

ON THE FORMALIST VIEW OF MATHEMATICS: IMPACT ON STATISTICS INSTRUCTION AND LEARNING

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In the paper, we argue that the persistence of students' difficulties in reasoning about the stochastic despite significant reform efforts in statistics education might be the result of the continuing impact of the formalist mathematical tradition. We first provide an overview of the literature on the formalist view of mathematics and its impact on statistics instruction and learning. We then re-consider some well-known empirical findings on students' understanding of statistics, and form some hypotheses regarding the link between student difficulties and mathematical formalism. Finally, we briefly discuss possible research directions for a more formal study of the effects of the formalist tradition on statistics education.

Wilensky (1993) has claimed that the failure in developing sound probabilistic intuitions is similar to other failures in mathematical understanding and is the result of deficient learning environments and reliance on “brittle formal methods”. It is, in our opinion, these same reasons that also cause the neglect of a number of other statistics concepts observed both in the curriculum and in the research literature. We argue here that deep-rooted beliefs about the nature of mathematics are imported into statistics, affecting instructional approaches and curricula and acting as a barrier to the kind of instruction that would provide students with the skills necessary to recognize and intelligently deal with uncertainty and variability. In this paper we give an overview of the literature on the formalist view of mathematics and its impact on statistics instruction and learning. While little empirical work has focused on examining the impact that formalism may have on student understanding, we use here examples from a number of studies in statistics that discuss student understanding in general and we illustrate how student difficulties may be related to mathematical formalism. Finally, we discuss possible directions for research on this important topic.

1 Formalist vs. Relativistic View of Mathematics

In recent years, the “formalist” tradition in mathematics and science has come under attack and a second agenda, which views mathematics as a meaning-making activity of a society of practitioners (Wilensky, 1993), has begun to emerge. The emergence of the new paradigm has been the result of developments in the history and philosophy of science which have caused a general shift, in the last thirty years, of virtually every social science and field of humanities away from rationalistic, linear ways of thinking. In the social sciences several critics have attacked formalist tradition in mathematics and science. Hermeneutic critics (Packer & Addison, 1989 in Wilensky, 1993) have criticized it for its detachment from context, its foundation on axioms and principles rather than practical understanding, and its formal, syntactic reconstruction of competence. Feminists have criticized it for alienating a large

number of people, especially women. Sociologists such as Latour (1987) have maintained that science can only be understood through its practice.

In response to criticisms following research findings and reports of the 1970s and early 1980s exposing students' impoverished understanding of mathematics and science, leaders and professional organizations in mathematics education are now finally promoting a relativistic view of mathematics (Confrey, 1980; Nickson 1981). They have come to believe that current teaching approaches are deficient in that they do not give students the chance to encounter different perspectives on the nature and uses of mathematics. Reformers argue that the culture of the mathematics classroom should change. Mathematics should be presented as open to discussion and investigation, as a socially constructed discipline which, even at the classroom level, "is not held to be exempt from interpretations that require 'reconsideration, revision and refinement' " (Nickson, 1992). The emphasis "should not be on mirroring some unknowable reality, but in solving problems in ways that are increasingly useful to one's experience" (Confrey, 1991). The teacher should encourage discussion, and allow students to generate and test their own theories. Nonetheless, as Wilson, Teslow, and Osman-Jouchoux (1995) warn us, while recent models of cognition are challenging our traditional notions of learning and teaching, changing long-held beliefs and attitudes towards mathematics is proving to be quite difficult. For people raised in the formalist tradition, it is very difficult to accept the fallibilist nature of mathematics.

2 Impact of Formalist View on Statistics Education

In the statistics domain, there has already been a move towards modernizing statistics education and a general acknowledgment that learning occurs most effectively when students engage in authentic activities. Although many students are still being taught in traditional classrooms, there is already a large number of statistics instructors who have adopted alternative approaches to their teaching and many statistics classrooms are experiencing wide incorporation of technology. But, as Hawkins (1997) points out, for reform efforts to be successful, it is "necessary not only to provide the infrastructure and finance to support technological innovations, but also to change attitudes and expectations about statistical education". Deep-seated beliefs of many people about the nature of statistics "as a branch of the older discipline of mathematics that takes its place alongside analysis, calculus, number theory, topology, and so on" (Glencross & Binyavanga, 1997, p. 303), hamper the reform efforts.

The linear and hierarchical approach adopted by statistical courses and syllabuses is testimony to the profound and continuing effect of the formalist mathematics culture on statistics. The structure of almost every introductory statistics course is to first start with descriptive and exploratory data analysis, then move into probability, and finally go to statistical inference. Biehler (1994) warns us that the danger of a curriculum with such a structured progression of ideas is that students get the impression that "EDA (Exploratory Data Analysis), probability and inference

statistics seem to be concerned with very different kinds of application with no overlap” (Biehler, 1994, p. 16). This leads to compartmentalization of knowledge: “The degree of networking in some students’ cognitive tool system seems to be rather low, otherwise the trial and error choice of methods that we observed quite frequently would be difficult to explain” (Biehler, 1997, p. 176).

In statistics courses, effort is often put on simplifying the process of learning by organizing it step by step, assuming that this helps to remove difficulties from students’ path by gradually leading them from basic to complex connections (Steinbring, 1990). However, the linear and consecutive structure of the course comes in sharp contrast with “the complex nature of stochastic knowledge which can only be understood as a “self-organizing process” (Steinbring, 1990, p. 8). The static image projected through the formalization of the chance concept to probability is misleading and hides the dynamic and complex nature of chance events. It is inadequate in helping students make the conceptual shift that is needed to understand the difference between long-run stability and variation in finite samples (Biehler, 1994).

3 Formalism and students’ difficulties with statistical reasoning

Despite the criticisms regarding the impact of formalism in statistics education, little empirical work has been done towards the better understanding of the difficulties students face that may or may not relate to formalism. Therefore, little is known regarding the details of how misconceptions are formed and how they may be prevented. We hereby attempt to touch upon this issue through a re-consideration of some well-known empirical findings on students’ understanding of statistics. We compiled a list of difficulties students face, as documented in the literature, and we formed some hypotheses regarding the link between students’ difficulties and the formalist tradition.

Over-reliance on sample representativeness. Statistical reasoning follows from two notions which, when seen from a deterministic framework, seem antithetical - *sample representativeness* and *sample variability*. Due to sample representativeness we can put bounds on the value of a characteristic of the population; due to sampling variability however, we never know exactly what that characteristic is (Rubin, Bruce, & Tenney, 1990). Balancing these two ideas lies at the heart of statistical inference. However, although recognizing that random selection always leads to variation, most students tend to underestimate the effect of sampling variability and, over-relying on sample representativeness, they search for patterns in the data with a certainty that such patterns exist – an outcome of their training in formalist tradition. Indeed, mathematics teaching with roots in formalist tradition often encourages this searching for patterns. In statistics however, when reasoning in terms of patterns, students often fail to conceptualize the chance variation involved in those patterns and hence to exaggerate the information provided. Often students view random fluctuations in data as causal and proceed to develop deterministic explanations.

Neglect of variation. A common finding in statistics studies is students' neglect of variation (e.g., Meletiou, 2000). Students tend to think deterministically and to have difficulties in differentiating between chance variation in the data and variation due to some form of underlying causality. Hoerl, Hahn, and Doganaksoy (1997), point to the gross inefficiencies that occur in industry because managers and technical personnel have a deterministic mindset and lack awareness of variation.

The results of a study conducted by Shaughnessy, Watson, Moritz, and Reading (1999) to investigate elementary and high school students' understanding of variability, indicated a steady growth across grades on center criteria, but no clear corresponding improvement on spread criteria. Indeed, most students treat the task of finding the center of a dataset as a typical mathematics task requiring the application of a simple formula for its solution. Ignoring the possibility of outliers, they rush into adding up all the numbers and dividing by the number of data values. Further, when asked to compare group means, students tend to focus exclusively on the difference in averages and to believe that any difference in means is significant.

We hypothesize that this over-emphasis on center criteria and neglect of variability is related to the emphasis of the traditional mathematics curriculum on determinism and its orientation towards exact numbers. Since centers are often used to predict what *will* happen in the future, or to compare two different groups, the incorporation of variation into the prediction would confound people's ability to make clean predictions or comparisons (Shaughnessy, 1997). The formalist tradition prepares students to search for the one and correct answer to a problem – a condition that can easily be satisfied by finding measures of center such as the mean and the median. Variation though rarely involves a “clean” numerical response. Standard deviation, the measure of variation on which statistics instruction over-relies, is computationally messy and difficult for both teachers and curriculum developers to motivate to students as a good choice for measuring spread (Shaughnessy, 1997).

Local representativeness heuristic - Perceiving patterns in random data. The research literature has identified a series of heuristics often subject to bias that humans develop in an effort to rationalize stochastic events. These heuristics indicate people's limited understanding of randomness, their tendency to reason deterministically and develop causal explanations for random fluctuations in the data. One well-documented in the research heuristic is *local representativeness*, the phenomenon where “people believe that a sequence of events generated stochastically will represent the essential characteristics of that process, even when the sequence is quite short” (Pratt, 1998, p. 37). For example, when tossing coins, people consider it less likely to obtain HHHTTT or HHHHTH than to obtain HTHTTH, because HTHTTH seems to better represent the two possible outcomes. Similarly, the fallacy of the gambler who, after a long sequence of red outcomes, expects the next outcome to be a black is, for Kahneman and Tversky (1973), the consequence of employing the local representativeness heuristic and perceiving a pattern in random data. The gambler's fallacy is also called the “law of averages” as

it describes people's tendency to believe that things should balance out to better represent the population distribution. This is the same idea as that which Shaughnessy (1992) calls active balancing strategy. For him, an active balancer is the person who, when given the problem "The average SAT score for all high school students in a district is known to be 400. You pick a random sample of 10 students. The first student you pick had an SAT of 250. What would you expect the average to be?" (p. 477), would predict the average of the remaining 9 scores to be higher than 400, in order to make up for the "strangely" low score.

Conventional instruction often fails to establish enough links between the learners' primary intuitions about the stochastic and "the clear cut codified theory of the mathematics" (Borovcnik & Bentz, 1991; in Pfannkuch & Brown, 1996). Students coming to the statistics class have already experienced the highly fluctuating and irregular pattern of random phenomena such as the occurrence of "Heads" and "Tails" in sequential coin tosses. The theoretical statement that $P(\text{Head on next toss})=1/2$, which describes the relative limiting frequency of an event, seems to students as being in sharp contrast to the intuitively felt inability to make specific predictions on this outcome (Borovcnik, 1990). Even if students understand probabilistic theory, they often fall back into the trap of causal thinking. In their urge to overcome uncertainty, to order the chaos, they might attempt to search for logical patterns, to develop "different mathematical "theories" and causal links. Such an approach "is highly interwoven with magic belief and astrology (the law of series, a change is overdue etc.), and the search for the signs to detect this early enough" (Borovcnik, 1990, p. 8); it leads to the development of heuristics such as the local representativeness heuristic.

Fischbein (1975) notes that although probabilistic intuitions exist from a very early age they are suppressed by schooling. In order to make his point, he reports on a study where participants were asked to predict the outcomes of a repetitive series of stochastic trials, and where even young children were able to make sound predictions based on the relative frequencies of the different outcomes. The reason that the intuition of chance remains outside of intellectual development is the emphasis of school mathematics on causality and determinism and its sole focus on deductive reasoning. Due to the lack of nourishment of probabilistic intuitions, or, as we argue in this article, due to the training in formalist traditions that discourage non-deterministic reasoning, learners develop a series of heuristics often subject to bias, in an effort to rationalize stochastic events.

The Outcome Orientation. This heuristic also describes students' tendency to interpret in deterministic terms phenomena that are actually stochastic (Konold, 1989). Lacking awareness of the stochastic dimension of such phenomena, students often make predictions based solely on causal factors (Pratt, 1998). Influenced by tasks posed in mathematics classrooms to which there is always a right answer, students tend to deal with uncertainty by predicting what the next outcome will be and then by evaluating the prediction as either right or wrong. A probability of 50%

is often assigned when no sensible prediction is possible. The information that there is a 50% chance of rain tomorrow sounds totally useless, a probability of 30% implies that there is no possibility of rain, whereas a probability of 70% means that it will definitely rain.

Disconnection from context – good mathematics is “pure” In the statistics classroom, concepts related to probability are most often taught through standard probability tasks such as throwing dice and tossing coins. However, this norm of using pseudo-real examples, borrowed from mathematics instruction, does not serve students well. As the research literature indicates, the ability to solve problems involving random devices does not transfer very effectively to more applied problems (Garfield & delMas, 1990). People’s understanding of probability is more limited in real-world contexts than in the contrived context of standard probability tasks.

According to Nisbett, Krantz, Jepson, and Kunda (1983), dealing with standard random devices makes easier the recognition of the operation of chance factors than in dealing with social events. Random devices have an obvious sample space and the repeatability of trials can be easily imagined. By contrast, the random nature of social events is often not as explicit and the sample space not well understood, since use of a real-world context increases the likelihood of prior beliefs and knowledge about the issues under investigation. Students comfortable thinking probabilistically when dealing with standard probability tasks, seem “oblivious” to probabilistic thinking for problems posed in real-world settings. Although, for example, students might be aware of the dangers involved when drawing conclusions from small samples, for problems posed in real-world contexts, they often ignore these dangers and do not hesitate to use small samples as a basis for inferences, erring thus towards the deterministic side (Pfannkuch & Brown, 1996; Meletiou, 2000).

4 Implications for further research

Even after having completed a statistics course, most students have poor intuitions about the stochastic and tend to think deterministically. In this paper, we re-considered some well-known empirical findings on students’ understanding of statistics, and asserted that student difficulties might stem from training in formalist mathematics traditions. As a consequence of their prior experience with traditional mathematics curricula and classroom settings that discourage non-deterministic reasoning, students have not developed adequate intuitions about the stochastic. Statistics instruction itself, also influenced by the formalist mathematics tradition, fails to build bridges between students’ intuitions and statistical reasoning. Although notions such as randomness and variation have a nature very much dependent on context and lend themselves especially well to the new perspective of mathematical concepts as social constructs, they are typically presented in the classroom as rigidly established bodies of mathematical knowledge without any reference to real-world context. As a result, instruction fails to convey to students the relationship between the knowledge they acquire in the statistics classroom and its uses in the real world.

Statistics education ought to find ways to help learners build powerful connections between formal mathematical expressions of the stochastic and everyday informal intuitions (Borovcnik, 1990). Longitudinal studies that trace the evolution over time of students' meanings regarding fundamental statistical concepts such as variation as a consequence of the interaction between mathematics and statistics curricula and instruction, would be extremely helpful towards discovering the sources of student difficulties with the stochastic. Such studies should investigate the relationship between stochastic and mathematical thinking, learning, and teaching, not only along the cognitive but also along the epistemological and cultural dimension (Metz, 1997). The epistemological dimension should look at the effect of students' beliefs about the place of chance and uncertainty on their emergent understandings. The cultural dimension should investigate how messages about the place of chance and determination, implicit in the practices and values of the classroom influence students' beliefs and ideas. Indicators of the classroom culture (Metz, 1997) that should be examined include choice of subject matter, structuring of problems, teacher's reaction to students' claims about causality, aesthetics of what constitutes a good solution or explanation, and teacher's willingness to accept multiple strategies and viewpoints.

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