GEOMETRY – THE RESOURCE OF OPPORTUNITIES

Frantisek Kurina

Univerzita Hradec Králové

CZECH REPUBLIC

Abstract:

In the first part I give two points of view on geometry and its teaching and two characteristics of thinking and problem solving (Holmes and Hadamard). In the second and third parts I inform about solutions of two geometrical problems and I formulate two didactical questions.In the fourth part I give a modification of the second example.

1. Introduction

Geometry is an inseparable part of mathematics. As a branch of finished area of science it is organised in axioms, definitions, theorems and proofs, in the same way as for example algebra or calculus. Geometry as a school subject should be considered as a nascent state of a branch of mathematics. According to *Raymond Duval* ([1], p. 38) geometry involves three kinds of cognitive processes which fulfil specific epistemological functions:

- *visualization processes with regard to space representation for the illustration of a statement, for the heuristic exploration of a complex situation, for a synoptic glance over it, or for a subject verification;*

- *construction processes by tools: construction of configurations can work like a model in that the actions on the representative and the observed results are related to the mathematical objects which are represented;*

- *reasoning in relationship to discursive processes for extension of knowledge, for proof, for explanation.*

In my opinion the „soft“ modifications of these functions are three arts (skills, abilities): (S**) Seeing,** (D) **Drawing,** (T) **Thinking** which are mutually intertwined in the process of problem solving.

The process of thinking, is of course very complicatd. *Jack A. Holmes* and *William Reitz* ([2], p. 570) gave an interesting characterization of it:

(1) An unregulated flow of ideas or stream of images, impressions, recollections and hopes.

(2) An undisciplined guessing that treads lightly and superficially over grounds and evidence in an effort to reach conslusions.

(3) The contemplation of ideas, or meditation, without any endeavour to control nature or experience.

(4) Reflective, cognitive, or critical looking into something for the sake of establishing belief and controlling action.

The process of problem solving is the process of creation which *Jacquaes Hadamard, Hermann Helmholtz* and *Henri Poincaré* characterised by four chronological stages ([3], p. X):

(P) *Preparation. You work hard on a problem, giving your conscious attention to it.*

(IN) *Incubation. Your conscious preparation sets going an unconscious mechanism that searches for the solution. The unconscious mechanism evaluates the resulting combinations on aesthetic criteria, but most of them are useless.*

(IL) *Illumination. An idea that satisfies your unconscious criteria suddenly emerges into consciousness.*

(VA) *Verification. You carry out further conscious work in order to verify your illumination, to formulate it more precisely, and perhaps to follow up on its consequences.*

Further I illustrate the above mentioned arts (seeing, drawing and thinking) in the process of solving two problems.

2. First example: Regular dodecagon

Find the area of the regular dodecagon inscribed in the circle with radius r.

Although this is a very simple routine problem for the students aged 15 years, in our experiment, the problem was solved by students aged 18 and 21 years (Secondary school and University). Is is not important for our investigation how many students were successfull, our aim is the study of students´ attitudes, methods and approaches to the problem.

All students saw that A = 12 A_1, where A is the area of regular dodecagon, A_1 is the area of the isosceles triangle ASB, where $\alpha = 30^o$ (Fig. 1).

The process of the solution of the problem is usually influenced by the main idea which emerges from the student´s mental world or Popper´s World 2 (see [4]) which consists of the student´s mathematical experience, images, patterns, theorems and examples.

Students´ solving processes

S_1 : Five students saw that triangle ABS is composed from two right-angled triangles, with area A_2 (Fig. 2).

Since $z = r \sin\beta$, $h = r \cos\beta$,

is $A_2 = \frac{1}{2} r^2 \sin\beta \cos\beta$, $A_1 = 2A_2 = r^2 \sin\beta \cos\beta$.

Applying the formula

$$\sin 2\beta = 2 \sin\beta \cos\beta$$

we get $A_2 = \frac{1}{4} r^2 \sin 2\beta.$

Since $2\beta = 30^o$

is $A_2 = \frac{1}{8} r^2$ and $A = 24 A_2 = 3r^2$.

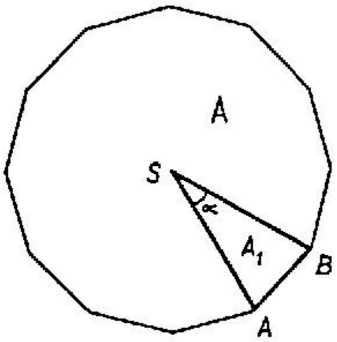


Fig. 1

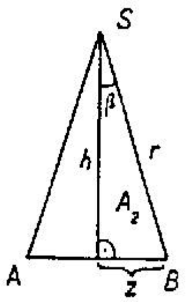


Fig. 2

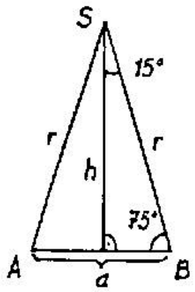


Fig.3

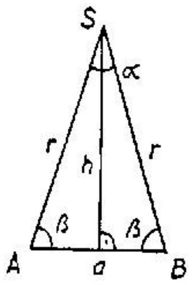


Fig. 4

S_2 : 16 students remembered that the area of the triangle (see Fig. 3) is

$A_1 = \frac{1}{2} ah.$

Since $h = r. \sin 75^o, a = 2 r \sin 15^o,$

they wrote $A_1 = r^2 \sin 75^o . \sin 15^o$

and $A = 12r^2 \sin 75^o . \sin 15^o$

S_3 : 6 students used for the finding the side a and altitude h of the triangle ABS the theorem of sinus and theorem of Pythagoras (Fig 4):

$$\frac{a}{r} = \frac{\sin\alpha}{\sin\beta}, \qquad a = r\frac{\sin\alpha}{\sin\beta}, \qquad h = \sqrt{r^2 - \frac{a^2}{4}}.$$

Some of them gave incomplete result for the area

$$A = 6r \cdot \frac{\sin\alpha}{\sin\beta} \cdot \sqrt{r^2 - \frac{r^2}{4} \cdot \frac{\sin^2\alpha}{\sin^2\beta}} .$$

S_4 : Two students found the side *a* by means of theorem of cosinus and the altitude h by the theorem of Pythagoras (Fig 4):

$$a^2 = r^2 + r^2 - 2r^2 \cos 30°, ..., a = r\sqrt{2-\sqrt{3}},$$

$$h^2 + \left(\frac{r\sqrt{2-\sqrt{3}}}{2}\right)^2 = r^2, \ldots, h = \frac{r}{2}\cdot\sqrt{2+\sqrt{3}}\ ,$$

$$A_1 = \frac{1}{4}.\ r^2.\ \sqrt{2-\sqrt{3}}.\ \sqrt{2-\sqrt{3}} = \frac{1}{4}r^2\ , A = 3r^2.$$

S_5 : Only one student had in his mental world the formula $A = \frac{1}{2}ab \sin\ \gamma$ for the area of the triangle (Fig 5).

He wrote

$A_1 = \frac{1}{2}r^2 \sin 30^o = \frac{1}{4}r^2, A = 3r^2.$

S_6 : One student knew the geometrical meaning of the vector product (Fig. 6):

$A = \frac{1}{2}\ |\mathbf{a} \times \mathbf{b}\ | = \frac{1}{2}\ ab \sin\ \gamma$.

He wrote $A_1 = \frac{1}{2}\ \mathbf{r} \times \mathbf{r} = \frac{1}{2}r^2 \sin 30^o = \frac{1}{4}r^2, A = 3r^2.$

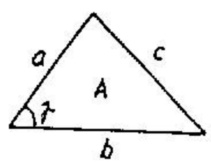


Fig. 5

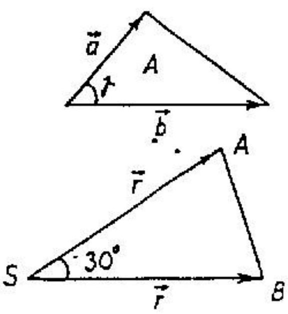


Fig. 6

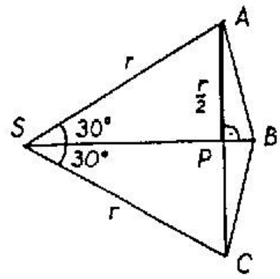


Fig. 7

S_7 : One student, recalled to his mind the Hero´s formula

$A' = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}\ (a + b + c)$. (Fig. 5)

His solution was:

$a = r\ \sqrt{2-\sqrt{3}}$ (see solution S_4),

$s = \frac{1}{2}\ r(2 + \sqrt{2-\sqrt{3}}),\ s - a = s - b = \frac{1}{2}r\ \sqrt{2-\sqrt{3}},\ s - c = \frac{1}{2}r\ (2 - \sqrt{2-\sqrt{3}}),$

$$A_1 = r^2 \sqrt{\frac{2+\sqrt{2-\sqrt{3}}}{2}\cdot\frac{2-\sqrt{2-\sqrt{3}}}{2}\cdot\frac{\sqrt{2-\sqrt{3}}}{2}\cdot\frac{\sqrt{2-\sqrt{3}}}{2}} = \frac{r^2}{4},\ A = 3r^2.$$

S_8: Only one student draw Fig. 8 and gave the following nice solution. Triangle SAC is equilateral and the altitude $AP = \frac{r}{2}$. For the area A_1 is therefore

$$A_1 = \frac{1}{2}r\,,\ \frac{1}{2}\mathrm{r} = \frac{r^2}{4} \quad \text{and } A = 3r^2.$$

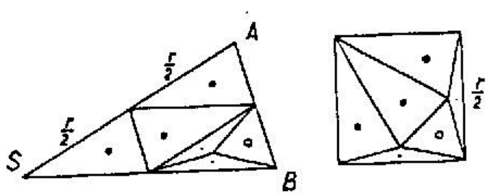


Fig. 8

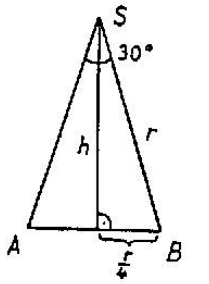


Fig. 9

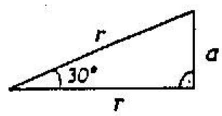


Fig. 10

According to Bolyai – Gerwien theorem from the result $A_1 = \frac{1}{4}r^2$ follows that the triangle *ASB* and square with side $\frac{1}{2}r$ can be dissected into mutualty congruent triangles. Such a decomposition is shown in the Fig. 8. This is a modification of known China solution from the third century A. D.

Some of our students made remarkable mistakes. Two of them I show here.

S_9 : One students draw the Fig. 9 and then found the altitude h according to Pythagoras theorem: $h = \frac{r}{2}\sqrt{15}$. His result $A = 3r^2\sqrt{15}$ is of course wrong.

S_{10} : Further student obtained similar result on the ground of Fig 10:

$$\sin 30^o = \frac{a}{r},\ a = r \sin 30^o,\ a = \frac{r}{2}.$$

Further failure were typical school imperfection in algebra, wrong knowledge of theorems, mistakes in teminology and mistakes thanks inattention.

From the above mentioned examples follows a very important practical question for me: How to cultivate the „black – box“ student´s mental world with the aim to develop geometrical thinking of students, how to teach them to solve economically mathematical problems?

3. Second example: Right – angled triangle

Here I bring a story of solving one geometrical problem. I conducted a workshop of 21 students and teachers where we solved six problems. One of them was:

Prove: If in the triangle ABC are the angles ACO, OCP and PCB congruent (CO is the median, CP is the altitude of the triangle ABC, O, P are points of the segments AB, Fig 11), then the segment AC is perpendicular to the segment CB.

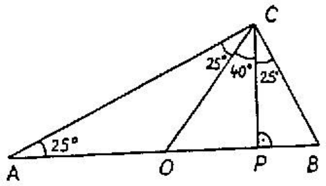


Fig. 12

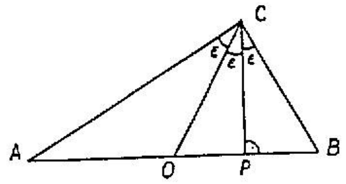


Fig. 11

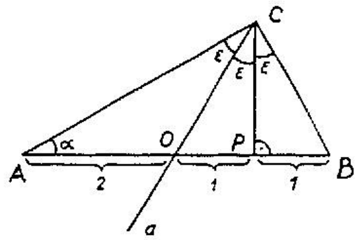


Fig. 13

The first reaction in the workshop was negative. Two students clamoured: „This is wrong! In the right – angled triangle *ABC* (Fig. 12) the angles *ACO* and *OCP* are not congruent!“ These students didn´t differentiate theorem and the converse of the theorem.

In the time we had during the workshop at disposal we found only one solution. One participant knew the theorem:

In a triangle, an angle bisector drawn from any vertex divides the opposite side into segments proportional to the sides adjacent to it.

Applying this theorem to the triangle APC (Fig. 12) we have

$$\frac{|CP|}{|CA|} = \frac{1}{2} = \cos\alpha$$

and therefore $\alpha = 30^{o}$. This means that $2\varepsilon = 60^{o}$ and $3\varepsilon = 90^{o}$.

I called on all participants of our workshops to continue in solving of the problem. For two of them the „incubation time“ of solving the problem was one day. I got two following solutions.

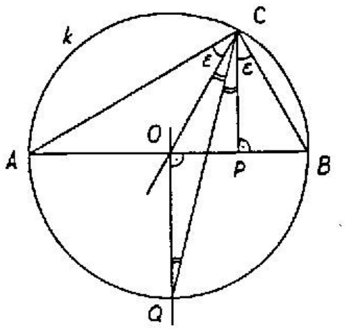
One teacher from our workshop knew the theorem:

In a triangle, an angle bisector drawn from any vertex and segment bisector of the opposite side intersect in the point Q of circle k circumscribed about the triangle.

Applying this theorem to triangle *ABC* (Fig 14) we have: The triangle *CQO* is isosceles triangle and therefore the circle k has the diameter *AB*. This means that *AB* is perpendicular *to BC.*

Further participant considered in this way (Fig. 15): Let A_1 is the area of the triangle *BCP*. The triangles *AOC* and *OBC* have the area $2A_1$. If we construct the altitude *OQ* of the triangle *AOC*, is $|AQ| = |QC|$, the triangles *AQO* and *CQO* are congruent and $\alpha = \varepsilon$. This of course means that $3\varepsilon = 90^o$.

Fig. 14



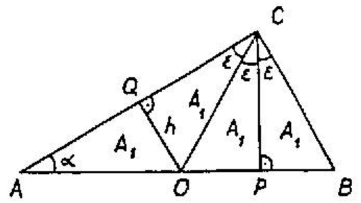


Fig. 15

Further student´s solution used trigonometry. From the relationships

$$tg\ \varepsilon = \frac{c}{4},\ tg\ 2\ \varepsilon = \frac{3}{4}c$$

(see Fig. 16) it follows

$$\frac{\sin 2\varepsilon}{\cos 2\varepsilon} = \frac{3\sin\varepsilon}{\cos\varepsilon}.$$

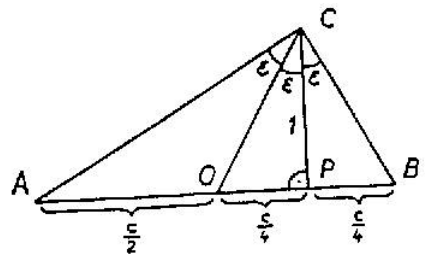


Fig. 16

Applying the formulas

$$\sin 2\varepsilon = 2 \sin\ \varepsilon\ \cos\ \varepsilon,$$

$$\cos 2\varepsilon = \cos^2\varepsilon - \sin^2\varepsilon$$

we have $\sin^2\varepsilon = \frac{1}{4}$ and $\varepsilon = 30^o$.

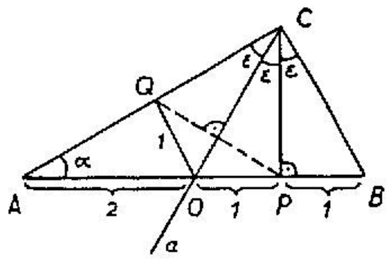


Fig. 17

The way to the following simple solution lasted two months (Fig. 17). In the line symmetry with axis *a* = *OC* is the image of the point *P* the point *Q* of the seqment *AC* and therefore

$$\sin\alpha = \frac{1}{2},\ \alpha = 30^o,\ 2\varepsilon = 60^o,\ 3\varepsilon = 90^o.$$

Why all the participants of our workshop including me didn´t see this solution? Seeing, drawing and thinking are the art. This is my belief.

4. A modification of the second example

Although the conclusion of second example was: *AC* is perpendicular to *CB*, we proved more: the triangle *ABC* has the angles 30^o, 60^o, 90^o . It seems that some of the conditions of our problem are redundant.

We obtain a new problem then:

Prove: If in the triangle ABC the angles ACO and BCP are congruent (CO is the median, CP is the altitude of the triangle ABC, O, P are different points of the segment AB, Fig. 18), then the segment AC is perpendicular to the segment CB.

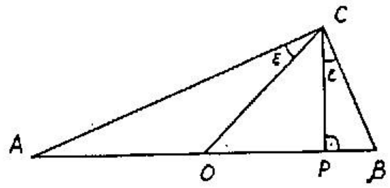


Fig. 18

In the Fig. 19 *k* is the circle circumscribed about the triangle *ABC*. As the angles *ACE* and *BCF* are congruent, the arcs *AE* and *BF* are congruent too and the segments *EF*, *AB* are parallel. This of course means: the segment *EF* is perpendicular to *FC* and midpoint *O* of the centre of segment *EC* is the centre of the circle *k* circumscribed about the triangles *ABC* and *EFC*. The segment *AC* is consequently perpendicular to *CB*.

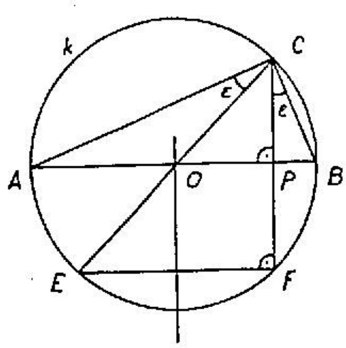


Fig. 19

Further solutions of this problem were given by two participants of our workshop: Aad Goddijn and Mária Bakó.

5. Conclusion

Geometry is the opportunity to cultivate the art of seeing, drawing and thinking, to develop creativity. Geometrical problems are for such activities a very applicable sphere. Geometry is of course also an area for cultivating various languages (nonverbal language of geometrical pictures, language of patterns and language of geometrical transormations.) In geometry it is possible to develop various methods (synthetic and analytic geometry, geometry of transformations, vector geometry, …). These opportunities are not entirely used in our contemporary education, just as cultivation of connection between mathematics, art, nature, sciences and technique.

References

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