THE DEVELOPMENT OF YOUNG CHILDREN'S NOTIONS OF PROBABILITY

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Task based interviews were conducted with 74 children aged four to twelve years from three schools. These children had not received any formal instruction in probability as it was not part of their school curriculum. The study confirmed the presence of three developmental stages, but also revealed two distinct transitional stages. This paper focuses on the characteristics of children's strategies for making probabilistic judgements in each stage, and on the implications for teaching.

1 INTRODUCTION

Piaget and Inhelder (1951) linked the development of probabilistic thinking to Piaget's general theory of cognitive development in three clearly defined stages. In more recent years, there has been a persistent body of researchers in several countries (for example; Fischbein, 1975; Fischbein & Schnarch, 1997; Green, 1983; Shaughnessy, 1981) who have added to what is known about probabilistic reasoning. However, their studies have tended to focus on quite specific concepts, a narrow age group of children, or on adult reasoning. Consequently, no complete and coherent theory of the development of probabilistic reasoning has emerged which could be used to guide curriculum designers and teachers. Indeed, there was still debate over the type of task to be used in studies with children.

Some Australian studies (Peard, 1995; J.Truran, 1996; K.Truran, 1996) have contributed to the growing body of knowledge, particularly in regards to the identification and clarification of specific probability concepts and research variables. Two significant studies took place simultaneously with the research study being reported here; Jones, Langrall, Thornton & Mogill (USA 1997, 1999) and Watson, Collis & Moritz (Australia 1997) and have made important contributions to the goal of formulating a comprehensive framework for the development of probabilistic thinking.

There is considerable evidence that probabilistic reasoning is linked to cognitive development and that children move through stages in their ability to make mathematically appropriate judgements in situations involving chance. However, the lack of agreement between researchers as to the nature of thinking in each stage and the age ranges that encompass each stage, suggests that there may not yet be an accurate description of the development of probabilistic reasoning.

The key to much contemporary mathematics educational research is engaging children in activities that reveal their reasoning strategies. Once the range of strategies children apply is understood, hierarchies and frameworks that map the development of strategies in relation to mathematical concepts and skills emerge. These frameworks can then be used to inform teaching practice that supports the

development of effective strategies. In probability, the range of strategies used by children has not been sufficiently described to allow the formulation of such a framework.

There clearly exists a need to investigate young children's probabilistic reasoning strategies across a range of ages and to test the notion of three distinct stages of development. The mathematical components of probability; randomness, sample space, likelihood, ratio and proportion, suggest the variables needed in designing tasks that are likely to elicit useful information about probabilistic thinking. The expectations of a range of primary curriculum documents, such as *The National Statement of Mathematics for Australian Schools* (1990), *Principles and Standards for School Mathematics* (USA, 2000,) and *Mathematics in the National Curriculum,* (UK, 1991) set the parameters for the interaction with children. The need to discover individual children's patterns of thinking necessitates an interview style of information gathering where children have the opportunity to physically interact with random generators and talk about what they are doing and thinking.

This study took advantage of the omission of the topic of probability in the New South Wales state curriculum for primary schools. This situation provided the opportunity to investigate children's intuitive understandings of probability concepts. Two research questions arose from the research literature and from trials of task-based interview protocols.

What strategies do children utilise for making judgements in different types of probability tasks?

1. In particular, tasks involving:
   - different types of random generators
   - various ratios in the construction of the sample space
   - different types of comparisons within and between sample spaces and random generators
   - responses in the form of both choices and explanation of reasons
2. Can these strategies be classified into the expected developmental stages of non-probabilistic thinking, emergent probabilistic thinking and quantification of probability?

2 METHOD

A series of five tasks were developed in the form of games using various random generators, to be played by individual children with the researcher. The researcher asked the children to make a series choices and decisions regarding probability and to explain their reasons. Two types of random generators were used, numerical (discrete items in up to 4 colours) and spatial (spinners with up to four colours). The ratios in which the colours were present were varied during the tasks. The children were required to make comparisons within the sample space (e.g. which colour is most

likely?) and between sample space (e.g. which spinner gives a better chance for red?). One task required the children to examine the relationship between a numerical random generator and a spatial random generator (e.g. put a number of objects into the box to match the spinner). The questions asked by the interviewer related to a range of concepts, such as, randomness, sample space, order of likelihood, impossibility and certainty, equal likelihood, simple conditional probability, proportions and quantification of chance.

The children's responses, in particular the explanations for decision-making, for each aspect of each task were classified according to common characteristics and the groups identified as strategies. Responses to questions dealing with the same concepts (e.g. equal likelihood) from different tasks were also gathered together and compared.

Annotated interview transcripts became individual profiles for each child. These were also grouped according to common characteristics and analysed in terms of the strategies identified through the task analysis.

3 RESULTS

Large amounts of data were collected about the specific tasks and the strategies they evoked, but the focus of this report is on describing the developmental stages revealed and confirmed through all angles of data analysis. Three age-related stages were identified, and unlike previous research, two transitional stages were apparent.

3.1 Key Descriptors of Stages

Analysis of the children's responses revealed three age-related stages of development, with two transitional phases, with following characteristics:

3.1.1 Stage 1: Non-probabilistic thinking

The key descriptors of Non-Probabilistic thinking are:

- Average age 5 years 8 months (range*:* 4 years 3 months to 8 years 2 months
- Minimal understanding of randomness
- Reliance on visual comparison
- Inability to order likelihood

The children at this level have an incomplete awareness of the concept of randomness.

Example: Sam, 6 years 11 months: "When you shake it you're probably going to put the red closest to me"

There is no clear connection made between the structure of the sample space and the likelihood of particular events.

Example: Jessica, 5 years 11 months "Because it's a good colour. Sometimes it's on flowers".

Although the children realised that the colour needs to be present in the sample space for that outcome to occur, they usually didn't realise that for an outcome to be impossible there must be none of that colour present at all in the sample space. That is, the children don't realise that even the least likely event can still occur.

Example: Albert, 5 years 7 months [Chose a spinner with a small sector of green as 'impossible for green]. "If it always lands here [points to red, yellow, blue sectors] green can't go".

At this stage there is a strong reliance on visual impressions for making comparisons. In some cases the children were able to identify *most likely* and *least likely* outcomes on a spinner and explain their choices in terms of the comparative size of the sectors.

Example: Jessica, 5 years 11 months "It's got the biggest yellow*"*.

When comparing two 'numerical' sample spaces, inappropriate reasons were used to justify choices.

Example: Samantha, 5 years 2 months "'Cause I won with that one before".

Children at this level are not able to order the likelihood of events in most situations. Equal likelihood is not recognised when presented in either numerical or spatial random generators.

3.1.2 Transition Phase from Non-Probabilistic to Emergent Thinking

Ten profiles contained almost equal numbers of Non-Probabilistic Thinking characteristics as Emergent Thinking characteristics. The average age of children in this phase was 7 years 9 months and the ages ranged from 5 years 9 months to 11 years 1 month.

3.1.3 Stage 2: Emergent Probabilistic Thinking

The key descriptors of Emergent Probabilistic thinking are:

- Average age 9 years 2 months (range: 6 years 11 months to 12 years 2 months)
- Recognition of sample space structure
- Ordering of likelihood through visual comparison or estimation of number
- Addition and subtraction strategies used in comparisons
- Concepts of equal likelihood and impossibility

The children in this stage exhibited recognition of randomness and its relationship to likelihood. For example, they could identify the most likely outcome of a random event and also understood that, because of the element of chance, this outcome may not occur. However, the occasional lapse in judgement still may occur, such as expecting a set of outcomes to conform to a pattern, or believing in their own ability to make a certain outcome happen through their actions or just because they want it to. Some children at this level were sometimes distracted by previous outcomes when making statements about *most likely* outcomes.

The recognition of the composition of the sample space is much stronger at this stage than at the previous stage, though this knowledge is inconsistently used to provide reasons for decision making. The children can both recognise and construct *impossible* sample spaces and *equal likelihood* sample spaces, and display an understanding of the idea of certainty (or otherwise) of specified outcomes.

Jacob, 10 years 7 months "Not necessarily [certain to lose] 'cause if the spinner stops there [on the smallest sector] more than the other colours it could win".

Their thinking includes the use of simple proportional reasoning to make generalisations.

Jacob, 10 years 7 months "Two of each [to make an equal-likelihood sample space]. It could be three or however many you wanted, as long as there's an equal number of each".

Visual estimations are still made and the comparative language of size (same, bigger, biggest, smallest) and number (more, less, none) used to describe the reasons behind probability judgements. Also, the more sophisticated strategy of visualising or finding *half* of the sample space is used to assist with spatial comparisons. The sectors of spinners are seen as fractional parts of the whole spinner.

Example: Jackie, 8 years 6 months "Each of them have a quarter of the colour"

These children experimented with strategies when comparing two *numerical* sample spaces and demonstrated an awareness of the need to consider and compare more than one aspect of the two sample spaces.

Example: Stacey, 7 years 11 months "It doesn't have as many yellow. There's both three red".

The children in this stage are beginning to manipulate numbers to support probability judgements. Additive and subtractive strategies are used to assist in comparisons of numerical sample spaces, but no multiplication or division are applied.

Example: Jacob, 9 years 3 months "There's four more yellow bears than red bears in Jar 2, but there's only two more yellow bears than red in Jar 1".

When attempting to put numerical values on spatial representations, the estimates were made through either *ordering* or *measurement* strategies, which generally lead to numbers that were quite close to the actual proportions.

Example: Adel, 9 years 3 months "'Cause that's bigger than all the others so I put the most in [5 red bears]. That's second biggest so I put four yellow, and those two looked exactly the same so I put the same amount [2 each]".

The children order the likelihood of outcomes through visual comparison (using spatial proportions) or through estimation of number.

Example 1: Adel, 9 years 3 months "You be yellow [car in the race]. It's got more area than blue or green".

Example 2: Jacob, 10 years 7 months "Number 1 [jar] because this one's got too many yellow for the red ones, but this one's got less".

3.1.4 Transition Phase from Emergent Thinking to Quantification

Ten profiles were identified as demonstrating evidence of the beginning of quantification of probability, yet the dominant type of thinking fitted with the characteristics of Emergent Thinking. The average age of children in this transitional phase was 9 years 5 months and the ages ranged from 7 years 6 months to 11 years 8 months.

3.1.5 Stage 3: Quantification of Probability

- Average age 11 years 3 months (range: 9 years 1 month to 12 years 7 months)
- Numerical comparisons
- Doubling and halving
- Proportional thinking
- Quantification of probability emerging or present

The relationship between randomness and likelihood is better understood than in the previous two stages.

Example: Julie, 12 years 3 months "It's [red] got the less part of the pie graph. It's [red] still got a chance because the colour's still there".

The link between the sample space and likelihood is explicitly made at this level.

Example: Serge, 11 years 8 months "There's more red than green so there's more chance of getting red".

These children were more focused on numerical comparisons, including the use of fractions, with both numerically and spatially presented sample spaces.

Example 1: Serge, 11 years 8 months "No, actually they're both the same. That's half of four, and that's half of eight, so they're the same [proportion]".

Example 2: Jack, 11 years 8 months "It's all split up into quarters - everyone's got the same percentage".

The ordering of likelihood at this level of thinking involves more careful estimation, subtractive comparisons, or quantification through multiplication and proportional thinking (mainly involving doubling and halving).

Example 1: Julie, 12 years 3 months "Well, if I put in one of each for blue and green - it's only small. That's twice the size [yellow], so two in there, and four times the size of that [blue], so I put in four".

Example 2: Serge, 11 years 8 months "It's got half the amount [of red than yellow] and it's got more red bears than that one, but this [jar] doesn't even have half [red compared to yellow]".

The quantification of the sample spaces, using a variety of strategies to make comparisons within and between sample spaces, indicates readiness for quantification of probability, though this was rarely observed. There were, however, a few instances of actual quantification of probabilities, stated as fractions or ratios.

Example: Jack, 11 years 8 months "This [jar] is one third [ratio of red to yellow] and this [jar] is one quarter".

The language of probability - words such as 'chance', 'more likely', 'probably' - is commonly used, whereas it was rare amongst children in the other stages.

4 Implications for Planning Teaching

The main characteristics of each of the three stages of development provide some guidance for the design of teaching/learning activities for each age group because they present a sequence of development of concepts and therefore indicate the types of activities that might be appropriate.

The following points of relevance to teaching arise from this study:

- The concepts of more/less likely, most/least likely and equal likelihood are more easily understood than the concepts of impossibility and certainty, the two extremes of probability, suggesting that particular attention should be paid to these latter concepts in teaching.
- Children under 6 years may possess some intuitive notions of probability but these are unstable. As the children depend largely on visual information, the use of hidden sample spaces in activities may not be productive.
- Children around the age of 9 years possess the basic probability concepts and are likely to be responsive to instruction that assists them to develop simple numerical strategies into more sophisticated proportional thinking.
- The concept of one half and the operations of doubling and halving emerged as highly significant mathematical skills. ‘One half’ is used by children as a stable marker or 'standard' to assist with both spatial and numerical comparisons, and for quantification in probability tasks. The application of ‘one-half’ thinking appears to be a crucial step in the development of proportional reasoning and hence the quantification of probability, and therefore should feature in teaching plans.
- There were substantial variations in the strategies invented and tested by the children. The degree of experimentation, inventiveness in novel situations, and the apparently instinctive selection of the simplest strategy, suggest that it would be detrimental to learning to impose general rules and taught strategies.

The suggestions made above in regards to teaching activities, warrant further investigation. The development, implementation and evaluation of sets of teaching/learning activities for each stage of probabilistic thinking could make an important contribution to mathematics education.

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