STRUCTURE SENSE[i]

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This paper presents an initial attempt to describe what is meant by algebraic structure, and by structure sense.Many students in high school, even bright ones, have difficulties with basic algebra techniques, i.e. transformations of expressions (expanding or factorizing), and solution of rational equations. Students often don't know "what to do" and "when to do it". For example: they may have difficulty recognizing when a substitution is called for, such as in the equation

$(x^2-4x)^2-x^2+4x=6$. Perhaps they think that brackets must always be opened as a first step to solving an equation; or they don't notice that the term "x^2-4x" appears twice, once inside brackets, and once multiplied by –1; or they are just not aware that it is possible to operate on a term as a single unit, substituting one symbol in place of the term to obtain a simpler equation. This could be an example of an inability to recognize algebraic structure - a symptom of a lack of "structure sense".

Sense - a "feeling": Number sense can be described as an intuition for numbers that includes such things as an eye for obviously wrong answers, and an instinct for choosing the arithmetic operation needed to solve a given problem. (Greeno, 1991). Arcavi (1994) suggested that symbol sense is a complex "feel" for symbols which would include an appreciation for the power of symbols; a feeling for when it is appropriate to use symbols; an ability to manipulate and to interpret symbolic expressions; a sense of the different roles symbols can play in different contexts. Acquisition of symbol sense is an important goal of teaching algebra.

Structure sense: A common mistake made by many students is similar to the following: they "simplify" the expression $4x+8$ by "dividing through by four". Are they confusing an expression with an equation, expecting an equivalent expression since "dividing through by four" in the equation $4x+8=4x^2$ yields an equivalent equation? It is easy to see why students might be confused by the external appearance of algebraic phrases or sentences. Could this be explained by a lack of "feel" for the underlying structure?

Linchevski & Livneh (1999) coined the phrase "structure sense", suggesting that students' difficulties with algebraic structure are in part due to their lack of understanding of structural notions in arithmetic. They explored and verified the assumption that the algebraic system used by students inherits structural properties associated with the number system with which students are familiar. However their conclusions are based on research on students just before and just after beginning algebra. The structure they examine is the order of operations in arithmetic expressions. They do not discuss structure sense in terms of what it might mean beyond the initial stage.

So far no consensus has been reached on what is meant by algebraic structure, and thus by structure sense. Esty (1992) discussed the grammatical structure of algebraic statements, giving as example three sentences in mathematics, which have apparent (surface structure) similarities: $(x+1)^2 = 5$; $(x+1)^2 = x^2 + 2x + 1$; let f(x) = $(x+1)^2$. Sfard & Linchevski (1994) discussed students'lack of awareness that "strings of symbols" might be interpreted in many different ways, depending on context. From the point of view taken in the present paper, "strings of symbols" form structures, the interpretation of which depends on the context. An awareness of such different interpretations is part of structure sense.

Data from a matriculation examination problem that required the use of algebraic transformations in a purely algebraic context led to the following observations about structure and about students' perception of structure. The expected methods to prove an algebraic identity were categorised according to two different classifications: sub-structure and logic. Each sub-structure was described by external appearance (pattern) and by the operations it "calls for". It was hypothesised which sub-structure would fit most naturally with which logic and the data seemed to verify this. A certain pattern, operations and logic seem to be connected in the students' mind. But does a student's written answer in fact reflect his/her thought processes? For example, many students didn't extract common factors. Was this because they didn't "see" the same terms appearing more than once, or was there some other reason? Students appear to recognize some common factors and not others. Does something in the structure of the expression cause them to recognize certain features? To what extent is recognition dependent on the visual pattern displayed by the item in question? Classification of objects and properties into structures seems to be dependent on the context and on the classifier's personal history. In order to discover what feature of the object triggers a specific response, appropriate tasks must be designed.

What does it mean for a student to have a sense for structure? I suggest that a student who chooses an efficient and elegant method to solve a problem is displaying good structure sense. A definition of structure sense could include an ability to recognise algebraic structure and to use the appropriate features of that structure in the given context as a guide for choosing which operations to perform.

References

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics 14(3),* 24-35.

Esty, W. W. (1992). Language concepts of mathematics. *Focus on Learning Problems in Mathematics 14(4),* 31-53.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education 22(3),* 170-218.

Linchevski, L. & Livneh, D. (1999)Structure sense: the relationship between algebraic and numerical contexts. *Educational Studies in Mathematics 40(2),* 173-196.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics 26,* 191-228.

[i] An expanded version of this article is available from the author at hochfam@bezeqint.net