PREFERENCE OF DIRECTIONS IN 3-D SPACE

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Abstract:

In the course of a previous research dealing with basic concepts like ***straight line*** *and* ***plane*** *and concepts of interrelation between them like* ***perpendicular*** *and* ***parallel*** *in 3-D geometry, I could clearly see that students tended to prefer certain typical directions, and to disregard others. These preferred directions were not necessarily the obvious ones – horizontal and vertical. They were often other directions, chosen relative to the given directions. Some of the students focused only on those preferred directions and were not able to see other options. For those students, awareness of their choices and the possible reasons for those choices could considerably improve their visual ability and flexibility in 3-D space. This study is an attempt to locate, analyze and classify those preferred directions among prospective teachers in a teacher education college.*

Introduction:

Basic concepts like *straight line* and *plane* and concepts of interrelation between them like *perpendicular* and *parallel* are usually known in the context of plane geometry. The extension of these concepts to three-dimensional space does not change their basic meaning, but it enlarges the variety of possible relationships between them. Awareness of these new possibilities requires an ability to visualize, which is often quite limited in students used to seeing everything in a plane.

In previous research, hereafter referred to as "the broad research", there was an attempt to examine, locate and analyze sources of misconceptions about visualizing and understanding the previously mentioned basic concepts (Cohen, 2000). One of the findings of the broad research was that when dealing with interrelations in 3-D space, students tended to prefer certain typical directions and disregard others.

For some of the students the preferred directions were their first choices, but they had no difficulties seeing other directions as well. However, there were students who focused only on those preferred directions and were not able to see, or at least, did not look for other options. Thus, they missed the variety of possible relationships between the basic concepts. For those students, awareness of their choices and the possible reasons for these choices can considerably improve their visual ability and flexibility in 3-D space.

The study discussed in this paper is an attempt to locate, analyze and classify those preferred directions among prospective teachers in a teacher education college and to examine their influence on mental images of relationship concepts.

Theoretical background

Most research concerning spatial ability in 3-D deals with solids and their representations. The emphasis of this research is on spatial *relationships* between geometric concepts not in the context of solids. Those relationships are treated in this paper as *figural concepts* as defined by Fischbein (1993). Fischbein sees geometric figures as mental entities, which possess conceptual and figural properties simultaneously. They have a strong image component, but they also have a conceptual component, which controls the aspects of formal definition, logical organization and abstraction. The relationships dealt with in this paper, although not being "geometric figures" in the ordinary sense, match this description. The development of a figural concept, according to Fischbein, generally is not a natural process. His advice is "to create didactical situations which would systematically ask for a strict cooperation between the two aspects up to their fusion in unitary mental object" (ibid p.161). The tasks in this study provide such didactical situations, and enable us to examine how those figural concepts develop.

The mental image associated with the relationship concepts requires visual abilities, such as *perception of spatial position* and *perception of spatial relationship*, as mentioned by Del Grande (1990). But, the visual process can be improved significantly by connecting it to a reasoning process (see Duval, 1998) and adding an analytical judgment that takes into account the possible misleading of visual judgment (see Hershkowitz 1989a). For example: The ability to see the infinitely many possible lines, which are perpendicular to a given line in the same point, is part of a conceptual understanding and not only a visual perception. For some people, the trigger for understanding this unexpected phenomenon can be their visual discovery of more than one line. For others, it can be the analysis of the meaning (definition) of "being perpendicular". Both will have to resolve conflicts (such as the fact that it seems to contradict a known theorem) and combine the two aspects in a new construct: the improved concept image (as defined by Vinner, 1983) of line $\perp$ *line*.

The preferred directions may also be connected to the prototype phenomenon, treated, among others, by Hershkowitz (1989b).

Concepts involved in the study

The tasks given to the students involved basic concepts: *point, straight line* (hereafter called *line*), and *plane,* and relationships between them: *parallel* and *perpendicular*. Usually, students' intuitive basic meaning corresponds to the mathematical meaning, but they have difficulties seeing the variety of possible relationships in 3-D space. It must be taken into account that unlike in a plane, here we have 3 different concepts of relationships for each of the terms *parallel* and *perpendicular:*

	line || line		line || line
parallel	line || plane	**perpendicular**	line || plane
	plane || plane		plane || plane

(In this study *line⊥ plane* is seen as a symmetrical relation although psychologically there is a difference between *line⊥ plane* and *plane⊥ line*)

Classification of preferred directions

As a first step, I would like to suggest a classification that may help us analyze students' tendency to prefer specific directions in specific situations. In the broad research, I observed over 200 students, usually during their discussions, over a period of 9 years. I noticed very clearly that they almost always chose typical directions when they illustrated interrelationship in 3-D space. It struck me that those choices were not at all random, nor were they necessarily the obvious ones – the horizontal and the vertical, but rather relative to the given directions. In fact, I found myself able to predict which directions the student would chose in a given situation. It became more and more clear that there were some rules governing those choices. This study was designed with the aim of identifying those rules and trying to understand their possible sources.

In General, we can see four **types of preference** for directions:

A. **gravitational**: Preference for horizontal or vertical directions (relative to the earth)

B. **plane thinking**: focusing on one plane at a time

C. **no conflicts**: Preference for "convenient" directions, in which there is no conflict between different concepts related to the same term: *perpendicular* or *parallel*.

D. **typical:** Preference for typical directions, chosen relative to the given directions:
- "with a balance", ◆ "mutually perpendicular", or ◆ "towards me"

(See explanation below, in section on analyzed examples.)

If the types lead to different answers, one type supersedes the other. For instance, type D is usually stronger than type A.

The study

The aim of this study was to observe students' choices of directions in various situations closely, and to try to analyze the reasons and sources for those choices. Seventeen students in a college of education in Jerusalem were interviewed and videotaped. Seven of them were interviewed singly, six in pairs and four were observed while working in two pairs with my "provoking" questions. All, except one, were in their second or third year in college, preparing to be mathematics teachers in elementary or junior high school. For most of them (13), the interview was the starting point of a course dealing with various topics involving visualization in 3D space. All the concepts mentioned in the interviews were taught systematically in the following lessons together with reflection and discussions about the students' own performance and conflicts.

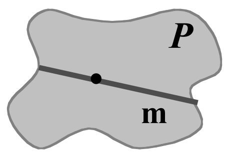
During the interviews, the students were given "direction tasks" (15 -19 tasks) and were encouraged to think aloud while performing them, defend their answers and opinions, and discuss them (if not working alone). The direction tasks were designed to examine students' first choices of directions in various situations, and check whether they were able to see other options.

Manipulative visual aids were used to illustrate *planes, straight lines* and *points*. For planes, students used very thin flat plastic surfaces, with random "cloud" shapes[1]. For straight lines, they used straws or thin rods. Some of the 'planes' had holes that enabled the 'lines' to pass through them. (Hereafter, when a clear distinction between the mathematical concepts and the visual aids representing them is needed, inverted commas will mark the later.) Little pieces of sticky plasticine were used to mark points or to attach 'lines' and 'planes'. Of course, I had to make sure that the students understood that those aids only represented infinite and "no width" planes or lines.

All the tasks were performed with 3D manipulative materials and not with drawings, because visualizing relationships from drawings requires additional abilities not relevant to this specific study.

Analyzed examples from the interviews:

Task 1: *Given a 'line'* ***m*** *situated in a 'plane'* ***P,*** *place at a given point of* ***m*** *another 'line'* ***b****, perpendicular to* ***m.***



We hold a 'plane' with a 'line' ***m*** stuck to it in a general direction (the term *inclined* or *general* will be used whenever it is neither horizontal nor vertical). The student has to place another 'line' ***b***, perpendicular to the line ***m***. After he puts ***b*** in his chosen direction, we ask if he can place it differently. If he does so, we continue asking if there are any more options. When the student is sure of his answers, we change the position of the given plane and line, and ask again.

As their first choice, all of the students chose one of the following two options:

1. They chose ***b*** to be **perpendicular** both to ***m*** and to the plane ***P***.
2. They chose ***b*** to be **in** the plane ***P***.

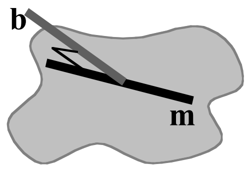
[1] In the some of the illustrative drawings of this paper, the "planes" were drawn rectangular for a better 3D feeling.

Many of those who chose one of these options gave the other option as a second choice. Only one of them even saw other options! (Some of them discovered the infinite number of options after performing further tasks in the interview.)

In a related question in the broad research questionnaire, 73% of about 300 students thought that if ***b*** was perpendicular to the line, it had to be perpendicular to the plane as well. Like the students in the interviews, most of them were probably trapped in their preferred direction 1, and could not see other options. (Those who saw option 2 are included here among the 27% of correct answers!).

The choices of direction in this task are not too difficult to explain: Those who see only **option 1** are probably unaware of the 2 different concepts related to the word *perpendicular*. In order to see other options one has to understand the difference between *line_line* and *line_plane*.

When it is not perpendicular in one way (e.g. ***b*** perpendicular to the line ***m***, but not to the plane containing ***m***) students conceive it as "not perpendicular". When examining such an example, most of them insisted that ***b*** was not perpendicular to ***m.*** The fact that ***b*** was not perpendicular to the plane prevented them from seeing the perpendicularity to ***m***



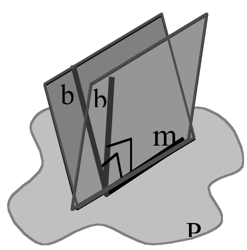
The preference in this case is probably of the **no conflicts** type: an unconscious choice of ***b*** perpendicular in every possible sense and not causing any conflicts between being perpendicular to the line and not being perpendicular to the plane.

The choice of **option 2** is probably explained by the tendency to focus on one plane at a time, and looking at it as a plane situation, thus being of the "**plane thinking**" type.

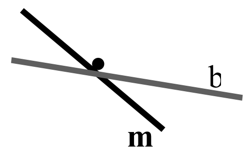
Among other positions, one was "***m*** *horizontal,* ***P*** *inclined"*, but this did not change students' answers. They did not choose ***b*** to be vertical as they would if the plane were not present, but rather perpendicular to the plane.

It seems that the given plane is conceived as a new floor (reference plane) **instead** of the horizontal (gravitational) one.

It is interesting to note that the only student who saw an infinite number of options was herself a "plane thinker": First, she chose option 1. When she was asked for other possibilities, she hesitated, took another plane, placed it so that it included ***m*** and found another line ***b*** perpendicular to ***m,* in the new plane**! Then, she changed the position of the new plane, each time finding another ***b*** in it, and said: "*there are infinitely many possibilities*". Clearly, she was creating a plane situation for herself.



Task 2: *Given two intersecting 'lines',* ***m*** *and* ***b,*** *place another 'line'* ***c****, perpendicular to* ***m****, at the intersection point.*



This task is quite similar to the previous one, but with no given plane, and therefore less sidetracking. I was amazed to discover that students behaved as if a plane determined by the two given lines was present in their mind. In fact, we can detect here the same two options as before, in addition to a third option, in few cases:

1. They chose **c** to be **perpendicular** to plane determined by **m** and **b**.
2. They chose **c** to be **in** the plane determined by **m** and **b**.
3. They chose a preferred direction referring only to **m,** and not to **b**. (An elaboration of those directions in general will be given in task 4.)

None of the ten students who did the task saw more than two options, usually 1 and 2, in the first position (*b and m inclined*).

When given the position "*m horizontal, b inclined*", two pairs of students discovered the infinite number of possibilities (but not easily), and then were able to see it in general positions as well. It is not surprising that the discovery occurred only in pair interviews, as a result of comparing their different choices for the same task.

Here is an example of such a process of discovery: Shay and Yaniv, after finding only options 1 and 2 in a general position, were given the "*m horizontal, b inclined*" position.

Shay shows option 1 and 2 with a rod. Then Yaniv shows option1, and option 3 – a horizontal line, that is, a line perpendicular to ***m*** *in a horizontal plane.*

Shay*:"It is really only approximate."*

Teacher *(to Yaniv): "Are there any more?"*

Yaniv*: "No, only two"*

Shay*:"May I?" (He takes the rod and chooses options 1 and 2)…"No, there are only two"*

Teacher*: "I want you to show me again - each one his own lines."*

Shay demonstrates again, very clearly, options 1 and 2. Yaniv demonstrates option 1 (not very clearly) and the horizontal line, which is distinctly different from Shay's choices.

Teacher *(to Shay): "Is it the same line as yours?"*

Shay*: "It is exactly the same line. It is only that his angle is not accurate." (He moves Yaniv's 'line' to option 2)*

At the teacher's request, Yaniv again places his horizontal line.

***Teacher** (to Shay): "Don't try to correct him, put another rod where you think is right." (He places one so that now they have in front of them both option 2 and the horizontal line.)*

***Teacher**: "Are they the same line?"*

***Shay**: "No, but his angle is not a right angle, it is not perpendicular."*

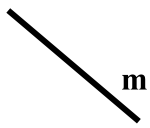
***Yaniv**: "That's how **I** see the right angle."*

***Shay** (looks carefully, and suddenly smiles): "Just a minute." (He starts turning the rod to different positions around **m**): "This is perpendicular, this is perpendicular, this is perpendicular…360°."*

*…Shay was still not sure if all those lines were legitimate when **b** was present, but when asked: "how many lines answer the given constraints?" he replied: "Infinitely many".*

This episode is a fascinating example of an observable process of learning. In brief: After trying to solve the problem with his existing construct (relating the differences to inaccuracy so that his structure of "only two perpendicular" remains), Shay becomes aware of the conflict (when he sees the two lines at the same time, remembering the third one). Only then, he needs a new construct, and suddenly has the insight of perceiving the infinite number of possible directions.

Task 3: *given a 'line' **m**,*
*place a 'plane' that includes **m**.*



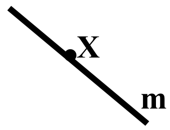
This task shows that even when given only one line, from students' behavior we can detect the presence of preferred planes in their mind. All of the students chose one or two of the following options as a first choice. Five did not see any other option, and most of the rest discovered the infinity of possible directions only after a long hesitation.

The typical choices in this task were three:

1. Choosing the **balanced plane -** a plane that has the same angle with the horizontal plane as the line. (To conceive this, imagine leaning a flat board on an inclined rod, as if it had risen from the horizontal floor by raising one side of the rod)
2. Choosing the **vertical plane** (which is also perpendicular to the balanced plane.)
3. Choosing the **towards me** plane. (a plane for which the line of intersection with the horizontal plane is transversal (perpendicular to the line from the eyes forward).

These preferences are of the **typical** type, and have a strong influence on other tasks.

Task 4: *given a 'line'* ***m****, with a 'point'* ***X*** *on it, place another 'line'* ***b,*** *through* ***X****, perpendicular to* ***m****.*



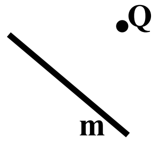
In the first general situation, nine out of fifteen students did not see more than two possibilities (six of those nine saw only one possibility). All of them gave one or two of the following typical directions (all perpendicular to ***m***):

1. a line **perpendicular** to the "balance plane"
2. a line **in** the "balance plane"
3. a line **in** the "towards me" plane
4. a line **perpendicular** to the **"**towards me**"** plane (Only once)

In subsequent situations, many students discovered an infinity of possibilities. Some of them discovered this when solving the special case of ***m*** being vertical, because in that case there is no "balance plane" (any plane is as "balanced" as the other, because the angle with the horizontal plane is 90^0). This fact makes it easier to see all possible directions for **b**. On the other hand, the "towards me" plane is very "attractive" to students, and those who stuck to options 3 or 4, found only one or two possibilities (one towards them and one to the side). In pair interviews, there were cases of students discovering the infinity of possibilities when comparing their choices. In all the above cases, as soon as students discovered the infinite number of possibilities in one situation, they went back and corrected their previous answers.

In a related question in the broad research questionnaire only 15% out of about 300 students answered the question: "How many lines through **X** are perpendicular to **m**?" correctly. Most of those who answered incorrectly thought there were one or two lines. Because this did not take place during a discourse situation, they probably stayed trapped in the preferred directions and did not discover the infinitely many possibilities as most of the students in the current study did.

Task 5: *given a 'line'* ***m****, with a 'point'* ***Q not*** *on it, place a 'plane', containing* ***Q****, and parallel to* ***m***



Here again at the first stage most of the students saw only one possibility, which was always one of the following typical choices.

1. A plane perpendicular to the plane including the line and the point. (The selected plane is **mutually perpendicular** to this **reference plane**.)

2. A plane which is **parallel to one of the preferred planes** containing the line: **a)** to the *balanced plane*, **b)** to the *vertical* one, or **c)** to the *towards me* one.

As before, most of the insights concerning the infinite number of possible parallel planes took place subsequent to a comparison of different answers. In one pair, for example, one student chose option **2a** and the other chose option **1**, (both were sure theirs was the only possibility). In their effort to understand one another, they felt the need for a clear definition of "*plane || line*". Once I gave them the definition, they saw all the possibilities.

The last example was not the only case of a need for a definition. Several times throughout the interviews, the students felt the need for conceptual organization of the concepts describing relationship, and could not rely only on their mental image. This can indicate the development of those relationships as **figural concepts.**

Another example of this development was observed in another task, in which they had to decide whether vertical planes, like those in figure 1, were perpendicular to the line. In many cases, the students looked for the meaning of *plane line*. The only relevant repertoire they had so far was the "right angle test" which works when checking perpendicularity between two lines. It is not surprising that they tried to apply it here, sometimes "proving" the perpendicularity by putting a vertical rod and showing the right angle with a rectangular piece of paper (figure 2).

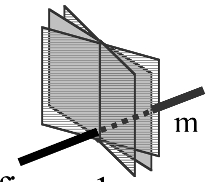


figure 1

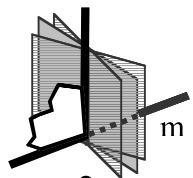


figure 2

This sometimes created a conflict between their (mistaken) conceptual meaning and the mental image ("it does not look perpendicular"). Different students reacted to this conflict in various ways depending on whether they were analytical types or visual types (see Krutetskii, 1976). However, the realization that either their method of checking or their visual image was wrong, was a big step towards the understanding of the difficult definition of *plane _ line.*

Conclusions

As illustrated in the tasks above, we can find regularity in students' behavior that can support the above suggested classification. The types of preferences depend, of course, on the given situation. Thus, the **no conflict** type often occurs when the task involves such a conflict (as in tasks 1 and 2). The **plane thinking** type can occur if it is possible to have at least one answer when reducing the situation to a reference plane – the one determined by the given or one of the preferred planes. When it is not possible, we often see the **typical mutually perpendicular** type – a choice of a plane perpendicular to the reference plane (as in task 5-option1). In the **typical** type, we can postulate the creation of new reference directions in the mind that replace the gravitational ones. In task 3, for example, we saw different kinds of typical preferences that affected performance in tasks 4 and 5–option2.

In addition, we can identify critical points in students' learning processes in the above analysis. Many times, students had sudden insights after becoming aware of conflicts arising from solving the same task in different situations, being exposed to different answers given by other students, or making a connection between tasks. A detailed analysis of these processes could be of great interest, but is outside the scope of this paper.

Moreover, a follow-up study examines the influence of student awareness of their own behavior on their future performance.

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