

THEMATIC WORKING GROUP 4**ARGUMENTATION AND PROOF**Rudolf vom Hofe

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The role and importance assigned to argumentation and proof in the last decade has led to an enormous variety of approaches in research. Historical and epistemological issues, related to the nature of mathematical argumentation and proof and its functions in mathematics, represent one focus of this wide-ranging research. Focus on mathematical aspects, concerning the didactical transposition of mathematical proof patterns into classrooms, is another established approach, which sometimes makes use of empirical research. Most empirical research focuses on cognitive aspects, concerning students' processes of production of conjectures and construction of proofs. Other research addresses implications for the design of curricula, sometimes based on the analysis of students' thinking in arguing and proving and concerns about didactical transposition. Recent empirical research is now looking at proof teaching in classroom contexts and addresses the question of curricular implications based on the results of these studies. The social-cultural aspects revealed in these studies motivate a more recent branch of research which is offering new insights. Comparative studies, trying to come to a better understanding of cultural differences in student's arguing and in the teaching of proof, especially as they relate to learning in different cultural contexts, can be seen as part of this new branch of research.

Papers collected in this section on "Argumentation and Proof" represent this diversity. Authors raise issues and questions about argumentation and proof from a wide range of positions and theoretical perspectives. These differences are reflected in the focus researchers take in their approach, as well in the methodological choices they make. This leads not only to different perspectives, but also to different terminology when we are talking about phenomena. Sometimes differences are not immediately clear, as we use the same words, even though we assign different meanings to these words. On the other hand, different categories that we build from empirical research in order to describe students' processes, understandings and needs are rarely discussed conceptually across the research field. Conceptual and terminological work is helpful in that it allows us to progress as a community operating with a wide range of research approaches. Differences in interests,

perspectives and terminology and their relevance become obvious when looking at the same data together. The experience we had in our working group, analysing the same video-data from different research perspectives, turned out to be fruitful and rich. It made the diversity of our approaches evident and valued this diversity at the same time. In keeping with this insight, one part of this introduction will give an overview of the contributions of this group, and another important part will honour the diversity that became so clear.

The papers deal with issues and questions in four main topics. From different positions and theoretical perspectives, the authors consider: (A) student's competence and experiences with proof (including argumentations, concepts of proof, proving processes etc.) (B) forms and uses of logical and mathematical reasoning and their relevance for understanding students' reasoning processes, (C) argumentation and proof in class – comparing different classroom contexts and (D) the role of mathematical problems and students' epistemological obstacles in proving.

In the first section (A) four papers discuss cognitive and epistemological aspects, concerning the processes of production of conjectures and construction of proofs. Nordstroem reminds us that many students lack the experiences with proofs that will help them become successful in mathematics in their later studies. Heinze and Reiss find as well that students at the end of upper secondary level have deficits in methodological knowledge about proof which are part of their problems with judging proofs. They describe three aspects of methodological knowledge about proof: proof scheme, proof structure, and logical chain, that they consider as important components of proof competence. Their empirical data support their claim that all three aspects of methodological knowledge are important when students validate proofs. Küchemann and Hoyles find in their long-term empirical study that there is a tension for students in retaining their intuitive sense of the mathematical problem given and producing deductive explanations fitting social norms. The authors come to the conclusion that teaching geometrical reasoning and giving students opportunities to progress in their reasoning requires not only clarifying effective heuristics, but also finding out ways to revisit and teach proof over time. Here more research has to be done. Misailidou and Williams point out the important role that visual elements and cultural contexts can play in argumentations. The authors conclude from their research that in order to foster students' argumentation competencies, the teacher requires not only content-specific knowledge, but a great richness of expertise that is local to the task and context of teaching, rather than general strategies. This points out the important role of teachers in ensuring that appropriate cultural tools are made available for students. Scimone describes the various difficulties students have in trying to prove a conjecture where it is not clear what approach will be fruitful. He takes a historical viewpoint in order to investigate mental representations of students in problem solving.

In the second part (B) logical, historical and epistemological aspects, related to the nature of mathematical argumentation and proof, and terminological aspects, based

on the differences between explanation, justification, argumentation and proof in mathematics education are discussed and used to describe students' activities and products. Durand-Guerrier uses the model-theoretic approach introduced by Tarski and distinguishes three dimensions: syntax (the linguistic form), semantic (the reference objects), pragmatic (the context, and the subject's knowledge in the situation), for a didactic analysis of mathematical reasoning and proof. She considers these distinctions as important in order to foster argumentation and the proving processes of students. Durand-Guerrier argues that the model-theoretic approach calls for continuity between argumentation and proof, in contrast with the discontinuity seen by researchers working in a cognitive approach (e.g., Duval). Pedemonte, although taking a cognitive approach, explicitly takes issue with Duval's assertion that deductive reasoning is not like argumentation, that there is a cognitive discontinuity between proof and argumentation. Pedemonte finds in her research both discontinuities and continuities in students' argumentations as they come to a conjecture and in the proofs they produce subsequently. She describes the transition from abduction to deduction in proving processes and illustrates that a gap between an abductive argumentation and a deductive proof is possible as well as a continuity between the two. Reid differentiates between different forms of abductive reasoning based on Peirce's early and late work. In analysing students' reasoning within a framework using these distinctions he demonstrates the value and relevance of these distinctive categories for a better understanding of students' reasoning processes. Yevdokimov questions the nature of proof and develops a typology of proofs. In particular he stresses the importance of intuition in mathematical reasoning processes and puts forward that intuition and proof are inseparable.

In part (C) different classroom contexts are discussed in which constructions of proofs and arguments take place. Comparing these different contexts helps to come to a better understanding of different teaching contexts of proof, especially as they relate to learning in different social and cultural contexts. Douek describes early argumentations of young (first grade) students learning argumentation in the process of writing. She shows how students' involvement in experimental situations rich in concrete experiences gives them the opportunity to develop important skills related to mathematical argumentation. The implicit assumption of Douek's work, that proof and proving processes are strongly linked to the discourse culture of the class, is made explicit by the research of Knipping. She identifies various argumentation structures and discourse cultures in French and German classes. Although the same mathematical topic is discussed in class, and the proofs seem to be close from a mathematical point of view, argumentation structures in the classes' discourses differ substantially, on a local and global level. She argues that these differences correspond to different functions of proof that are recognized in the class' culture.

In the last section (D) reality-related thinking, basic ideas and epistemological obstacles in argumentation and proof are discussed. Here mathematical and educational aspects, concerning the didactical transposition of mathematical proof into classroom, and implications for the design of curricula are discussed. Blum

argues that reality-related applications provide interesting contexts for proof and enable pupils to gain non-formal insights. Vom Hofe offers descriptions of different argumentations based on conceptual understanding and stresses the importance of a genetic concept development. The two papers elaborate the role of reality-related thinking for argumentation and proof. In particular, the authors are interested in the relation between modelling processes and proving processes. They explore how basic ideas, individual concept images and epistemological obstacles are relevant in argumentation and proving processes and situations. Detailed case studies and long term empirical studies are envisaged to explore the development of student's concept images and their role in students' argumentation and proving skills. Analyses of the empirical data are expected to give theoretical insights into the relation between modelling and proving processes. The authors expect these studies to be helpful in identifying levels in reality-related arguing and proving.

Another element in the work of the group, not represented in the papers, was the video session. As noted above, this experience involved collective analyses of video data and transcripts from different theoretical perspectives, which revealed and acknowledged the diversity in our approaches.

The video segment showed three 14 year old students working on the problem of determining the number of handshakes that occur when n people shake hands. In the case of 6 people they had found the answer by adding $1+2+3+4+5$. Faced with $n = 28$ they were attempting to find a formula. The process of hypothesising formulae and verification of the formulae was the focus of the segment. The scene can be characterised as an unguided problem solving situation.

The working group participants broke into small groups to discuss the students' mathematical activity, guided by a set of questions based on theoretical and methodological elements of the papers presented. Not surprisingly, when the small groups reported back they had chosen very different foci for their discussions and raised different questions for further discussion.

The first group discussed the benefit of the three categories *syntax*, *semantic* and *pragmatic* for analyses of students' argumentations given in the video data. In particular, the categories were considered for describing differences in students' argumentations and for a better understanding why students in the problem solving process did not come to one collective argumentation. Further, the question arose in how far the use of figures in the problem solving process could be described in these categories. In particular, the group tried to find out in how far these categories would be helpful to describe the motivation for shifts between different figures and the motivation for shifts between figures and arguments.

The second group discussed the transition from abduction to deduction as described in the paper by Pedemonte. The process of attempting to analyse the reasoning of the three students involved in solving a problem was contrasted with the more structured mathematical activity of the students in Pedemonte's paper, who were looking for

geometry proofs. This allowed the members of the group to come to a deeper understanding of the use of the category of abductive reasoning in both Pedemonte's work and in Reid's paper.

The third group discussed the characterisation of proof given by Heinze and Reiss. The focus of their paper is that three different aspects of *methodological knowledge* (considered as an important component of proof competence) may be distinguished: *proof scheme*, *proof structure*, and *logical chain*. In particular, these three aspects were used to analyse the students' argumentations given in the video data. The difficulties met in this analysis allowed the development of a critical discussion about the terms and definitions referring to proof in the paper.

The fourth group analysed the video transcripts from the perspective of the process of mathematical modelling and translating between reality and mathematics. In particular, the role of '*Grundvorstellungen*' (mental models) was considered. These can be seen as mental links between the real world and mathematics. From this point of view, reality related proving can be described as the reverse process of mathematical modelling: While the usual way of modelling takes a real world situation and transforms it into a mathematical situation, reality related proving goes the other way, i.e., a mathematical context is transposed into a corresponding real life situation which forms a new basis for argumentation.

In addition to the different observations reported by the groups, the experience of examining the same data raised some important methodological questions. We discussed questions such as:

- To what extent do results of our empirical studies reflect (or not) the teaching and learning experience the students went through?
- How do our research contexts (multiple-choice tests, tests based on open tasks, interviews, classroom observations, etc.) affect students' performances, answers and explanations?
- What are possible implications for our methodology and the applicability of our results in teaching or teacher training?

It is difficult to specify conclusions for a group that spend considerable time exploring the value of diversity. What is presented here reflects themes that emerged, rather than points of unanimous agreement.

In our discussions it became clear that there is a need to specify the meaning of the terminology we are using to describe types of argumentation and proof, but equally importantly there is a need to understand better their inter-relations and relevance in the context of learning. Furthermore, the processes by which types are transformed, from private to public argumentations, in varied contexts, through teaching, etc. need to be a focus for research.

Our discussions led us to consider several questions for further research:

How can we explain the differences in proving that are observed in different contexts?

How can we deepen our understanding of the relationship between argumentation and proof?

How can we address the methodological and theoretical challenges we face?

What are the implications of this research for school practice and how can the challenges in school practice be addressed by research.

The reader will be confronted with a wide range of positions and perspectives in the following papers. We hope that this introduction helps both to establish this diversity and to reveal connections between the papers. The heterogeneous research foci represented here and the complexity of the outcomes of this research need further analyses. These analyses cannot be done here, but questions and issues that emerged from our working group show directions for future research. Diversity is a richness and challenge at the same time; therefore, we very much hope that the culture of future conferences continues to support the diversity we found at CERME 3.

[List of contributions](#)

[List of Thematic Groups](#)