to 4 (= mathematical proof). The presented proofs were classified in different kinds of inductive, respectively deductive argumentations. The results show that only 10% of the students strictly rejected (= 1) all inductive argumentations, 80% gave a positive evaluation (= 3 or 4) to at least one inductive argumentation. The deductive arguments were, in general, better evaluated then the inductive ones. It is remarkable that many students accepted both, inductive and deductive arguments.

Vinner (1983) described an investigation with 365 students from grades 10 and 11. In a regular mathematics lesson they proved the statement, that each number of the form $n^3 - n$ is divisible by 6 (n an integer). The next day the students were presented three solutions of the problem "Prove that $59^3 - 59$ is divisible by 6." The first solution was a simple calculation, the second was the proof of the general statement with $n = 59$, and the third solution was a reference to the general statement and its proof. About one third of the students preferred the second solution. Moreover, 23% of the students were not able to accept the universal validity of a proven mathematical statement.

2.3.2 Proof structure

Selden and Selden (1995) investigated students' abilities to construct or validate a proof structure of a mathematical statement. They analyzed data from tests and examinations of 61 students, which attended introductory mathematics courses at the university level. They presented items to them, which aimed at the identification of the logic structure of informal written statements. There were 8.5% correct answers, and these answers were given by only 13.5% of the students. Selden and Selden (1995) conclude that deficits in identifying the logical structure of a statement will entail deficits in constructing a proof structure for these statements.

[2] In Harel & Sowder (1998) the notion „proof scheme" has a wider definition: "A person's proof scheme consists of what constitutes ascertain and persuading for that person" (Harel & Sowder, 1998, p. 244). However, their results give some ideas about students' difficulties with the proof scheme (as defined in Section 2.2.).

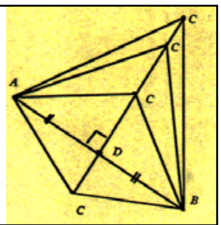
2.3.3 Chain of conclusions

It is difficult to investigate students' abilities to evaluate or construct a logical conclusion in a mathematical proof. Mistakes may be due to deficits in the logical competence as well as to other reasons like deficits in the conceptual knowledge. There are some empirical findings concerning the converse of a logical implication (e.g. Heinze & Kwak, 2002; Küchemann & Hoyles, 2002). The results show that students have difficulties with this special aspect of mathematical proving if it is isolated from a realistic context.

2.3.4 Multiple choice items measuring methodological knowledge

In the last few years, several empirical studies were published in which the methodological knowledge was investigated using a multiple choice test developed by Healy and Hoyles (1998). This test consists of simple mathematical statements, each of them followed by at least four different "proofs". There is an empirical-inductive solution, a solution in a formal style with a circular argument and two correct solutions, one in a formal style and one in a narrative style. The students are asked to validate these "proofs" and to answer questions in a multiple choice format for each of these solutions, e.g. they are asked whether the solution is faulty, whether the solution shows that the statement is universally valid or whether it is valid for special cases only. Moreover, the students are asked which of the solutions they like most and which solution would get the best mark from the teacher.

Example (empirical): C is any point on the perpendicular bisector of AB.
Proof: Triangle ABC is always isosceles.
Sylvia's answer:
I moved C to different places on the perpendicular bisector and measured AC and BC. They were always the same, so the triangles were isosceles.



In a nation wide survey with nearly 2500 high achieving students of grade 10 Healy and Hoyles (1998) found out that most of the students were able to recognize correct proofs. However, the evaluation of the proof depended on different factors like the formal presentation of the arguments. Students assume that a correct proof in a narrative style has a lower chance to get a good mark from the teacher than a formal solution. In contrast, the students preferred a narrative style, when they constructed proofs. Moreover, they often used empirical arguments though they were aware that these arguments were not correct.

Our investigation with nearly 700 students of grade 8 in Germany showed that it was much more difficult for students to judge incorrect "proofs" than correct proofs: while 67% described the correct proofs as correct, only 35% gave correct answers for the incorrect "proofs" (Reiss, Hellmich & Thomas, 2002). Moreover, in the last case the students performed better in evaluating the circular argument than in recognizing the problem of an empirical argument. They had better results concerning validating the correct proof in a narrative style than concerning the correct proof in a formal style.

Findings of a study with 81 students of grade 13 in Germany revealed that even at the end of the upper secondary level students have deficits in all three aspects of the methodological knowledge (Table 1, Reiss, Klieme & Heinze, 2001). Comparing these results with our findings from grade 8 one can say that in grade 13 more students recognize the problem of an empirical argumentation. However, the percentage of students which reject the circular solution is still about one third. A further analysis of the data has shown that the essential parts of the proof competence are methodological knowledge, declarative knowledge, and metacognition (Reiss, Klieme & Heinze, 2001).

Type of solution	Correct answers
Empirical solution	51 %
Circular argument	33 %
Formal proof	57 %
Narrative proof	30 %

Table 1: Correct evaluation - results

3 Methodological knowledge – an interview study

3.1 Research question and design of the study

As already mentioned a multiple choice item is restricted to certain answers. There is hardly any information why a student selects a specific answer. Moreover, it is not clear whether students understand a multiple choice item as expected by the researcher. Therefore, we wanted to get detailed information about students' methodological knowledge. Our research questions were the following:

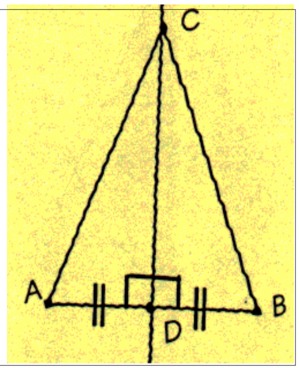
- What are the main deficits in students' methodological knowledge?
- What are the students' ideas and thoughts when solving the multiple choice items in the domain of the methodological knowledge?

These research questions were addressed in an interview study. Eleven grade 8 students were chosen from a sample of 700 students according to their achievement in a written test on geometry items (Krell, 2002; Reiss, Hellmich & Thomas, 2002). In the interview study, each student was asked to think aloud when evaluating four

Example (circular): C is any point on the perpendicular bisector of AB. Proof: Triangle ABC is always isosceles.
Jens' answer:

Statement	*Reason*
Angle ADC = 90°	*Perpendicular line*
Angle BDC = 90°	*Perpendicular line*
Angle CAB = Angle CBD	*Base angles of an isosceles triangle are equal*

Conclusion: AC = BC. The statement is true.



given solutions for a geometrical proof item.

There was one empirical solution, one solution containing a circular argument and two correct proofs, one in a formal style (using a congruence argument) and one in a narrative style (using a reflection argument). In each case the students had to evaluate the solution and then to answer three questions: whether the proof is universally valid or whether it is only valid for special cases and whether the proof contains a mistake. After finishing all items the interviewer asked the student to think a second time about the question he/she answered wrong. If the student did not change the answer the interviewer gave a hint which part of the proof might be wrong and repeated the question.

3.2 Results

In the following we present a part of the results of our interview study. In Table 2, we give an overview about the main results. The symbol "v" means that the item was finally solved correctly (without hints, but after giving a second thought whether the first answer was wrong). If the student was not able to evaluate the solution correctly, the numbers 1, 2 and 3 classify which kind of mistake he or she made (cf. Section 2.2). The symbols +, o, – give information about the students' achievements in the written test on geometry (+ means high, – means low).

Student	Geometry Achievement	Empirical Argument	Circular Argument	Formal Proof	Narrative Proof
Robert	+	v	v	v	v
Malte	+	v	v	v	v
Kristian	+	2	2	v	v
Anja	+	1	2	2	v
Maria	+	2	2	v	v
Leonie	o	1	2	v	v
Swantje	o	1,2	2	v	v
Tobias	o	v	v	v	v
Michael	–	2	2	v	v
Katrin	–	1	2	2	3
Cornelia	–	1,2	1,2	v	v

Table 2: Results of the interview study

The results in Table 2 show that nearly all students were able to evaluate the formal and the narrative proof correctly. However, most of the students found the formal proof using the congruence argument to be more complicated than the narrative proof using the reflection argument. Katrin, who had difficulties with both proofs, did not fully understand the structure of the formal proof. It seems that she did not understand the congruence arguments. In the case of the narrative proof she was not sure, whether there was a mistake. The proof is based on the fact that point B is a mirror point of A where CD is the axis of reflection. Katrin noticed that also A is a mirror point of B. Since this is not mentioned in the proof, she thought that this was maybe a mistake:

Interviewer:	If there is a mistake, what do you think where it is?
Katrin:	Yes, eh, because C … because CD is the perpendicular bisector of AB … yes, the mirror point, somewhere there is the mistake. I don’t know, because …
Interviewer:	Do you think it is not true that B is the mirror point of A?
Katrin:	No. One can do this also the other way round. It doesn’t matter...

Several students accepted the empirical argument in the first solution. We found out that they used inductive proof schemes in their explanations. Anja argued: “I thought that I could do the same with other triangles, too. (…) In my opinion this should work.” For the solution with the circular argument only one student, Cornelia, used an empirical-perceptual argument when she evaluated this “proof”. She said, that one can see that the line CD (the perpendicular bisector) is perpendicular to AB. However, she also said that there is another reason, because in the drawing there is the sign for the right angle at the crossing of AB and CD.

Most of the students’ mistakes in the evaluation of the solutions are related to problems in the proof structure. First of all, some students (Maria, Swantje, Michael, Katrin, Cornelia) did not notice the circular argument in the second solution. Even when we asked them directly whether the third step with the circular argument was correct, Maria and Swantje agreed. Michael, Katrin and Cornelia changed their opinion:

Interviewer (points to the circular argument):	(…) Do you think that this argument is o.k.?
Cornelia:	Hm, I think so.
Interviewer:	Hm. Which statements are acceptable in a proof?
Cornelia:	I don’t know.
Interviewer:	You should show that this triangle is isosceles. Is it allowed to use the fact that these angles are equal?
Cornelia:	No, because maybe it is not true that the triangle is isosceles.

Finally, many students used a circular argumentation when they evaluated the solutions (particularly in the case of the incorrect solutions). The students argued, that these solutions are correct, because the statement is correct and they accepted these solutions as proofs. Some of these students explicitly said that these "proofs" are only valid for isosceles triangles. In the case of the empirical solution, the interviewer asked whether measuring is a correct proof procedure in mathematics, which was denied by these students: "No, because one can measure something a hundred times and in the hundred-and-first case it does not work. One has to develop some formulas or something like this. But … I think the triangles she measured were always isosceles. Maybe there exists something else, but I know that this cannot happen" (Maria).

Another problem was the understanding of the meaning of "universal validity". For example, many students said that the correct proofs are correct, but they are only valid for isosceles triangles. Therefore, they are not universally valid. "It is a proof only for this triangle. For other triangles which are not isosceles it is not adequate" (Kristian).

4 Discussion

The results of the interview study indicate that multiple choice items rather tend to give surface information. Several students marked wrong answers in the test, though their ideas about the solutions were correct and vice versa. Moreover, in several cases the information about students' mistakes as measured in the multiple choice format did not adequately reflect the three aspects of the methodological knowledge. This is particularly the case for the empirical argumentation. As mentioned above, several students considered the empirical solution as a proof because of mistakes related to the second aspect (proof structure). They stated that the empirical solution is correct, because the assertion is true. Most of these students knew (and said so explicitly) that empirical arguments do not form a proof. The students who took part in our study were younger than the participants of other studies we reported about. Probably young students have more difficulties to understand these items.

Our interview study shows that all three aspects of the methodological knowledge are important when students judge proofs. It seems that the third aspect (chain of conclusions) is not problematic, since the correct proofs were mostly described as correct. However, in some cases it was not clear whether the students really understand each step in the proof. Problems with the first aspect (proof scheme), in particular, the preference of inductive arguments, are often fostered by the use of inductive argumentation in elementary school. Students have difficulties to bridge the gap between empirical argumentation and formal argumentation. This is confirmed by a study which shows that the problems of Taiwanese students are different: here, the transformation of incorrect or improper formal arguments to correct formal arguments can be observed (Lin, 2000).

All three aspects of the methodological knowledge have to be taught in the mathematics classroom. Students' difficulties with the proof structure indicate that this aspect is dealt with insufficiently. First results of a classroom video study, which was carried out recently in our research group, indicate that teachers regard the aspect of proof structure as less important in comparison to the other two aspects. When teaching proofs, most of the teachers pay too little attention to the proof structure or to the ideas of a proof. In general, the proof the teacher has in his mind determines the instruction. The students are supposed to follow this particular proof step by step. Rarely, they are encouraged to develop their own ideas.

When discussing the results of the previous section we have to take into account that the eleven students in our interview study had to solve only one geometry problem (with four solutions). Regarding these restrictions it is possible that the described students' problems are influenced by the context of this item. A follow-up study with different items is necessary to clarify this question.

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