THEMATIC WORKING GROUP 6

ALGEBRAIC THINKING

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INTRODUCTION

This group was announced as revolving around two major themes to be discussed: the learning and the teaching of algebra. We proposed to prospective authors to address the following issues:

Discussion of the *learning of algebra* both from theoretical and practical perspectives.

The "cognitivist" view of misconceptions was challenged from within its own perspective ("misconceptions reconceived"[1]), but also by alternative theoretical views such as socio-historical or anthropological perspectives. In which sense/s do these approaches differ, complement or contradict each other? How are these approaches reflected in the mathematical topics (and the school level) they choose to focus on, and what are their research findings? How do these approaches relate to historical and epistemological issues? In what sense the three (or more) decades of research on algebra learning under different perspectives can be synthesized? What do we know now, that we did not know 10, 20, and 30 years ago?

The following are some examples of the more specific issues to address:

- evolution of students' algebraic concepts and process,

- algebraic thinking,

- uses of various representations in working with algebraic tasks,

- alternative views on student difficulties,

- the role of technology in the learning processes.

[1] See, for example, Smith, J.P. III, A. A. diSessa and J. Roschelle (1993) "Misconceptions reconceived: a constructivist analysis of knowledge in transition" *The Journal of the Learning Sciences* Vol 3(2), pp 115-163

Discussion of the *teaching of algebra* both from theoretical and practical perspectives.

There has been a lot of emphasis on research on teaching mathematics in general, especially following Shulman's seminal papers[2] published in the 80's. What is the specific progress, which has been made on the issues of algebra teaching? Have the different theoretical approaches to learning (as described above) influenced research on algebra teaching, and if so, how? (theoretically?, practically, e.g. curriculum design principles?) What do classroom studies contribute to the teaching of algebra (again theoretically and practically)?

The following are some examples of the more specific issues to address:

- What is the role of the teacher, the context, different representations, and technology in promoting students' understanding?
- What theoretical frameworks could be used for designing instruction aiming at promoting student learning?
- What curricular innovations could be suggested? What could be promising new ways to teach specific topics in algebra?
- What should be the place of "technical manipulations" in the teaching process?

PLANNING AND CONDUCTING THE WORKING GROUP

As organizers of this Working Group, we decided to stay as close as possible to the spirit of CERME - an environment for the interaction among participants over an extended period of time for the purpose of professional discussions and learning, providing unique opportunities to meet colleagues with whom one does not have a chance to exchange ideas so intensively elsewhere. Our main goals were a) to be as inclusive as possible, and b) to offer a setting in which the main ideas of **all** the papers submitted are honored and reflected upon. Thus, we forwent the usual publication process (sending the papers to reviewers and then accepting/rejecting them or accepting pending revisions). Instead, the four of us read the twenty-four papers submitted and we accepted all of them as they were. But, we invested time and work in a) grouping the papers into four main sub-themes[3] and b) proposing a main idea from each to be posed as an issue for discussion (see next section below).

Therefore, we proceeded to prepare leading questions in order to carefully structure the discussion and provide coherence to each sub-theme. A main question was drafted for each of the papers submitted (according to the main issue raised), and sent, together with all the papers, to all the participants in advance. During the meeting, authors (as well as the participants at large) had the opportunity to refine or

[2] See, for example, Shulman, L. S. "Those Who Understand: Knowledge Growth in Teaching." *Educational Researcher*; v15 n2 p4-14 Feb 1986.

[3] Our proposed classification was only one among many other possible ways of grouping the papers. When papers were relevant to more than one sub-theme, we invoked them accordingly during the discussions.

extend the questions proposed, to supply others, and to clarify or elaborate on certain points of their own papers (or experiences) in order to "fuel" the dialogue. Since the number of papers in each sub-theme was not the same, the time allotted to the discussions for each sub-theme varied. The group as a whole and also small subgroups (in parallel) discussed all the sub-themes.

The group was very active in discussing ideas, and authors of papers went away with some new food for thought about their papers – which eventually could be re-written and submitted either to other conferences or to journals. In order to have a proper representation of the work done, we decided to publish in these proceedings:

- the "structure of the sessions" document, and
- a two-page abstract of a version of the papers submitted.

In the following, we provide the description of the sub-themes and the questions for discussions as they were sent to the participants, and the abstracts the participants wrote after the conference taking into account both their original papers and the discussions in the group.

SUB-THEME 1: THEORETICAL ISSUES AND APPROACHES

The papers grouped under this category addressed theoretical foundations as their main topic. As with any theory, we proposed to analyse their strengths and weaknesses regarding: description, explanation, prediction and scope (e.g. what is this a theory of?). The following were the questions we proposed to discuss as emerging from the papers.

- Cross-theoretical fertilisation seems to rely on the assumption that no single theory or approach may be able to capture the complexities of algebra as an intellectual activity. In what sense the issues which concern this group can benefit from the bridging of the fields of epistemology and didactics? What about ontology?
- What knowledge is required for students in order to become competent in algebra? Would the three orders of knowledge – a) contents, b) rules (semiotic rules, rules governing 'truth and validity', rules of the game) and c) metaknowledge - provide a comprehensive view of the pre-requisite knowledge underlying algebra?
- What may be the role of the three main intellectual functions: orientation, executive control and correcting, in the learning of algebraic skills? To what extent these functions may be helpful to describe, explain and predict the learning of skills?
- "Institutionalization" seems to be a crucial stage in learning. What exactly does this construct entails? What is/should the teacher role be in this process?

SUB-THEME 2: UNDERSTANDING AND NURTURING INCIPIENT ALGEBRAIC THINKING

It is often claimed that arithmetic is the epistemological and curricular predecessor of algebra. This raises several issues to consider, for example, can algebraic ideas (or algebraic modes of thought, or the intellectual predispositions to algebra), be "seeded" during the study of arithmetic? If not, why? If yes, we need to elaborate on a) what aspects of algebra? b) how much of it, c) when?, d) at what cost?, e) what may the teacher role be?. More specifically, we propose to discuss in depth the meaning and implications of the following ideas

- "algebraic awareness", "algebraic thinking", "dimensions of possible variation"
- "potentially algebraic task"
- "purpose", "transparency"
- "consistency of strategy"
- "measurability of algebra readiness"
- "Multi-directional mathematical thinking"
- "algebraic babbling"
- "arithmetic vs. algebra interplay"
- Grade 7 as a starting point?
- role of teachers in the use of technological tools

SUB-THEME 3: REPRESENTATIONS, CONTEXT, MODELING

It is claimed that a) symbolizing, b) generalizing (and contextualizing), and c) selecting and switching representations are among the most important intellectual activities of algebra.

- To what extent providing an operational context (regardless of its artificiality) as an aid to understand and generalize, may help to learn some of the most essential features/activities of algebra? In what sense can a context be instrumental? Would any context be useful? Why or why not?
- What may the inherent difficulties and benefits of incorporating modeling as a central activity in the teaching and learning of algebra in secondary schools?
- Sometimes changing one representation for another highlights concepts and strategies which were not apparent before. What can we learn from the use of graphs when dealing with symbolic manipulation of equations? And why is it important?
- How can algebraic approaches enrich geometrical knowledge and viceversa? How can pictorial features, if at all, support (or hinder) symbolic competence?

SUB-THEME 4: INTERESTING (AND LESS FREQUENTLY ADDRESSED) TOPICS IN ALGEBRA

The learning of some topics and ideas of algebra are less frequently discussed than others. It is fortunate that five papers submitted to the Group can be classified under this category and thus providing us with an opportunity to address them in detail.

- What are some of the limitations, imperfections, or the particular ways in which technological tools operate? In which ways can these be harnessed, rather than rejected, to foster meaningful learning, especially in the area of "syntactic" knowledge?
- Proficiency with solving equations may overgeneralize to the solution of inequalities. When observed with appropriate tasks, this overgeneralization reveals difficulties with the very notion of inequalities. What is the nature of those difficulties? Which theoretical model can we choose in order to make sense of these difficulties? Are these models exhaustive, complementary, contradictory?
- Theoretical frameworks to identify and interpret student difficulties with inequalities may help us understand the sources of those difficulties. However, how, if at all, can these frameworks be harnessed to support instruction in inequalities? It would seem that a different kind of theoretical approach may also be needed to study the teaching (rather than the learning) of inequalities. If so, what can this frames be? And, how and where the focus on learning and the focus on teaching meet and complement each other?
- Much of algebra is about "structure". Also, being knowledgeable in a certain domain consists of "having a feel" for it, or "sensing" it. What may the construct "structure sense" mean and how can it be useful to understand learning and support instruction?
- To what extent and why students common syntactic difficulties may persist throughout secondary school? How may the teacher's role be in studying and coping with these difficulties?

List of contributions

List of Thematic Groups