ELEMENTARY GEOMETRY SPLIT INTO DIFFERENT GEOMETRICAL PARADIGMS

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Since several years, we have developed a scientific investigation on the teaching of Geometry, especially for pre-service teachers. This research is based on a specific theoretical frame. We have chosen to give some elements of our frame and we hope to present more consistent examples of our empirical studies during the working group. For us elementary geometry appears to be split into three various paradigms: natural geometry (Geometry I), natural axiomatic geometry (Geometry II) and formalist axiomatic geometry (Geometry III). We confront our approach with the approach of van Hiele, which to date is not used in France. We obtain a synthesis that we currently study in teachers training.

During their school career, students are faced with different mathematical worlds, at least with a numerical one and a geometrical one. In the numerical world, the objects (e.g. the numbers) are represented by "abstract signs" (juxtaposition of digits), that do not evoke the quantity they refer to. In contrast to this, in the geometrical world representations of objects often remain spatial objets. And in fact, the way is very long from a real spatial object to the notion of "figural concept" described by E. Fischbein (1993).

Well known researchers, like van Hiele (1986), have based a pedagogical approach to Geometry upon the development of the conception of the figure and of its processing. Students come along from a global and perceptive approach to a structural way to see Geometry. The crucial point of this development is the appearance of deduction, which allows the transition from "seeing to knowing" (Parzysz 1988)

For us, this way to Geometry is to a great extent correct but too strictly linear and univocal especially if we want to understand the obstacles met by adults who want to become teachers. Indeed, after Bachelard[1] and Koyré[2], several thinkers have shown the illusion of a peaceful evolution of scientific concepts even in mathematics. A kind of culmination of this conflicting view of the history of the ideas is reached with Kuhn's works. For him, there are scientific revolutions that replace the old paradigms with new ones[3]. We retain and develop this idea of different paradigms for Elementary Geometry. Before presenting these paradigms, let us begin with an example.

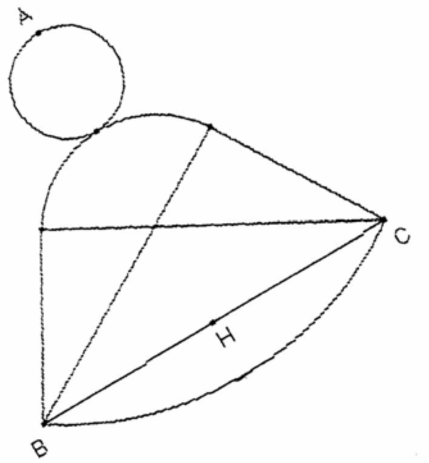
1. Rouen's Bell

The problem was given to students in a pre-service teachers exam in Normandy.

[1] Bachelard G (1938). *La formation de l'esprit scientifique*. Paris :Vrin 1983 (transl Formation of the Scientific Spirit, Clinamen P)
[2] for instance Koyré A (1957) *From the closed World to the Infinite Universe* John Hopkins Baltimore.
[3] For the case of mathematics, see Gillies Ed (1992) *Revolutions in Mathematics*. Oxford University Press

These students wish to become teachers in primary school; they have finished university studies but not necessarily in mathematics.

The bell: We wish to enlarge the figure below (ABHC) into (A'B'H'C') so that the length A'H' is the double of AB.

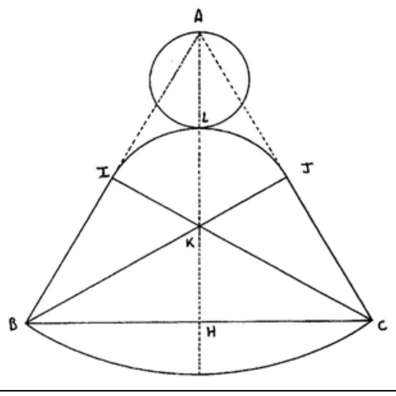


1) Make this enlargement with ruler and compass. Leave the lines of construction visible.
2) Some pupils say that the area of the final figure is four times bigger than the one of the initial figure. Are there correct? Justify your answer.
   If they are wrong, find the exact proportion between the two areas.

The drawing given during the exam is made with CABRI and the candidates must draw the figure with ruler and compass on white paper.

1.1 On the analysis of the figure

No hypotheses are explicitly given; students must find properties needed for the construction at different scale. Immediately the problem requests a perceptive analysis of the drawing, based upon the intuition in its first meaning: apprehension of an object by vision. This intuition is related to a first typology of the geometrical objects depending on knowledge of the individual who proceeds to the analysis.
Let us give a certain number of the possible hypotheses. Let I, J, K and L be the points in the drawing below (not given on the original figure).



H1 A is on the straight lines (BI) and (CJ)
H2 H is the midpoint of [BC]
H3 Angles IBC and JCB are equal to 60°.
H4 Angles BIC and CJB are right angles.
H5 Arc IJ is an arc of a circle of centre K and radius KI.
H6 The circle of diameter AL is tangent in L to the arc IJ.
H7 BC is an arc of a circle of centre A and radius AB.

1.2 On the validation.

How can one validate all these hypotheses? Here we meet with two levels related to two different conceptions of Geometry. In the first level, it is allowed to use

measurement tools and to experiment in the sensible world. In the second one, reasoning relies on the mathematical properties of the abstract geometrical figure.
In the sensible world, the following tools play the main role: the ruler to verify co-linearity, the set square (with angles of 90° and 60°[4]) to check the measure of angles and the compass to confirm assertions about arcs of circles. In fact, in the last case, the use of the compass invalidates H7: A is not the centre of an arc from B to C; the respective centre is the midpoint of the segment [AL].
In the world of geometrical figures, we have common configurations like equilateral triangles and, in this world, ruler and compass define the set of figures, which can be constructed. In the elementary school, this set is not very large and it gives important information about the relations underlying a figure. In our example, it leads to think that angles measure 60° and curves are actually arcs of circles.

1.3 On the construction.

The effective construction of this drawing depends on the tool-kit that is used. If we keep the tools used to check the hypotheses on angles and co-linearity, the set square plays a fundamental role. Indeed, it is easy to construct the bell: draw AH, then the perpendicular line to (AH) in H and drag the 60° angle of the set square, so you obtain the equilateral triangle. In that case, the problem is solved in a homogeneous paradigm where all the devices act in the sensible and measured world. We call this first paradigm where reasoning is naturally close to experience and intuition: natural geometry (Geometry I).
But the exercise asks for a ruler and compass construction. In that case, reasoning on the drawing is not enough: we must connect figures with standard constructions using mathematical properties. According to the chosen construction, it would be necessary to apply Thales' theorem or properties of medians in an equilateral triangle. The paradigm has changed and a new Geometry appears that favours different ways of reasoning and a new link to experience and intuition. We call this new one natural axiomatic geometry (Geometry II).
In this exercise, the change of paradigms is not explicit and causes some sort of misunderstanding. The problem is given in Geometry I and the test givers expect a solution in Geometry II. This confusing play between two paradigms may be obvious for an expert but not for a lot of students: we think that it is useful to make explicit the existence of these different paradigms, above all in teachers' training.
In this article we try to advance in the understanding of the complexity of the geometry.

2. Three geometrical paradigms

In France, the term Geometry is present in all the mathematics curricula from kindergarten to secondary school and university. Obviously it cannot have the same signification: the drawing, for instance, does not play the same role, the figure can even be an obstacle to certain type of geometry (Parzysz 1988), it can also disappear

[4] In France there are two sorts of set square: one with angles 90°, 30° and 60° and the other, less usual, with angles 90° and 45°.

in other problems which favour the use of vectors. Our research aims to better understand the different meanings determined by the same term of Geometry. In this paper, we only consider Elementary Geometry defined as a theory of space, which tends to represent the local properties of the real space. Its more elaborate form is $\mathbf{R}^3$ with the structure of a Euclidean space.

Our research[5] puts in evidence three different paradigms, what brings us to distinguish various forms of geometry. To clarify these paradigms we used the forms of knowledge of the space put in the evidence by Gonseth (1945-1955): intuition, experiment, deduction. We revisited them in the light of recent contributions of the historiography of mathematics and also in a perspective of teaching, which gives a different sight on this knowledge.

Geometry I (Natural Geometry). The source of validation is the sensitive. It is intimately related to reality. Intuition is often assimilated to immediate perception, experiment and deduction act on material objects by means of the perception and the instruments. The backward and forward motion between the model and the reality is permanent and allowed to prove the assertions. For example, dynamic proofs are accepted in this Geometry.

Geometry II (Natural Axiomatic Geometry). The source of validation bases itself on the hypothetical deductive laws in an axiomatic system. A system of axioms is necessary but the axioms are as close as possible to the intuition of the space around us. The axiom system can be uncompleted, but the demonstrations inside the system are necessary requested for progress and for reaching certainty.

At last, **Geometry III (Formalist Axiomatic Geometry**). In this Geometry, the umbilical cord is cut between reality and axiomatic: axioms are not any more based on the sensitive. The system of axioms can be without any relation to reality, what Wittgenstein (1918) illustrated by the sentence: *« Les axiomes d'une géométrie peuvent ne contenir aucune vérité.* »[6]. The type of reasoning is the same as inside Geometry II, but the system of axioms is complete and independent of its possible applications to the world. The only criterion of truth is consistency (i.e. absence of contradictions).

Our fundamental principle is that the various proposed paradigms are homogeneous: it is possible to reason inside one paradigm without knowing the nature of the other. Students and professor, and it is a source of educational misunderstanding, are not necessarily situated in the same one. We summarise different aspects of various 'Geometries' in the following table.

[5] The theoretical frame which we developed leans on an epistemological approach, based on the study of philosophic, mathematical and didactic texts, as well as on mathematicians' papers on geometry. Our research is not only theoretical but also empirical and we have verified the existence and problem set by geometrical paradigms all along the school career and in the teachers training (Houdement & Kuzniak 1996, 1999).

[6] Wittgenstein L (1964) 1975 *Philosophische Bemerkungen*. XVI.177 Gallimard p205. "*The axioms of a geometry can contain no truth".*(translation Houdement&Kuzniak)

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near of the Real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms.
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties et demonstration	Demonstration and links between the objects. Structure.

3. Look at a drawing through our three paradigms or the role of the drawing.

Let us consider the well-known problem of the construction of a triangle with the length of its three sides given, for example lengths are 4 cm, 8 cm and 10 cm.
This problem can be given to young students, for instance if they dispose of many different sticks of the three lengths. A first natural solution takes place in Geometry I. The same problem can be given later using ruler and compass. The students realise an experience in plane and the task is accomplished if the triangle exists really on the table or on the paper. A closer look can show that certain triangles (i.e. certain combination of lengths) are 'strange' or that others do not exist. Consequently, some questions will emerge: does the triangle (4, 4, 8) exist or not? Why is it impossible to draw the triangle (4, 4, 10)? In Geometry I a deduction linked to an experience can solve the second question: the length of a side is longer than the sum of the two others.
But the first question conducts to the general question of existence of a triangle: then it is necessary to make a decision and to introduce a precise definition of a triangle. This general problem of existence of a triangle with its three lengths can be resumed by "If A, B and C are three points in a plane, the inequality AB $\leq$ AC + BC is always true." But this affirmation is an axiom that means a point of departure into Geometry II. In this way an experience in Geometry I can contribute to give sense to axioms in Geometry II.
Another interpretation can be offered in Geometry III and produce the Chasles theorem (about the sum of vectors): "If A, B and C are three points in a plane, the three vectors verify the property: vector (AB)=vector(AC)+vector(CB)". In this case, the inequality on the lengths is a consequence of a calculus true in a wide variety of spaces and not specially related to our real space.
It is easy to see that the same physical object (the drawing of a triangle) could permit different types of thinking depending on the type of questions (the geometric

paradigm) it can help to answer. The first change of paradigm, the passage from Geometry I to Geometry II is really sensitive because it is the first time in mathematics that the mental perspective on the object has to change drastically, without any 'visual' change, symbol or pictorial aid.

4. Confrontation with the approach of Van Hiele.

To clarify and to deepen our paradigmatic conception of Geometry, it seems helpful to connect our vision with that of Van Hiele. The way we follow was introduced by Parzysz[7], the first attempt of a synthesis of our approach with that more classical one by Van Hiele.

To be clear, we roughly summarise the Van Hiele levels:

- Level 0, visual level. Geometrical figures are recognised by their shapes; the student recognises the external form, but cannot justify. Van Hiele speaks of "spatial thinking" (Van Hiele 1986).
- Level 1, descriptive level. Properties of the figures permit to recognise them. The student possesses a network of relations on the subject. Van Hiele speaks of "geometric spatial thinking".
- Level 2, informal deductive level. The third level is a theoretical level that studies logical relations between properties of the figures. This level needs a new language (definitions...). Van Hiele speaks of "mathematical geometrical thinking".
- Level 3, axiomatic deductive level. This level is a formal logical one, a study of the nature of relations between certain theorems inside an axiomatic theory. Van Hiele speaks of "logical mathematical thinking".
- Level 4, structural level. Different axiomatic structures are envisaged.

We freely use Van Hiele's levels outside his theory to give us good benchmarks about the levels of the mathematical thinking of the students. In fact, it gives us a different view, maybe more easily recognisable, on intuition, experiment and deduction.

In our conception, it is indeed necessary to distinguish between the individual student who gradually discovers geometry from the individual adult who is supposed to master all the levels. If he (or she) is an expert, for him (or her) the use of levels depends on the problem to be solved and also on the paradigm in which the problem can be solved.

To cross geometrical paradigms and Van Hiele's levels and to take into account for the interplay between the paradigms, we introduced a two dimensional table[8]:

[7] Parzysz B. (2001). Articulation entre perception et déduction dans une démarche géométrique en PE1. *Actes du XXVIII Colloque COPIRELEM de Tours*. IREM Université de Tours

[8] Kuzniak et Rauscher (forthcoming 2003). Autour de quelques situations de formation en géométrie pour les professeurs d'école. *Actes du XXIX Colloque COPIRELEM de La Roche sur Yon*.

	Geometry I	Geometry II	Geometry III	
Level 0 Visualisation				Empirical pole (Intuition and experiment)
Level 1 Analyse				
Level 2 Informal deduction	Transition			
Level 3 Deduction demonstration		Transition		Theoretical pole (deduction)
Level 4 Abstract Structural				

Technologic horizon Formal horizon

This table should be considered more as a dynamic plan of work in progress than a fixed point of view. In particular, it will be necessary to clarify the nature of each field of the table. For example, level 4 is not a part of Geometry II and when it occurs in Geometry I, it is a sort of very refined geometry where tools developed in Geometry II justify the empirical practices of Geometry I. There are indeed abstract developments from Geometry I not shown at school but which were the objects of theoretical works like Geometrography (Houdement and Kuzniak 2002). By using Chevallard's terminology (1999), our paradigms can be interpreted as different praxeologies of Geometry. Here we find an important difference with Van Hiele's levels that present *a hierarchy of thinking* whereas our geometric paradigms try to keep an internal coherence and are based on homogeneous theories.

Geometries do not pursue the same long-term objective and have different horizons of preoccupations: a technological horizon for the Geometry I and a formal and structural horizon for the Geometry III.

Geometry I integrates level 1 and 2 that send back to an empirical pole, a sensitive geometry which contains intuition (insight), experiment and deduction on material objects, that means objects only considered under their physical aspect.

Geometry II contains level 3 in its component deduction and its axiomatic system. But level 3 remains a level of transition. Geometry II's relation to reality remains important.

Geometry III contains level 4. The reality does not play a role any more. But for many of us, opposite to Van Hiele, figurative representations offer important help for investigations in this Geometry (Chartier 2002).

In Geometry I, expertise goes, in our table, top down from the empirical pole towards the theoretical one. In Geometry III, it goes rather bottom up and the empirical pole appears as a heuristic tool.

At school, the privileged way (marked in grey in the table) to advanced mathematical and geometrical thought is that one which passes through transition on our table.

5. Conclusion

The passage from one type of Geometry to another is really complex: it comes to a change of theory. This change can be seen as a revolution or as a dialectic and progressive evolution.

At least, two transitions are not of the same nature. The first (from Geometry I to Geometry II) concerns the nature of the objects and of the space. The second (from Geometry II to Geometry III) is more of an epistemological character. During elementary school, the first transition is certainly the more crucial one and one could think about the opportunity to teach Geometry II soon and to many middle school students.

As we have said before, we first developed our theoretical frame for teacher training. We try to make pre-service teachers sensible to these problems and to make them explicit the different paradigms (as we have described in our paper for CERME II on pretty (good) didactical provocation). In this perspective, our recent synthesis helps us to understand the rapport to the geometry of the students by describing their work with the help of the classification. The table also offers us hints for acting on the students' knowledge.

At least we also used our general frame to analyse various pedagogical misunderstandings at middle and high school and even during the teacher training.

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