CHILDREN'S UNDERSTANDING OF FAIR GAMES

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This paper analyses the responses given by children from two samples (n=320, n=147) to two test items concerning fairness in a game of chance. We study the influence of age and mathematical ability on the percentage of correct responses. Interviews with a small sample of pupils serve to describe children's conceptions of fair games.

1 BACKGROUND

The concept of probability may be developed through games and experiments with dice, coins or spinners, which help children in acquiring concepts such as chance, independence and mutually exclusive events (Amir, 1998). Games of chance are one major context where children meet random situations, becoming aware of their unpredictability and realising the need for probabilistic estimates. These games form part of children's culture outside of school, and, as Peard (1990) showed, help children acquire probabilistic knowledge, even before any formal instruction on the topic.

The study of children's conceptions about fair games has recently increased. Watson and Collis (1994) found that many children thought that some numbers were more likely to appear than others were, even in fair die. These children showed anthropomorphic conceptions about probability or were only guided by the physical features of random generators to decide about their fairness. Some of them also realised the need to resort to experimentation to decide about the fairness of a die. Moreover, results of Lidster at al. (1995) support the view that children begin to develop the notion of fairness before starting school.

Vahey at al. (1997) examined the probabilistic reasoning used by middle-school students with a technology-mediated inquiry environment that was developed to engage students in analysing the fairness of games of chance. Their research demonstrates that students employ productive probabilistic reasoning when participating in this task and that commonly reported heuristics such as representativeness do not fully describe students' reasoning about games of chance.

In a cross-cultural study, Vidakovic, Berenson and Brandsma (1998) examined students' beliefs about fairness with 16 eighth grade students. The study revealed that students have a wide spectrum of intuitions and ideas about fairness involving chance, probability and sample space. Amir (1998), Kauffman & Bolite (1998) and Maher (1998) report parts of a study carried out in Israel, Brazil and the USA. Amir suggested some children (5^{th} and 6^{th} grades) believe the first to throw the die has an advantage and better chances of winning and that the player who throws the dice may affect the result in his favour. In an interview with two fourth grade students, Kauffman & Bolite suggested children were able to recognise that the game was unfair if one player had more chance of winning. They were also able to change the rules to equate the probabilities of winning for both players. Maher reached similar conclusions and, moreover, found that social influences are essential to student learning about statistical ideas.

Fairness may be established in one of two ways: either if all the players have the same probability of winning and obtain the same amount of money, or by balancing out the expectations when players have unequal probabilities. The abovementioned authors never ask the children to balance out the players' expectations since they are only interested in finding out whether children are able to perceive fairness as a balance of winning probabilities. Here, we try to extend these findings by taking into account how children apply the idea of expected value to determine the game's fairness. For each player, the product of the prize and his probability of winning that prize gives the expected value.

Scholttmann & Anderson (1994) studied 5 to 10 year-olds' intuitions about expected values using games with one or two prizes. However, they did not apply this idea to determine children's conceptions of fairness in a game of chance. They conclude that even very young children hold correct intuitions of expected values and consider both the probability of winning and the prize value to arrive at decisions. However, both in assigning probabilities and in relating the prize to the probability of winning, children often use additive strategies, which are unsuitable for estimating expectations.

2 EXPERIMENTAL STUDY

We analyse 10 to 14 year-olds' conceptions about fair games, to complement our previous studies into children's beliefs about probability and their influence on probability assignment (Batanero & Cañizares, 1998; Batanero, Serrano, & Garfield, 1996; Cañizares et al., 1997). The results in this study were obtained by analysing written responses to two items given by two samples of pupils, aged from 10 to 14 (n = 320 and n = 147). We study the percentages of correct answers and the arguments provided by the pupils to justify their answers. Finally, a number of interviews were conducted with a sub-sample of pupils; these serve to describe different conceptions of fair games. Below, we reproduce the two items on the fairness of a game, taken from Fischbein and Gazit (1984) and Green (1982):

ITEM 1. Eduardo has 10 white marbles and 20 black ones in his box. Luis has 30 white marbles and 60 black ones in his box. They play a game of chance. The winner is the child who pulls out a white marble first. If they each take out a white marble simultaneously, no one wins and the game has to continue. Eduardo claims that the game is not fair because in Luis' box there are more white marbles than in his box. What is your opinion about this?

ITEM 2. María and Esteban play a dice game. María wins 1 peseta if the dice comes up 2, 3, 4, 5 or 6. If the dice comes up 1, Esteban wins some money. How much should Esteban win when he throws a 1 if the game is to be fair? Answer_________ Why?

2.1 Global Results

In item 1, a high percentage of pupils considered the game to be fair, although not all reached this conclusion through correct reasoning (see Table 1). Amongst the pupils' strategies that we observed, most compared the number of favourable cases, which led them to believe that the player with more white marbles had an advantage; only a quarter of the children used a relevant strategy (correspondence).

Table 1. Percentage of strategies in Item 1

	Fair game		Unfair game		Incomplete
	Correspondence*	Other strategies	Favourable cases	Others	
Sample 1	32.9	31.5	32.9	2.7	
Sample 2	26.6	9.1	42.7	9.8	11.9

* Correct response

In general, pupils believed that fair play is synonymous with equiprobable outcomes. Therefore, for most pupils the difficulty in this item arises not from judging whether the game is fair, but from establishing whether or not there is equiprobability.

Over half of the responses to item 2 were correct (see Table 2). 46% supported the correct response by quantifying the two opponents' possibilities, as in Ricardo' response (12 years;1 month): *"María has 5 more chances, therefore, I think it is fair to give 5 pts to Esteban"*. 19.6% of the arguments admitted Maria's advantage, though they did not quantify it explicitly.

These arguments justified the correct response, or any amount of money greater than 1 pts. Thus, Ginés (11 years and 3 months) answered: *"Esteban should win 6 pts, because he has less chance",* and Triana (11 years and 3 months) used the same argument to justify another response: *"Esteban should win 2 pts because otherwise it would not be a fair game. He must win more money because María has more chance of winning"*. Carlos (12 years and 4 months) also used the same argument, but this time associated to a correct response: "*Esteban must win 5 pts, because María has greater possibilities, and she is more likely to win, therefore, should Esteban win, he ought to win more money"*.

The second most common response was to assign the same prize (1 pts.) to each player, regardless of the probabilities of winning.

Table 2. Percentage of responses in item 2

	1 pts	2, 3, 4 pts	5 pts*	6 pts	Other	Blank
Sample 1	17.7	7.0	51.4	10.8	9.2	4.4
Sample 2	11.9	9.1	58.0	9.1	7.0	4.9

* Correct response

2.2 Influence of age

In general, the correctness of the answers improved with the age of the respondent. In item 2, the most frequent incorrect answer in younger children is to balance the two players' odds and assign them the same amount of money (1pta.), ignoring the unequal probabilities of the events involved. We should conclude that, although most children are conscious of the fact that payment can balance unequal odds, some younger children are unable to co-ordinate the different variables in the problem. Instead, they compare only one variable in the events involved, either the probabilities or the prize assigned to each player, but they do not consider both at once.

2.3 Influence of mathematical ability

To study the influence of this variable, we assigned each pupil a *mathematical level*, using a score provided by his or her teacher. This was based on the child's performances in mathematics during the previous school year. This score takes three values: *high, middle* and *low mathematical ability*.

In both items, as mathematical ability increased, we observed an increase in the proportion of correct answers as well as in the proportion of quantitative justifications. In item 2, the percentage of pupils giving a quantitative justification for their correct response was 34.9% for low mathematical ability, 46.6% for middle and 57.1% for high mathematical ability.

2.4 Children’s reasoning about the fairness of a game

Finally, we found a variety of interpretations for the concept of fairness, which suggests that it might be appropriate to include fairness in the teaching of probability. We carried out interviews with two pupils from each age group in the second sample; these pupils were selected according to the level of proportional reasoning shown in their responses, following Noelting's classification (1980). Thus we classified the children's conceptions into the following categories.

2.4.1 Pupils who do not differentiate between equiprobable and non-equiprobable events, due to equiprobability bias (Lecoutre, 1992).

Carolina (13 years and 7 months, level IA) considers games to be fair when both players have the same chance of winning. However, because she has difficulty in establishing whether two compound events are equiprobable, she considers both games to be fair until the interviewer questions her. When there is no equiprobability, and different payments are established, she finds a certain balance in the winnings, though she considers the game to be unfair, as shown below:

I: What do you think we mean when we say that a game is fair?

C: That both players have the same chance of winning.

I: Do María and Esteban have the same chance of winning this game?

C: I think they do. Well, one of them has more chance, but, I think so. The other one might win as well.

I: Yes, but, María has five numbers to win and Esteban only one. Do they have the same chance?

C: Not, María has more chance.

I: Do you believe that we could change Esteban's prize to make the game fair?

C: Well, it is not fair, because they still do not have the same chance. Esteban wins more money, when he wins, but the possibilities are not the same.

Alejandro (10 years and 5 months, level IA) is a pupil with equiprobability bias, who does not consider the possibilities in item 2 to be unequal. His idea of fairness is associated with playing with the same elements (the same cards, the same balls...):

I: What do you think a fair game is?

A: Oh... you should have the same amount of marbles (referring to item 1). If one of the players has 10 white and 20 black marbles, then the other should have 10 white and 20 black marbles.

I: I'll tell you another game, and you tell me whether it is fair or not: With a pack of cards, we pick a card out without looking. If the card is a heart, you win. If it is an ace, I win. Do you believe that this is fair?

A: No, because both of us should have to pick the same card.

2.4.2 Pupils without equiprobability bias.

We found four different types of reasoning as regards the fairness of a game.

In item 1, José Antonio (13 years and 3 months, level IIB) does not recognise the boxes as having equiprobability. He has difficulty with proportional reasoning and uses additive comparisons to solve the item. He distinguishes between "equal probability" and "equal difficulty" for deciding whether a game is fair. This distinction makes him say that, even when one player has more chance than the other does, the game is fair:

I: What do you think we mean when we say that a game is fair?

J.A: That the two players have the same chance of winning.

I: Then, is this game fair or not? (Item 1)

J.A: Yes, it is.

I: Then, the two players have the same possibilities...

J.A: Yes, but one (Luis) has more difficulty than the other (Eduardo). Difficulty is not the same as possibility. It is fair, because they both have more or less the same chance, but Luis has more difficulty because he has got much more black marbles than white *marbles. Eduardo has a difference of 10 and Luis has a difference of 30. So it is harder for Luis to win.*

I: So, if it is more difficult for Luis to win than Eduardo, how can the game be fair?

J.A: Well, it seems fair to me.

In more familiar contexts, such as dice and cards, Jose Antonio is capable of determining the equiprobability, or lack of it, for two compound events. He does not correctly assign the payment to make the game fair, since he does not understand the inverse proportion between the favourable cases and the prize.

I: I'll give you another game: With a pack of cards, we are going to play with the following rules: We draw out a card. If it is a heart, you win 1 pts. and if it is a different card, I win 1 pts. Is this fair?

J.A: No, because there are more cards of other kinds, and so it is easier for you to win.

I: Then, how would you change the rules to make the game fair?

J.A: Since I must draw out a heart, I should win more money than you do.

I: How much more?

J.A: I do not know ... four pts., for example, or more. Just more than you.

I: But... tell me how much.

J.A.: Well... do the eights, nines and tens count?

I: No. Spanish cards only have up to seven, and then the figures...

J.A: Then, I would win thirty (this is the number of outcomes for the teacher, whilst the number of outcomes for the pupil is 10).

Rafael (12 years and 9 months, level IIA) considers that a game is fair when there are equal possibilities for all the players. When we propose a new game with cards to Rafael, he can appreciate the lack of equiprobability between the compound events, and he proposes two new equiprobable events, but he is not able to decide how the payment should vary to balance the winnings and make the game fair.

I: Then, how could we change the game so that it would be fair?

R: By giving half the cards to one and half the cards to the other.

I: But if we keep the hearts for me and the rest for you...

R: Then it is unfair.

Alberto (12 years, level IIIA), despite assigning the advantage to Luis in item 1, changes his strategy during the interview, from comparing only the favourable cases to using correspondence. He then establishes the equiprobability and compares the fairness of the game using a "balancing" argument.

I: (Reading item 1 and Alberto's response): "You say that the game is not fair, since Luis has more white marbles than Eduardo; however, Luis also has more black marbles than Eduardo, therefore he also has less chance of winning".

Al: Of course, because there is the same proportion, half 20 is 10, and half 60 is 30. However you have to draw out a white marble to win, and Luis has 30 white marbles, whereas Eduardo only has 10 white marbles. Of course, Luis might also lose, because he has 60 black marbles, whilst Eduardo has just 20. They are balanced. I think it is fair, because there is the same proportion...

Alberto considers that we would have to guarantee a greater number of attempts so that the game would be fair. This idea is in line with representative heuristics (Kahneman et al., 1982), which Alberto showed in other items, related to a lack of understanding of independence of trials.

I: (Item 2)... Do you think that it is possible that, in a fair game, a player has more probability of winning than the other?

Al: It depends. If we gave Esteban 5 pts, it would be balanced, because Maria only receives 1 pta for each number. But, if we only give them three chances to draw numbers out, it is sure that we will obtain some of Maria's numbers, since Esteban only has one possible winning number. Well… if he draws out that number, we would give him all this money, but he if does not draw out that number, and the game finishes after three trials. María is going to win 3pts. and the other one will have nothing.

Juan Manuel (10 years and 11 months, level IIIA) differentiates between equiprobable and non-equiprobable events, and between fair and unfair games. He is also capable of modifying the payment in a game in which the players have different advantages to make it fair. Similar responses were found in Pablo (11 years and 10 months, level IIB) and Juan (12 years and 7 months, level IIB).

J.M: (Item 1) Yes, it is fair.

E: Why? Does one of them have an advantage?

J.M: No, because 90 divided by three is 30 and here there is a third (he indicates 30) and here there are two-thirds (he indicates 60). Thirty divided into thirds is ten, and here there is a third (he indicates 10) and here there is two (he indicates 20).

Juan Manuel is capable of determining the equiprobability of events and changing the prize to balance the winnings in Item 2.

I: I'll give you a game, and you tell me if it is fair or not. With a pack of cards, we are going to play with the following rules: We draw out a card. If it is hearts you win 1pta. and otherwise, I win 1pta. Is this fair?

J.M: No, because it is easier for you. If anything other than hearts comes out, you win.

I: Then, how would you change the prize to be fair?

J.M: Then, you would win 1 pta. and I would win 3 pts.

3 IMPLICATIONS FOR TEACHING PROBABILITY

New mathematics curricula for elementary and secondary education propose active learning of probability where children experiment with games of chance. According to Shaughnessy (1997), when teaching probability we should not only help children to develop understanding, but also address psychological issues related to chance. Thus it is important to research children's intuitive understanding and beliefs, including their perceptions of probabilistic games of chance. As stated by Truran (1998), some research shows evidence of children's belief in animistic influences (other than randomness, strategy or skill) on chance outcomes. Most pupils in our research demonstrated an adequate conception of fair games, and were also conscious of the existence of external factors influencing fairness, such as the idea of "cheating". This might be an argument for starting to teach probability concepts while children are still at elementary school, a change which may have a crucial impact on children's development of probabilistic reasoning.

Our study also demonstrates children's conceptions, from considering a fair game only when you play with the same result (Alejandro), or the idea of fairness as an equal chance for both players (Carolina) to the need to modify the prize if both players have different probabilities. Most children find it easier to determine whether two compound events are equiprobable in contexts involving cards and dice than in urn contexts, because they only need to compare favourable cases. However, the co-ordination of task variables was achieved by only 4 out of the 8 pupils interviewed, with a level between IIIA and IIB on Noelting's classification. One pupil only considered the game to be fair in the long run. The teacher must consider this variety when teaching probability to children.

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