EXPERIMENTATION AS A TOOL FOR DISCOVERING MATHEMATICAL CONCEPTS OF PROBABILITY

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This paper puts focus on students' ability to handle the component of probability, while acting under uncertainty in an experimental environment. In particular, we are interested in the extent to which seventh grade students are able to develop secondary intuitions of probability during interaction with mathematical modelling in a co-operative setting. To catalyse this kind of learning situation, involving consciousness and reflective processes, a competitive game is designed and introduced to the students. We argue that this kind of experimental mathematical activity supports the desired development towards secondary intuitions.

1 Background

A major goal of school education is to prepare people for everyday life, where one is often confronted with situations that contain an element of uncertainty. To be prepared to handle such situations, one needs to develop general qualities, such as the ability and will to plan, experiment and discover in the world around. It will therefore be essential to confront pupils with simple mathematical models to stimulate such qualities and to develop an ability to carry through, understand and use logical reasoning (Skolverket, 2002). According to this, it can be argued that insight into aspects of probability can be a relevant factor as probability is both a part of mathematics – dealing with ideal, abstract operations and entities (Fischbein, 1991) – and a relevant part of everyday situations; *...practical experience with probabilities provides an ideal way of familiarising children with the fundamental concepts of science, such as prediction, experiment and verification, chance and necessity, laws and statistical laws, knowledge through induction, and so on* (Fischbein, pp. 93, 1975).

In light of this the current study is, from a mathematical point of view, going to investigate how students handle components of probability, while acting under uncertainty. In particular it will focus on situations involving compound events, as they can be assumed to stimulate and cover several interesting phenomena related to probability. These are issues that will be discussed within the framework of this paper.

2 Two perspectives, towards learning of probability

There are two research perspectives seen in the area of learning probability. First there is the psychology/cognitive perspective including Kahneman and Tversky's work, quoted and developed in Gilovich et.al (2002), with focus on analysing patterns in order to identify thinking misconceptions and judgmental heuristics.

The second perspective is that of mathematicians and mathematics educators, with focus more on learning probability from a mathematical point of view (Keeler and

Steinhorst, 2001). According to Shaughnessy (1992) mathematics educators are concerned with improving students' knowledge of probability; … *they wish to change the students conceptions and beliefs about probability and statistics* (Shaughnessy, pp. 469, 1992).

Many implications from the results of the psychologists can be seen in the research of mathematics educators, as the psychologists have provided a theoretical framework in considering judgmental heuristics. An extensive mapping from this view, and from the related area of misconceptions, can be identified in the literature, including *representativeness and availability* (Gilovich et al, 2002; Shaughnessy, 1992), the *outcome approach* (Konold, 1991; Li & Pereira Mendoza, 2002) as well as the *equiprobability bias* (Cañizares & Batanero, 1998; Li & Pereira Mendoza, 2002).

However, this sharing seems so far to have been only in one direction, as the psychological view seems to be that misconceptions are difficult to overcome. Keeler and Steinhorst (2001) believe that the dualism between the two perspectives is supported by the design of the different settings in which the most common task utilized by the psychologists is to measure a forced-choice response, in contrast to the educators, whose tasks allow a more open-ended exploration of the participants thinking.

If the psychologists' view is interpreted as focusing *how* we act in probabilistic situations, then the perspective of mathematics and didactics should, in an opposite sense, be considered from the view of instruction, regarding how we *should* act.

The works of Piaget and Inhelder (1975) and Fischbein (1975) are two classic texts supporting this latter view as they describe the development of probabilistic cognition in individuals, particularly in children. Even if several aspects in the work of Fischbein are to be found in the theory of Piaget – such as emphasizing the complexity in developing probabilistic concepts, schema dependence in cognition and adaptation to environment (Greer, 2001) – some essential differences can yet be seen. The most fundamental difference is that Fischbein considers intuition to be the origin of the development of probabilistic thinking, while Piaget links this origin to his cognitive framework and relates probabilistic thinking to proportionality based on operational terms, such as combinatorial abilities (Piaget & Inhelder, 1975; Fischbein, 1975). Fischbein's assumption regarding the role of intuition, together with his opinion that concepts and intuitions can be socially mediated, modified and developed – in contrast to Piaget (Hawkins & Kapadia, 1984) – constitute the foundation when he makes the distinction between primary intuitions and secondary intuitions. He defines primary intuitions as cognitive acquisitions, derived from individual experiences, without systematic instruction, and secondary intuitions as formed by education and linked to formal knowledge (Fischbein, 1975). However, he concludes that formal instruction and convictions do not make the primary intuition disappear. Instead, he argues that primary intuitions are present in all minds, including the greatest of mathematicians. The important part is therefore to understand when they are improper for the situation or influence decisions in a negative way (quoted in Greer, pp. 26, 2001).

These two major bodies of research, together with the heuristic view, have both contributed several important results. Even so, it could be argue that some aspects need further study in order to explain and understand how people act under uncertainty. First, there are some philosophical conflicts, which give rise to difficulties in defining what probability really is and thus, what to teach? Secondly, there has been comparatively little investigation into the pedagogical setting, with respect to the relation between the subject's internal resources and the external, structural resources offered by the environment (Pratt, 2000).

Regarding the first issue several attempts have been made to classify various positions. Hawkins and Kapadia (pp. 349, 1984) summarize their contribution as *a priori probability, frequentist probability, subjective and intuitive probability and formal probability*. But, despite philosophical controversies, there are some underlying, specific elements of mathematics, which can be accepted by most positions (Borovcnik et al., 1991). Considering specifically the literature of compound and independent events the study relates its mathematical approach to the following elements (Batanero et al., 1998; Truran, 2001; Borovcnik et al., 1991; Jones et al., 1999);

- Randomness – concerning pupils' ability to identify and handle the randomness of a situation.
- Sample space – with respect to control and ability to identify elementary events.
- Probability of an event – establishing the likelihood of a certain event.
- Probability between events – concerning a developmental stage of the previous component, with respect to symmetrical features and the mathematical concept of proportionality.

Considering the second aspect, regarding the interactions between the individual and the situation, it could be argued that Fischbein with colleagues believes that instruction can catalyse the adaptive processes of assimilation and accommodation, in a truly constructivist spirit: *We see the process of learning and understanding as a reciprocal process of communication between a referent situation and the mathematical structure* (Fischbein et al., pp. 531, 1991). In the same study they establish that children seem to develop a natural intuition regarding the probability of an event, on the basis of the corresponding sample space, and that this ability seems to be strengthened when the interaction takes place through a task addressed in a general form. However, though several researchers agree with Fischbein that there are positive effects of instruction in the learning of probability, they are aware of difficulties in changing students' conceptual understanding. Yet, in the literature a common opinion is that students best develop understanding of probability events by working with so called hands-on activities, that is, letting the students act in events containing an element of uncertainty (Greer, 2001; Shaughnessy, 1992). Konold (1991) also stresses the use of interactive processes in learning probability and argues that the following steps are crucial:

- test own beliefs against the beliefs of others,
- test own beliefs against own beliefs about other related things,
- test own beliefs against empirical evidence.

Pratt (2000) claims that Fischbein's theory lacks sufficient details to guide the development of learning environments that support the development of secondary intuitions. In connection with this, Pratt (2000) introduces an approach in which the components of interaction are separated into internal and external resources. The external resources involve aspects of the setting – such as tools, the task, the teacher, and so on – in contrast to the internal, which are identified as individual resources: *The development of knowledge is taken to depend upon the interaction of external resources with the child's dynamic knowledge, incorporating all forms of intuitional and formal thinking, to which I shall refer generally as internal resources* (Pratt, pp. 3, 2000). He then makes a further distinction of the internal into local and global resources. These two resources can be seen as counter-balancing each other, the local related to short-term behaviour and the global to long-term behaviour. According to Pratt (2000) a crucial step is to develop an understanding of long-term behaviour and in particular of the features that regulates this. We see that this view has close links to Fischbein's ideas about the role of intuitions and that of social interaction to mediate, modify and develop new concepts and intuitions.

Synthezising the discussion so far, we distinguish between primary and secondary intuitions and refer primary intuitions to intuitions that are individual experienced, derived from every-day life, and secondary intuitions to intuitive ideas that are abstracted from interaction and linked to formal knowledge, involving defining and structural characteristics.

3 Object of study

The study investigates the extent to which seventh grade students handle components of probability, which have not been formally presented to them in school, during interaction in an experimental situation. In particular, it will focus on how students approach advanced mathematical concepts of probability, regarding compound events, by discovering functional relations via mathematical modelling during experimental activities.

Two major questions of concern are:

1. What primary intuitions – linked to heuristics and biases – appear in the interaction?

2. What secondary intuitions – supported by the activities and linked to our mathematical approach – are to be intended in the students' modelling strategies?

4 Method

To fulfil the aim of study, it is based on empirical examples gathered within a classroom situation in a compulsory school in Växjö, Sweden. The activities have been both observed by myself and videotaped. Discussions are also tape-recorded. The tape-recorders were placed to catch the communication of the groups estimating ”winning strategies”, while the video was used to document their acts at the moment of play. In connection with the co-operative activities, follow-up interviews were conducted, in order to clarify and further investigate appeared phenomena.

To analyse and describe cognitive phenomena, obtained from the experimental activity, intentional analysis is used, a method by which you ascribe meaning to students actions in terms of intentions (von Wright, 1971).

In the didactical situation the pupils are mixed into four groups of two, that is a total of eight participants. Within this setting they are confronted with a competitive dice game, framed by a design introduced to us by Ulla Öberg, Malmö högskola (the design originating from the Mathematics Task Centre Project, Australia). Dice are used as random generators as they can be seen to generate interaction between every-day experienced knowledge and the desired mathematical elements, supported by the design of the setting, and thereby stimulating students' primary and secondary intuitions; *Some RG, such as, dice, coins and urns, are common in western culture. All three are devices where a number of outcomes are clearly specified, and which can be manipulated in a well-defined way to produce one of these outcomes without its being possible for anyone or any machine to predict which outcome* (Truran, pp. 94, 2001)

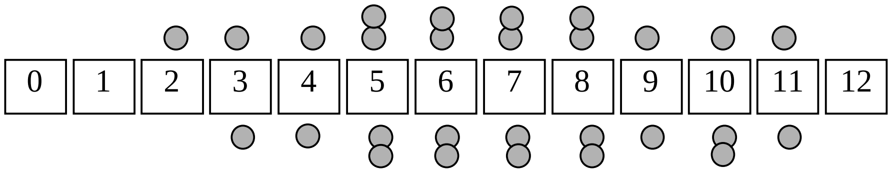
5 Preliminary study

First I will give an overview of a preliminary study (Nilsson, 2002) and summarise some conclusions and suggestions for redesign, which are used in the current study.

The aim of the study was to make an overall inventory of cognitive phenomena, appearing in an experimental environment through co-operative activities. Although the object of study was quite general, the study resulted in a qualitative description of how individuals handle situations of uncertainty and further what affects interpretations and actions.

According to the approach described above, two teams played against each other in a competitive dice game. They had a board with areas marked 0-12. Each team also got 14 markers, which they were asked to distribute, as they liked among the 13 areas. When they were done, they took turns on rolling two standard dice. If one team or both of them had at least one marker in the area, which was marked with the sum of the dice, each of the teams removed exactly one marker from this area. The team who first removed all its markers from the board won.

First set-up of two teams:



The areas 0 and 1, which will never be a sum of two standard dice, were included to induce the concept of sample space.

Among the conclusions from the study are that all groups successfully identified the restriction of the sample space, before beginning to play and without question. What also appeared, yet with some differences, was an intuition of certain numbers having greater or lesser ”chance”. In several groups, the members agreed, without explanation, that seven is the number with the greatest chance as opposed to two and twelve, which generate a lower chance. However, although they were able to identify two and twelve as having equal chance, we did not find any further tendency towards thinking about proportionality or symmetric relations in their discussions. Instead a tendency towards the equiprobability bias is common as several set-ups articulate a rather uniform distribution in their strategies, at least among the middle numbers. On the whole, there was very little explicitly reflective work done by the participants and in cases of reflection there was just an obvious tendency to apply the recency fallacy intuition (Fischbein 1975). An example of the *positive* recency effect is represented by the following extract, in which we enter the dialogue after Anna and Albin had succeeded in their first play and were starting to model their next set-up;

Anna: What shall we talk about now? That we should not have so many at seven next time? (*laugh*)

Albin: Yes, not so many at seven or maybe one at seven!

One at seven then. Which one did we get plenty of?

Anna: We got five and eight and stuff like that.

Albin: Mmm, and nine we got often.

We claim that, in order to induce attention to mathematical objects in the situation, some guidance and restrictions should be applied to the setting. Further, a distinct regularity of the medium, in terms of stability of frequencies, over which the participants are supposed to interact, should be embedded to provide for benefits in discovering meaningful and useful structures to encourage thinking of phenomena from a probabilistic point of view.

6 Current study

On the basis of the conclusions from the preliminary study, together with implications from the literature, the current study focuses on students' approach to advanced mathematical concepts of probability by discovering functional relations via mathematical modelling. In addition, the design puts attention on mathematical objects without being too limited, keeping the students active and thinking on their own.

Regarding the tool of analysis, we continued working with the intentional analysis as we considered that to be a suitable tool, which fulfilled our purposes (von Wright, 1971)

The current study is in many parts in accordance with the preliminary one. With respect to independent events and the total of two dice we confront the students with an experimental, co-operative and competitive situation with hidden mathematical concepts, designed in such a way that discovering and developing useful relations will give benefits in winning strategies. Differences can be seen in the way the task is conducted, including redesign of the medium, specifically the dice, which are designed in several different ways.

In an attempt to capture several important mathematical aspects of probability regarding compound and independent events, four pair of dice were presented to the students in the following order.

1. The yellow setting – Here the faces are marked with one and two eyes, distributed as (111 222) and (111 222).
2. The red setting – Includes two different dice, each with a distribution of two outcomes among the faces as (222 444) and (333 555).
3. The blue setting – Similar to the yellow with the difference that there are now four sides marked one and two sides marked two, that is (1111 22) and (1111 22).
4. The white setting – Similar to the red setting but now with the distribution shifted towards the lower numbers as (2222 44) and (3333 55).

All four settings included a board marked from 2 to 12. This, together with the yellow and the red settings was meant to support a developed understanding of the restriction of the sample space. Considering the third and fourth settings the third was designed to put focus on probability, related to combinations and proportionality, while the fourth one was aimed to connect the two former approaches. Involving consciousness and reflection, the design leads students' to interact with a medium with a unfamiliar structure, giving more distinct feed-back, to encourage the desired development of secondary intuitions.

7 Results and analysis

Regarding the sample space we recognised that the design encourages students to focus on its restriction. After some of them had failed in their red set-up, putting markers on the impossible totals 6 and 8, the following episode took place in discussion of the white set-up.

Patrik: Two, no three…no four sides with the number of two.

Sara: Four of two! Here it is four of three…

Patrik: These were the same?

Sara: …now there are more of three and two. Therefore it cannot be…?

Patrik: Now we put more on two and…

Sara: It cannot be two! The lowest it can be is five, and the highest nine. Six and eight cannot appear cause this we learned last time.

However, failures related to the restriction of the sample space seem to be conditioned culturally and connected to ordinary dice. Recognising that the designs differ from everyday experience, almost all participants were able to exert control over the sample space.

Furthermore, we also see from several discussions that strategies are related to the area of heuristics and biases. At a first glance, the *equiprobability* bias is obvious in all groups, particularly during the first two settings, exhibited in the students' uniform way of distributing markers. Their intentions seem to be framed by *fairness*. When dealing with the two first settings, the fair representation of numbers on each die seems to support an intuitive thinking that generates equiprobability for the total. In the last two settings, the students still use fairness as an internal resource. However, the shifted distribution of numbers on the dice stimulates thinking towards structural characteristics, making the former bias concerning equiprobability vanish. According to Fischbein (1991), this means that we can identify here an intuitive intention, namely that more frequent pairs in the compound event produce sums with higher probabilities. That the correctness of this intuition is in accordance with the heuristic of *availability* can be seen from the following episode, extracted from an interview over the yellow dice sample space, regarding the order of the dice.

The interviewer writes down the elementary outcomes for each die on a paper and then asks the student to figure out all possible totals.

Mats: Yes, now I see, it is easier to get three! …1+2 or 2+1, this you can get 18 times. …and between ones and twos, if you go to the same, gives nine.

After that, Mats correctly computes all elementary outcomes within all settings. We conclude that systematisation is important to the availability of transparency and thus to stimulate a development of a new secondary intuition regarding the order of included elements in a compound event. As Mats articulate this;

Mats: It is a little tricky to see if you haven't drawn!

8 Discussion and Conclusions

The aim of study was to investigate how students approach advanced mathematical concepts of probability, regarding compound and independent events, during interaction in an experimental environment. For this aim, it is suggested to create situations in which the participants have to overcome primary intuitions in order to develop secondary intuitions that are linked to formal knowledge.

From the analysis over the students modelling strategies it seems that the variation within the design both supports the identification of primary intuitions, especially the equiprobability bias, and that of forcing the participants to organise their gambles with the basic probabilistic concepts used as decision tools. Even if such abstracted tools not immediately can be regarded as completely acquired and stabilised secondary intuitions we recognise how the environment stimulate thinking towards the mathematical structure with respect to:

- sample space control,
- stochastic independence,
- comparisons of probabilities for compound events,
- estimation of probabilities by frequencies.

Therefore we argue that the study shows the importance of situations in action, where children have to solve problems that involve, and supports a development of, such secondary intuitions.

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