THE EMERGENCE OF PROBABILISTIC KNOWLEDGE

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In this paper, I summarise a theoretical framework for the growth of probabilistic knowledge. Through reflection on two theories that seek to model sense-making activity at quite different grain sizes, a synthetic view is proposed that draws its power from three sources: (i) its connection with the two original theories, (ii) its ability to model the behaviour of children working with a particular computer microworld, and (iii) its consistency with work in the literature. In particular, the theory offers a coherent way of thinking about inconsistency in children's responses and proposes principles that could underpin effective teaching or curriculum development.

1 Introduction

Pratt and Noss (2002) have set out a detailed theoretical framework for the *micro-evolution* of mathematical knowledge. (Henceforth I will refer to Pratt and Noss jointly in the first person plural). In that paper, we connect a small grain-size theory of sense-making (diSessa, 1993) to a larger grain theory (Noss & Hoyles, 1996). By *micro-evolution,* we refer to changes in the mathematical thinking of individuals working on specific tasks over relatively short spaces of times, in contrast to overall longer-term development.

Such a focus tends to highlight changes from minute to minute. A sense of inconsistency emerges and this is nowhere more pronounced than in the research on probabilistic thinking.

The next section sets out a summary of our model in relation to randomness. The reader should refer to Pratt & Noss (2002) for more detail of the theory and for illustrative data taken from Pratt (1998). The third section relates that theory to some of the classical research on probability and chance.

2 Theoretical framework

Our theoretical framework brings together theories of sense-making as proposed by diSessa (1993) and Noss & Hoyles (1996). This synthesis however is not a unification in the normal sense, since it is acknowledged that the former theory operates at a considerably lower level of grain size than the latter. The relationship between the two is more akin to examining a phenomenon (in this case sense-making) through a microscope (small grain image) and then examining the same phenomenon with the naked eye (large grain image).

2.1 A microscopic view of sense-making

diSessa's theory is complex; within the scope of this paper, I am only able to set out some critical aspects. diSessa argues that knowledge is essentially piecemeal, at least in the first instance. Very small pieces of knowledge, called p-prims (short for phenomenological primitives) are abstracted directly through experience. An example of a p-prim might be characterised as "I push – it moves". P-prims are initially disconnected, each possessing a cueing priority, which determines how likely it is to be triggered. In addition, diSessa argues that each p-prim has attached to it a reliability priority, which is modified post-hoc. Thus, if a phenomenon triggered a particular p-prim, and the subsequent sense-making activity appeared to support the validity of the p-prim then its reliability would increase. Similarly, reliability would be affected by its internal consistency with other p-prims. Reliable p-prims are more likely to be used than less reliable ones in subsequent sense-making activity. Over a period of time, certain p-prims will be found to be highly reliable and simultaneously triggered by specific phenomena. In this sense a whole set of p-prims becomes connected, forming what diSessa calls a co-ordination class, but which we might roughly think of as a concept. The process of structuring p-prims is referred to by diSessa as "tuning towards expertise". diSessa's ideas raise three questions:

i) diSessa's work focuses on physics, which inevitably leads him to consider causality as the phenomenological basis of p-prims. Would a focus on probability lead to the same conclusion?

ii) What is the relationship between the setting, and in particular the tools being used, and sense-making?

iii) Mathematicians see power and rigour in lack of reference to context. How can a theory based on direct abstraction of phenomenological experience explain that perception of mathematical knowledge?

2.2 A top-level view of sense-making

Noss & Hoyles (1996) regard sense-making mental tools as internal resources within a web-like structure that also embeds external resources such as visual aids, computational tools, prompts from a teacher, and discussion with peers. The child's sense-making activity, referred to by Noss and Hoyles as *webbing,* involves the child in forging and re-forging links across those resources, both internal and external. There is indeed a dialectical relationship between internal and external resources, in the sense that, for example, the computational tools shape the structure of the child's internal resources, whilst at the same time the child's internal resources shape the web itself through the choices made.

A particular construct within the theory of webbing is that of *situated abstraction*. Accordingly, when children engage with tools (and I should mention that the data is taken from situations involving the use of computational tools), they articulate sense-making in terms of accessible tools and structures. Those articulations signal knowledge drawn out of the activity and yet apparently specific to that activity. Two questions emerge from this perspective:

- What is the relationship between the top-level and microscopic perspectives on sense-making?
- As with diSessa's theory, we must still ask how this view of contextualised abstraction can be compatible with formal abstraction, which derives its power from its removal from the specific situation?

I will attempt to address these questions and those emerging from the microscopic view by summarising our own theoretical framework below.

2.3 A synthetic view of sense-making

We should emphasise at this point that we are proposing a model that places some emphasis on the role of the pedagogical setting in shaping the growth of probabilistic knowledge, in contrast to frameworks that claim to study such development in more naturalistic settings. We propose that such a model for the growth of probabilistic knowledge must contain five elements:

1. A description of naïve probabilistic knowledge.
2. A description of the setting including the structuring resources.
3. An elaboration of the nature of new probabilistic knowledge.
4. An account of the relationship between new knowledge and the setting.
5. A proposition of how prior knowledge illuminates sense-making in unfamiliar settings.

Data that illustrates our elaboration of these five elements is set out in Pratt & Noss (2002). Here I restrict myself to summarizing the conclusions. These conclusions are drawn from the activity children of age 10 working on a computer-based task, designed to probe into their emergent sense-making for randomness.

2.3.1 Element 1: Naïve knowledge

The following conclusions about naïve knowledge were deduced from interviews about real-life situations, which might feasibly be regarded as having random elements, and from the early work with simulated random generators such as dice, spinners and coins (see *Element 2* below). Naïve knowledge consisted of four separable resources for articulating randomness. Children judged whether a gadget's behaviour was random by…

1. …whether they were able or not to predict its next outcome *(Unpredictability)*.
2. ...their ability to control the outcome *(Unsteerability)*.
3. …their ability to find a pattern in a sequence of outcomes *(Irregularity)*.
4. …whether the gadget looked fair *(Fairness)*.

These naïve resources contrast with the nature of p-prims. Whereas p-prims are causal in nature, the four resources for making sense of randomness have no consequences. As mere attributes, they have limited value for making sense of phenomena beyond simple classification. In common with p-prims, we suppose that these resources have been abstracted in a fairly direct way from experience. They were linked to the specificities of the situation. Thus, the appearance of the dice might trigger the fairness resource, whereas a focus on throwing the spinner might trigger the unsteerability resource.

The lack of any consequential aspect to the naïve resources allowed more than one such resource to be triggered by the same phenomenon even when occasionally the two were in apparent contradiction. For example, a non-uniform spinner sometimes triggered both the fairness resource – suggesting that, since in this case fairness was not apparent, the spinner was not random – and the unsteerability resource – with its apparent implication of randomness. Such contradictions were often problematic for the children, who were happy to shift seamlessly from one perspective to another.

2.3.2 Element 2: The setting

The children were presented with a series of computational devices, called *gadgets*. Each gadget was in fact a simulation of an everyday random-number generator, enhanced with some tools that proved critical in the children's evolution of probabilistic knowledge. For example, the coin gadget was *thrown* with a strength determined by the child. The effect of using the strength control was in fact to increase the length of time that the coin spun but the strength had no impact on the actual outcome. The trial could also be replicated, thus enabling the coin to be re-thrown with exactly the same strength. The coin gadget (like all the gadgets) could be opened up to reveal further tools. Inside the coin were tools to facilitate the execution of many trials, the scrutiny of past outcomes and the graphing of results in the form of a pie chart or a pictogram.

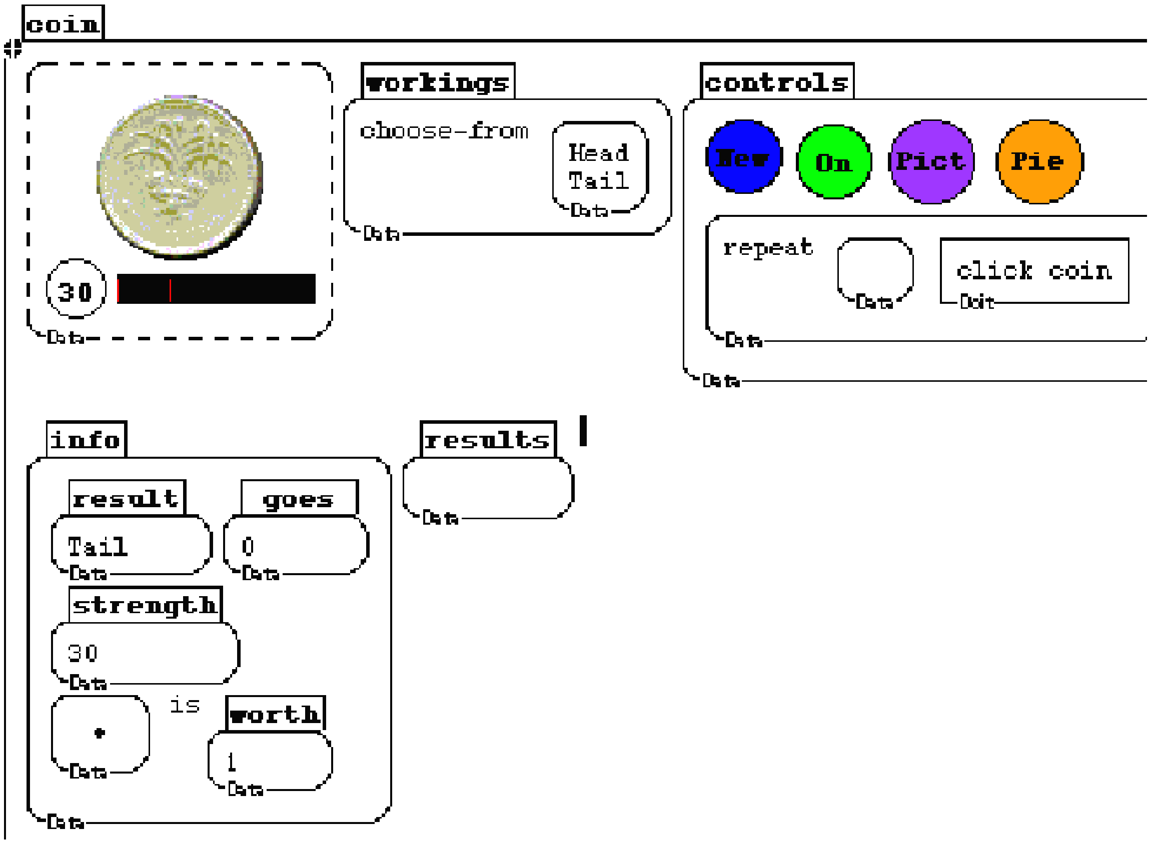

Figure 1: Inside the Coin gadget

This in Figure 1, the information box contains data about the most recent result, the strength of the latest throw, how many times the coin has been thrown and a key for the pictogram. Pictograms and pie charts can be generated by clicking the corresponding keys on the controls box. Other controls allow the graphics to be switched on and off (crucial when doing long run experiments) and the experiment to be initialised when starting afresh. The final control in the controls box facilitates the repetition of experiments so that many trials can be executed automatically. In Figure 1, the results box is empty. When at least one trial has been executed, the results are listed in this box and can be browsed by the child.

A particularly important tool was the *workings box*. In Figure 1, the workings box is shown as `choose-from [head tail]`. For the dice, the workings box in Figure 2 reads: `choose-from [1 1 1 2 3 4 5 6]`. This workings box is in effect a non-standard representation of the distribution for that gadget.

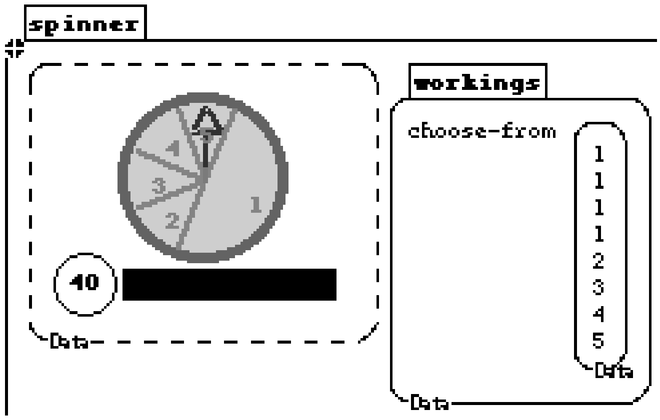

Figure 2: The workings box of the Spinner gadget

If one focuses on the verb, *choose-from*, then one sees the workings box as an action that determines the outcome of the dice. If one instead focuses on the set of possible outcomes, then one recognises an urn-like representation of the dice's probability distribution. The children were able to edit the workings box in order to modify the behaviour of the gadget. In this sense, the children initially saw the workings box as another form of control, though later they began to see it more as a representation. The tools, although instantiated within gadgets, were identical across gadgets (other gadgets included the spinner, Figure 3, the two-spinners and the two-dice gadgets).

The children were initially challenged to identify which gadgets were working properly. Some gadgets were set by default in non-standard ways. For example, the dice gadget was initially biased towards sixes. The notion of "working properly" was intentionally not defined. Once a gadget was regarded as perhaps not working properly, the children were shown the tools and asked to "mend" the gadget. As well as being intrinsically purposeful, the task of first identifying faulty gadgets and then mending them was seen as providing a window on the children's sense-making activity. We presented no view about which gadgets were working properly; rather we were interested in how the children conceived that coins, spinners and dice should work.

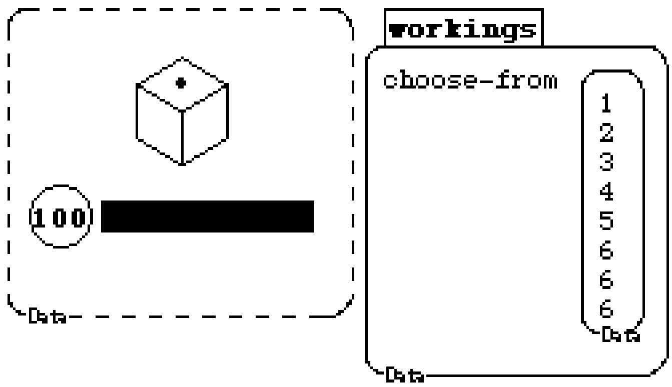

Figure 3: The workings box of the Dice gadget

2.3.3 Element 3: New knowledge

Before discussing in the later elements how new knowledge emerged, I wish to summarise here the nature of that new knowledge. We observed during the interaction with the external resources, two new internal resources. They took the form of situated abstractions, referred to as N and D.

The Large Number Resource, N, emerged out of the children's natural inclinations to repeat trials and graph the results. The facility to generate many outcomes quickly was crucially important in encouraging children to test and re-test their ideas. The facility to accumulate results was important in enabling them to move to larger-scale experiments. Initially the children were surprised that the uniform spinner generated an “unfair” looking (i.e. non-uniform) pie chart. Their natural inclination was to attempt to redress the imbalance by editing the workings box. However, any such attempts failed to produce the uniform pie chart that they sought. Eventually the children discovered, sometimes with encouragement from the researcher to try more trials, that something very surprising happened when a large number of trials was used. The children typically expressed this idea as, “The larger the number of trials, the more even the pie chart”, a statement that we regard as an intuitive basis for the Law of Large Numbers.

This articulation represents a typical situated abstraction, embedded as it is in the children’s setting. It also appears to relate closely to diSessa’s *Ohm’s* p-prim, “more effort implies more result”. The number of trials is seen as the effort and the degree of evenness is the result. N is noticeably different in nature from the naïve resources in that N is causal in nature, whereas the naïve resources were not.

The Distribution Resource, D, emerged out of editing and re-editing the workings box. The children found that when the workings box contained the same number many times, that number would be represented by a large sector in the pie chart. The situated abstraction in this case could be schematised as, “the more frequent an outcome in the workings box, the larger its sector in the pie chart.”

Children tended to over-generalise N, in the sense that they thought the pie chart would appear uniform as long as they used a large number of trials irrespective of the shape of the workings box. Thus, they were surprised when the dice gadget, with its biased workings box, refused to generate a uniform pie chart no matter how many trials they used. Some children however appeared to co-ordinate N and D, in a proposition that might be schematised as, “the more frequent an outcome in the workings box, the larger its sector in the pie chart, provided the number of trials is large”.

When the children, having already articulated N and D in the context of working on their second or third gadget, began to interact with the next gadget, they behaved as if they had no resources available to them from their prior work. They seemed to need to re-invent N and D in the context of the new gadget. It appears that N and D, at least at that stage, had quite narrow scope. Schematically, we think of a situated abstraction as being surrounded by a *contextual neighbourhood* that describes the essential features under which it was constructed.

It is important to appreciate how diSessa’s framework informs our thinking about the relationship between N, D and the naïve resources, fairness, unsteerability, unpredicatability and irregularity. We do not imagine that these new resources in any sense *replace* the naïve resources. These pieces of knowledge remain as helpful means of characterising short-term randomness. They can however, through the tuning towards expertise within this sort of pedagogical setting, become embedded within a more elaborate structure that incorporates N and D. The naïve resources will help to identify phenomena as interpretable in terms of randomness but the simultaneous triggering of N and D will provide more explanatory power for long-term situations.

Earlier, I mentioned how we can view p-prims as a microscopic view of what is seen at top-level in terms of situated abstractions. Although we would not want to suggest that there is any simple mapping between the two perspectives, it is interesting to note that in some sense Ohm's p-prim appears to underpin both N and D. Furthermore, we see a connection between the priorities attached to p-prims and our notion of a contextual neighbourhood. Certainly both relate to the cueing mechanism. The notion of a contextual neighbourhood would also seem to imply the necessity for a rather complex context-based model for cueing priority.

2.3.4 Element 4: The role of the setting

This element attempts to address how the children moved from the articulation of naïve resources to the situated abstractions, N and D, with particular reference to the resources in the setting. Initial attempts by the children to make sense of the behaviour of the gadgets was in terms of their naïve resources, such as unpredictability, unsteerability and fairness. Such resources were of much higher priority than resources related to longer-term aggregated behaviour. Indeed, we suspect that for some children such resources might not have existed at all at the outset of this activity.

N began to emerge because the pie chart and the pictogram focussed attention on the aggregated results in a way that everyday experience with spinners and dice is unlikely to do. As discussed above, the tools also encouraged the accumulation of results. Crucially, resources such as unpredictability were insufficient to account for the emergence of patterns in randomness, and so other heuristics needed to be found.

D began to emerge when the children started to work on a non-uniform gadget. By setting up some of the gadgets in this way, we provoked the use of the workings box, which prompted the emergence of D. However, by focussing on the workings box, the children appeared to forget N. It was as if N's contextual neighbourhood did not incorporate this activity with the workings box. It was only when some children found that D was insufficiently powerful that N was triggered, resulting in a co-ordination of N and D and an increased priority, or as we prefer to think of it, a broadened contextual neighbourhood, for N.

There were, according to our analysis, certain key affordances in the setting, which facilitated the construction of N and D. Randomness was instantiated as a manipulable computational system. Randomness was *phenomenonalised* in a setting that encouraged children to test out their personal conjectures. Resources with narrow contextual neighbourhood, typically ones that had only recently been constructed, would not be used in favour of longer established resources, unless the latter could be shown to lack explanatory power. The conjecturing environment enabled the children to recognise the paucity of their naïve conceptions, and encouraged them to seek other ways of making sense of the phenomena. As part of that philosophy, the children were given *redundant controls*, such as the strength control, in order that they could see for themselves the lack of explanatory power of those tools.

2.3.5 Element 5: Unfamiliar settings

N and D were initially constrained to a narrow highly specified contextual neighbourhood. When the children worked with new gadgets, their lack of familiarity seemed to demand starting afresh. The nature of their knowledge appeared highly situated as has been observed in many other studies of mathematical activity (for example, Lave, 1988). Indeed the lack of transfer was entirely predictable from a situated cognitionist perspective.

What was perhaps less predictable was what ensued. When resources such as unpredictability, with relatively high priority, were seen as lacking explanatory power, resources such as N and D were “remembered” and tried out in a tentative and conjectural manner. The consequence of N and D proving themselves as valid in more than one context was that they were triggered much more readily when working with subsequent gadgets. It appears that the priorities associated with N and D increased as they proved themselves to be reliable and, as a result, the contextual neighbourhood widened.

3 Research on probabilistic thinking

The above theoretical framework addresses the micro-evolution of probabilistic knowledge. One way of judging a theory is to assess its ability to describe and predict behaviour. Of course, the theory fits well the data from which it is drawn. How does it connect though to other research in this domain?

The seminal piece of work on chance and probability is that of Piaget and Inhelder (1951). Although most definitely aimed at the macro-evolution of knowledge, we can nevertheless recognise connections between the two theories. Piaget and Inhelder experimented with random mixtures and found that young children were unable to make sense of this type of phenomenon as it conflicted with their operational mode of making sense of the world. In particular, the notion of irreversibility, inherent in random mixtures, was inconceivable. Piaget and Inhelder claimed that the organism invents probability as a means of operationalising chance.

The children we observed appeared to set aside random phenomena by attributing naïve descriptors to such phenomena that allowed them to recognise and so marginalize random phenomena. In this sense, the naïve resources of unpredictability, unsteerability, fairness and unreliability provide an important role. They enfold a process of contextualisation that identifies random phenomena. However, the naïve resources lack the explanatory power, which only merges through consideration of longer-term aggregated behaviour, or, for some, through consideration of mathematical theory. We see the phenomenonalisation of randomness as providing an intuitive route towards the operationalising of chance to be taken before following a more conventional theoretical route.

A strength of this theory is that it provides a coherent way of thinking about the inconsistencies, which are readily observable in the classroom and have been recorded by many researchers of probabilistic thinking. For example, Konold et al (1993) discussed the inconsistent nature of children's responses. Another example is in the work of Nisbett et al (1983), who point out the sensitivity of children's responses to the situation. At a low grain size, we see notions of randomness as disconnected pieces of knowledge, with different resources generated by changes in the setting. Furthermore, our theory predicts how a teacher or curriculum developer might begin to help children widen the contextual neighbourhood of normative resources. Such a setting must provide, through a conjecturing environment, opportunity for them to recognise the limited nature of their current means of making sense of phenomena, together with high affordance to use tools that are likely to result in increased priority for new constructions, that are hopefully intuitive bases for powerful stochastic concepts, such as the Law of Large Numbers. We see no reason why the notion of providing such a setting should not extend to moving beyond the computer to other classroom settings that provide the necessary conjectural environment.

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