# THE QUALITY OF STUDENTS' EXPLANATIONS ON A NON-STANDARD GEOMETRY ITEM

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*We report on the types of explanations that students gave for their answers to a non-standard geometry item. We suggest that these response-types form a partially ordered hierarchy on the basis of mathematical quality of explanation. Students were given the item in Year 8 and again one year later, and a comparison of the frequency of response-types in Years 8 and 9 suggest that many students evaluated the mathematical quality of the responses in an alternative way.*

## Introduction

The analysis presented here forms part of The Longitudinal Proof Project (Hoyles and Küchemann: http://www.ioe.ac.uk/proof; Hoyles and Küchemann, 2000), which is analysing students' learning trajectories in mathematical reasoning over time. Data are collected through annual surveying of high-attaining students from randomly selected schools within nine geographically diverse English regions. Initially 3000 Year 8 students (age 13) from 63 schools were tested in June 2000. The same students were tested again in June 2001 using a new test that included some questions from the previous test together with some new or slightly modified questions. Altogether 1984 students from 59 schools took both the Year 8 and the Year 9 test. The same students have again been tested in June 2002 with the similar aims of testing understandings and development. Each test comprised items in number/algebra and in geometry, some in open format and some multiple choice. The first step in the process of devising the items was to review the research literature in order to identify the major issues students are likely to face when learning to prove in each domain. Subsequent steps involved discussion with teachers and piloting in six schools. Following analysis item by item each year and longitudinally, the final stage of the research will be to draw together these analyses to suggest more general trends in development in a domain.

In geometry, we devised items for each annual test that set out to distinguish if students reasoned from a basis of perception or from geometrical properties, to find out if they were able to perform a series of angle calculations and to give reasons for each step, and to assess whether they could decide what was or was not an adequate proof of a simple conjecture. In this paper we focus on responses to one geometry item, G2b, in which students were asked to determine the area of an overlapping region and then to explain their answer. The item was part of each of the three annual tests so we are able to trace any changes in explanations over time. We report here on the Year 8 and 9 responses.

Pedemonte (2003), in her paper in this volume, describes how two groups of students working on a geometry task involving squares and triangles, come up with a conjecture concerning the areas of the triangles. The students have relevant geometric knowledge (eg, about congruence) and experience of argumentation and of writing proofs. The students justify their conjecture by means of a series of abductions (which one might represent as J→K, I→J, H→I, etc, where K is the conjecture)[1] and manage, more or less, to *re*-present the argumentation in the conventional form of a written proof (ie H→I, I→J, J→K). Our students were younger than Pedemonte's with less knowledge and experience. Another obvious difference between our studies is that we only asked for an explanation, not a proof, and few, if any, of our students represented their argumentation in proof form. On the other hand, many of the explanations can be interpreted as abductions, which fits with Reid's (2003) suggestion in this volume that abductive arguments are often used to explore and to explain. However, as we discuss later in this paper, the number of abductive steps, in particular connected steps, were usually fewer than exhibited by Pedemonte's students.

Douek and Pichat (2003), in their paper in this volume, report that with focussed and systematic teaching even very young children can learn to produce precise, carefully structured written accounts of familiar scientific/mathematical situations. At the same time, and perhaps closer to our students' experiences, Anderson et al (1997), in a study of naturally occurring arguments in 4th grade classrooms, found that the students' utterances were often vague and with no explicit conclusion, and that they were usually missing, or seemingly missing, explicit warrants to authorise conclusions.. They suggest that this is because students take the shared knowledge of the participants as given and not needing to be spelt out, and they go on to suggest that the underlying arguments are usually perfectly sound. Reid (1999), on the basis of observing grade 10 mathematics classes, suggests there are several modes of explaining, including non-explanations (where, for example, students refer to their own or the teacher's authority), explaining how, explaining why, explaining to someone else (spontaneously, or in response to a question) and explaining to oneself (in an attempt to come to a personal understanding).

Completing a written test, for researchers that the students do not know, is clearly different from the classroom activities considered by Anderson et al and by Reid. Nonetheless, it is possible that some of the factors that they identified are operating with the written test. In particular, students might assume, through the habits of everyday discourse or from a lack of familiarity with the conventions of mathematical argument, that some of the knowledge that they share with the researchers (whom they don't know, but who presumably know 'everything') does not need to be made explicit. In this regard, a particularly interesting feature of many students' responses to item G2b is that their explanations were vague, often just repeating the information that was given. This may in part also be due to a lack of familiarity with the thorny issue of how far 'back' one needs to go to justify a mathematical explanation and with the need to use 'transformational reasoning' (see for example Simon, 1996) to transmute the givens in the question into something more explicit. However, as we hope to show, some students also seem positively to value certain characteristics of such answers despite the obvious shortcomings when judged by standard mathematical criteria.

**Students' responses to G2b**

Item G2b was developed from a question used by Frant and Rabello (2000), and our Year 9 version is shown in Figure 1 (the Year 8 version was almost identical except for a slight difference in wording due to a difference in part a) of question G2).

We were attracted to the question for a number of reasons. First, it is non-standard[2] and therefore, rather than simply calling-up a known procedure to solve it, students would be more likely to consider the structure of the situation in some way. At the same time, it does not require a great deal of formal geometric knowledge, so that our students are unlikely to fail to find the required area through a simple lack of knowledge. Further, it is amenable to a dynamic approach (involving rotation) and we were curious to know how readily students would work in this way.

We were interested primarily in the nature of students' explanations for their answer, rather than the answer per se. Nonetheless, we were surprised by the high proportion of students giving the correct answer of 1/4 for the overlap: 86 % and 93 % of the total sample (N = 1984) in Years 8 and 9 respectively. However, when it came to students' explanations the situation was perhaps less impressive, - as well as being more complex, as we shall see.

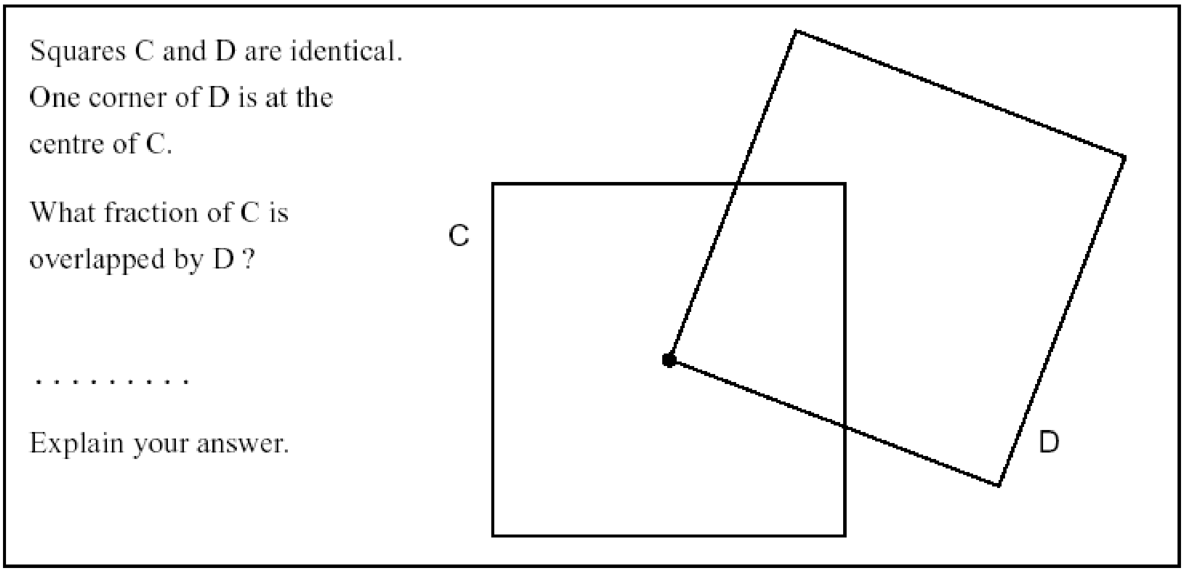



**Fig 1: Item G2b (Y9 version)**

After examining numerous students' scripts we came up with the coding scheme for students' explanations shown in Table 1 below.

A code with leading digit 1 (codes 11, 12 and 13) was given to students' arguments that appeared merely to be based on perception ('It looks like a quarter', or indeed, 'It's about one third'), or which arose from an attempt to measure (for example by drawing a grid and counting squares). We also included correct answers with no explanation under code 1.

A code with leading digit 2 (code 20) was given to the numerous explanations which, though not incorrect, seemed to consist of little or nothing more that a rehearsal of the givens (typically, 'It's a quarter because the corner is at the centre and is a right angle')[3].

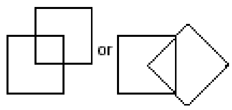
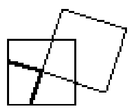
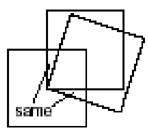
| code | |
|---|---|
| 11 | *Specific estimate, close but wrong*<br>Answer = 1/3 or 1/5 (or decimal equivalent) + any or no explanation. |
| 12<br><br>13 | *Correct value but no structural explanation*<br>Answer = $\frac{1}{4}$ + no explanation, or perception ("it looks like a quarter"), or spurious reason ("the overlapping sides are halved and half times half is a quarter").<br>Answer = $\frac{1}{4}$ + actual, valid measuring (eg draws grid and counts, or measures right angled triangles and calculates). |
| 20 | *Correct value but only implicit reasons*<br>Answer = $\frac{1}{4}$ + sensible but only partial explanation (if obviously *not* sensible, then code 12).<br>Could involve just one property ("corner is 90°") but might involve several properties, and/or valid operations ("90° is a quarter of 360°"; "You can divide the square into 4"); might include some reference to **turning** (but not as for code 31 or 32). |
| 31<br><br>32 | *Correct value; structural explanation involving rotation to salient position*<br>Answer = $\frac{1}{4}$ + refers to turning square D so that it is oriented as in one of these diagrams or draws one of the diagrams (eg turn it 'to the side' or 'to the bottom' or 'till it is parallel').<br><br><br>*Correct value; structural explanation involving partitioning or full turn*<br>Answer = $\frac{1}{4}$ + claims that "the overlap fits 4 times", by referring to turning square D through successive 90° turns, or to partitioning the square into 4 equal parts, as in the diagram; or draws diagram.<br> |
| 40 | *Correct value: explanation of 1/4 in general case using compensation*<br>Answer = $\frac{1}{4}$ + uses 'compensation' argument to explain **why** rotating from simple case (code 31) conserves the area of overlap ("on one side it is covering slightly more of the square and on the other the same amount less").<br><br> |
| 91<br>92<br>93 | *No response*<br>*Informative no response*<br>*Miscellaneous wrong response* (but not 1/3, 1/4, 1/5) |

**Table 1: Coding scheme for answers and explanations in G2b**

Arguments that we considered were more 'structural', ie based on geometrical properties, were given a leading digit of 3 (codes 31 and 32). A code 31 response involves rotating D until it has the same orientation as C so that the overlap becomes a square a quarter the size of C, or, less usually, rotating D until its sides are at 45° to the sides of C, so that the overlap is identical to one of the four equal regions that would be formed by drawing the diagonals of C. It could be argued that the students giving a code 31 response are merely presenting one case and claiming it as a proof (see Movshovitz-Hadar, 2002, for descriptions of the ‘because, for example’ phenomenon). However, we feel a code 31 response goes beyond a mere empirical argument, such as one based on measurement, since the size of the overlap has been determined on the basis of geometric properties, even if the approach is 'visual-contemplative' (Knipping, 2003) rather than truly analytic. A code 32 response involves partitioning square C into 4 equal parts congruent to the overlap. The partitioning might be seen in a 'static' way (by simply extending the two sides of D that go through the centre of C on the given diagram), or it can be viewed in terms of a succession of 90˚ rotations of square D (or, at least, of the given overlap).

Mathematically, a code 31 response is less complete than a code 32 response, since it does not in itself explain why the area of *all possible* overlaps is 1/4 (or, if the diagram is interpreted as representing a specific situation, why the *original* overlap is 1/4). However, in devising the coding scheme we had no evidence to assume that students choose a code 32 response, rather than a code 31, for this reason, and we therefore decided to include both kinds of response under the same broad code (ie code 3). From this cognitive point of view, we are still inclined to regard the two response-types as equivalent rather than ordered.

We used the code 40 for responses which would otherwise have been coded 31 but which included some kind of compensation argument to show that turning D does not change the area of the overlap. A minimal code 40 response might state that the overlap is 1/4 when D is turned so that it is 'parallel' to C, with the additional explanation that as D turns, 'the overlap gained is the same as the overlap lost'. More typically, students would draw D in its original and 'parallel' positions and state that the triangular region that was newly overlapped was the same as the triangular region that was no longer overlapped. Sometimes students gave, say, a code 20 or 31 response but included in their explanation the claim that the overlap was *always* 1/4 but without any further statement to justify this claim. We noted such cases by adding the letter A (for 'Always') to the response code (eg, 20A or 31A).

The claim that the overlap is always 1/4, and the justification for this in terms of compensation, can be viewed as a series of abductions. Where the compensation argument refers to the two small triangles mentioned above, a third abduction would involve some kind of justification for the claim that they are the same, perhaps by comparing some of the corresponding sides and angles (ie a congruence argument). It was extremely rare for students to take this third step, and we did not include it in our coding scheme.

Regarding the *mathematical quality* of students' responses, we would argue that the codes in Table 1 are more or less hierarchical. Thus for example, a code 20 response is generally better than a code 1 response (as it is concerned with mathematical properties, even though it might not

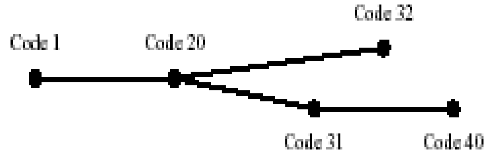



**Fig. 2: The response codes ordered by 'mathematical quality'**

be saying anything 'new'), but a code 20 response is less informative or 'revealing' than a code 3 response. Similarly, a code 40 response is better than a code 31 response, as it goes a step 'further back' or is more fine-grained by explaining why the area of the overlap is conserved under a rotation. Conservation of area is not relevant for code 32, and it might be better to think of this code as being on a different branch of the hierarchy than codes 31 and 40 - thus giving a partially ordered set as illustrated in Figure 2, right.

Table 2 shows the frequency distribution of the codes for the students' responses in Year 8. As mentioned earlier, the vast majority of students (86 % in Year 8) could find the correct value for the size of the overlap. However, as can be seen from Table 2, only about half of the students who gave a correct value (and about 43 % of the total sample) supported this with an explicit structural reason (codes 31, 32 and 40), whilst a substantial minority (28 % of the total sample) gave code 20 responses.

| **Code** | **Code description** | **Number** | **Percent** |
|---|---|---|---|
| code 11 | Close but wrong estimate | 90 | **5** |
| code 12 | Correct value; no structural explanation | 297 | **15** |
| code 13 | Correct value; valid measurement | 15 | **1** |
| code 20 | Correct value; only implicit reasons | 548 | **28** |
| code 31 | Correct value; rotation to salient position | 519 | **26** |
| code 32 | Correct value; partition or repeated rotation | 236 | **12** |
| code 40 | Correct value; compensation | 94 | **5** |
| code 9 | No correct value; miscellaneous | 185 | **9** |

**Table 2: G2b Year 8 code frequencies (N = 1984)**

Table 3 compares the Year 8 frequencies with those of Year 9. We found these frequencies quite puzzling at first. In general, students made quite clear and substantial progress from Year 8 to Year 9 on most of the items on the proof test. However, progress on item G2b seems to be quite modest: as can be seen, there are slightly fewer 'perceptual' (code 1) or miscellaneous incorrect (code 9) responses in Year 9. However, the clearest sign of 'progress', if that is what it can be called, is the increase in code 20 responses, from 28 % to 35 %.

| **Code** | **Y8 %** | **Y9 %** |
|---|---|---|
| code 11 | **5** | **2** |
| c12, c13 | **16** | **15** |
| code 20 | **28** | **35** |
| code 31 | **26** | **31** |
| code 32 | **12** | **8** |
| code 40 | **5** | **5** |
| code 9 | **9** | **5** |

**Table 3: G2b Y8 and Y9 code frequencies (N = 1984)**

Not only are these 'gains' small, there is a high degree of inconsistency in students' responses, as can be seen from the adjacent two-way table (Table 4). Thus, for example, only 14 of the 94 students who gave a code 40 response in Year 8 gave a code 40 response in Year 9.

| G2b | Y9 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Code** | c11 | 12,13 | c20 | c31 | c32 | c40 | c9 | Total |
| Y8 | c11 | 6 | 19 | 20 | 22 | 11 | 2 | 10 | 90 |
| | 12,13 | 6 | 65 | 103 | 79 | 27 | 15 | 17 | 312 |
| | c20 | 5 | 70 | 270 | 134 | 33 | 18 | 18 | 548 |
| | c31 | 3 | 52 | 162 | 240 | 22 | 25 | 15 | 519 |
| | c32 | 5 | 36 | 68 | 63 | 43 | 11 | 10 | 236 |
| | c40 | | 13 | 22 | 37 | 6 | 14 | 2 | 94 |
| | c9 | 5 | 43 | 45 | 42 | 14 | 9 | 27 | 185 |
| | Total | 30 | 298 | 690 | 617 | 156 | 94 | 99 | 1984 |

**Table 4: G2b Y8 by Y9 code frequencies (N = 1984)**

If one ignores any code 9 responses, but assumes that the other codes are partially ordered as illustrated in Figure 2, then, from Table 4, the numbers of students who progress and regress are 508 (26 %) and 479 (24 %) respectively. The net progress is then just 2 % !

This propensity to switch strategies from one year to the next, or indeed to offer more than one kind of explanation on a given occasion, which was not uncommon, has echoes of the source-like argumentation structure described by Knipping (2003) in this volume. The seeming dramatic lack of progress on item G2b has made us question whether the mathematical hierarchy that we have assigned to the codes, chimes with the way the students see these different kinds of responses, in particular with respect to codes 20 and 3.

As part of our case studies of certain schools in our sample, we have interviewed individual students about their Year 8 and 9 (and sometimes also Year 10) written responses to G2b. In these interviews, we looked particularly at two response patterns (both of which are quite common, as can be determined from Table 4):

Pattern A, where students gave a code 20 response in one or more years and did not give any code 3 (or 4) responses;

Pattern B, where students switched from a code 20 response to a code 3 (or 4) response, or vice versa.

In the case of Pattern A, we were interested in whether we could get the students to elaborate on their code 20 responses, ie to explain *why* the givens (a 90˚ corner at the centre of the square) mean that the area of overlap is a quarter. In other words, could they shift to a code 3 or 4 response? Some students seemed unable to do this, which fits our view that the codes are mathematically hierarchical. On the other hand, others seemed able to move to a code 3 (or 4) response quite easily. In such cases, our interested shifted to finding out which kind of response they preferred. This was also our interest for Pattern B. However, when we tried to probe students' views on this, their responses were often not very revealing. In part this can be explained by the fact that our students were not very experienced in providing explanations (as opposed to answers) - at least in geometry - and thus had no clear models to go by, in general, and in our particular test/interview situation. Nonetheless, having produced both kinds of response, we were surprised that the students often had difficulty describing their characteristics. We were even more surprised that, with both kinds of response in front of them, students did not always, and immediately, express a clear preference for code 3 over code 20 responses.

From a mathematical point of view, a typical code 20 response is unsatisfactory, because it does not reveal anything: it simply reiterates the givens and as such is essentially circular, as it is a condensed version of 'If the corner is a right angle and is at the centre, then it's a quarter because the corner is a right angle and is at the centre'. This shortcoming may seem glaringly obvious to an experienced mathematician. On the other hand, it could be argued that this is just an example (albeit an extreme one!) of the difficult and ever present issue of how far one need go in unpacking mathematical properties in order to prove a statement. (Thus for example, as we discussed earlier with regard to the code 40 compensation argument described in Table 1, is it enough to state that the two triangular regions in the diagram are the same, or does one need to justify this?). It is also worth pointing out that in this particular item the givens do play a decisive role. The area of the overlap would *not* be constant if C and D were rectangles, say, rather than squares, or if the corner of D was not at the centre of C.

The deductive steps in a mathematical argument are tautological (Toulmin, 1958), in the sense that the relationships used to construct the argument follow from the givens; as such, though the steps may reveal what is hidden, essentially they say nothing new. No wonder students have difficulty deciding what depth of explanation is required, even if they can apply the transformational reasoning (Simon, 1996) necessary to reveal fruitful relationships. In their paper in this volume, Heinze and Reiss (2003) discuss the methodological knowledge involved in constructing proofs. In the light of the difficulties described above, it is quite possible that students who are just beginning to come to grips with this knowledge (for example, with the distinction between empirical and conceptual proofs), may find code 20 responses appealing because they seem general and concerned with mathematical properties. Moreover, a basic code 31 response, say, can seem quite specific (in that it is concerned with the overlap when the squares are in a particular orientation); as mentioned previously, it might appear as more of a demonstration ('Look, here it is clearly a quarter') than a proper, structural, explanation.

The argument here is that some students, at least, are content to give a code 20 response rather than a code 3 or 4 response, not because they are mathematically unable to give a 'higher level' response, but because they value the characteristics of code 20 responses described above. To probe this further, we compared students' responses to item G2b with their total score[4] on the national Key Stage 3 mathematics tests that English school students are required, by statute, to take towards the end of Year 9. Table 5, below, shows the average KS3 score for those groups of students giving particular response-codes in Years 8 and 9. (Students for whom we do not have an appropriate KS3 score have been omitted, which has reduced the sample slightly, from 1984 to 1901.) As can be seen from the table, the pattern of average KS3 scores is quite similar for Years 8 and 9. In particular, students who gave code 11 and code 9 responses have average KS3 scores well below the sample average, while students who gave code 20, code 31 and code 4 responses have very similar (and above average) average KS3 scores -although the average for code 32 is markedly less than for code 31 and in Year 9 it is less than the average for the sample as a whole. Thus the data support the

| | **Year 8** | | **Year 9** | |
|---|---|---|---|---|
| **Code** | **No. of students** | **Average KS3 score** | **No. of students** | **Average KS3 score** |
| code 11 | 84 | **66.3** | 27 | **69.8** |
| c12, c13 | 296 | **74.8** | 286 | **76.3** |
| code 20 | 539 | **83.0** | 670 | **81.9** |
| code 31 | 494 | **83.8** | 589 | **81.6** |
| code 32 | 224 | **80.3** | 148 | **76.2** |
| code 40 | 91 | **84.5** | 87 | **82.7** |
| code 9 | 173 | **69.9** | 94 | **68.7** |
| Total | 1901 | **79.7** | 1901 | **79.7** |

**Table 5: G2b Y8 and Y9 average KS3 scores for different codes (N = 1901)**

conjecture that students giving code 20 responses are not necessarily mathematical less able (as measured by the KS3 test score) than those giving code 3 responses.

| | Year 9 | |
|---|---|---|
| **Code** | **No of students** | **Average KS3 score** |
| code 20 | 446 | **80.2** |
| code 20A | 224 | **85.4** |
| code 31 | 497 | **79.3** |
| code 31A | 92 | **94.2** |

**Table 6: G2b Year 9 average KS3 scores for some 'Always' codes**

A feature of our coding scheme briefly mentioned earlier is that we also noted whether, in their explanations, students explicitly stated that the overlapping area would *always* be a quarter. We did this by adding the letter A (for 'Always') to the code. The code 20A and 31A frequencies turned out to be quite high, especially in Year 9, and it is therefore interesting to look at the average KS3 scores for these codes, which are shown (just for Year 9) in Table 6, right.

As can be seen, the A codes have (markedly) higher average KS3 scores than the corresponding non-A codes, which fits the notion that higher attaining students are more concerned with generality.

## Conclusion

The students in our sample, though relatively high attaining, are unlikely to have had much experience of providing mathematical explanations in geometry, especially in written form. This lack of experience can manifest itself in various ways, depending on the item. For example, in a question involving a three step calculation to find the size of an angle (Küchemann and Hoyles, 2002), most of our students could evaluate the angle successfully, but when asked to explain each step, rather than give a mathematical justification (such as 'The angle sum of a triangle is 180˚'), many students gave procedural explanations (such as 'I took 40˚ from 180˚ to find the remaining angles'). In the case of item G2b, it is perhaps not surprising that students' explanations were often less explicit than would conventionally be deemed desirable and that progress was therefore not very evident. However, we were surprised by the substantial number of students whose explanation were not only vague but essentially circular, and in particular by the finding that the frequency of such explanations increased rather than decreased from Year 8 to Year 9. Further consideration of the data lead us to conclude that some students may have chosen to give such explanations, not because they did not have access to more structural explanations, but because they valued certain characteristics of these explanations, namely their generality and reference to mathematical properties.

It is interesting to consider how one might help students to see the need to go beyond the givens when constructing a mathematical argument, especially as the stopping point in this process is essentially arbitrary. One heuristic which might help to bring out the 'consequences' of the givens is to consider how one can transform a given problem in such a way that the result still holds (for example, in the case of G2b, by increasing the size of square D) or so that it no longer holds (for example, by changing C and D into rectangles).

In general, we suggest that presenting students with unfamiliar questions such as G2b can provide a rich context for classroom discussion as to the norms expected in a mathematical argument. What are the consequences of the givens, how far should reasons go beyond perception and how far should an argument be elaborated? Such a situation has a strong didactical purpose, as it involves a result which students know but which they need to explain and justify.

**Acknowledgement**

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**Notes**

[1] Pedemonte represents the argumentation in a more detailed way, using notions of claim, data and warrant taken from Toulmin (1958).

[2] Strangely, a very similar task has since found its way into the government guidelines for teaching lower secondary school mathematics (DfEE, 2001).

[3] Most code 20 responses simply repeated some or all of the givens. However, the code included responses where students did say something new, for example by referring to turning or partitioning, but in too vague a way to be classed as code 31 or code 32 responses.

[4] Most students in our sample took either the Level 5-7 KS3 tests or the Level 6-8 KS3 tests. We used a conversion table kindly provided by the QCA to convert students' total score on the 5-7 tests to an equivalent total score on the 6-8 tests. A small minority of students took the Level 4-6 tests, for which we did not have a conversion table and so these students were omitted from the KS3 score analysis.

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