THE PROCESSES AND DIFFICULTIES OF TEACHER TRAINEES IN THE CONSTRUCTION OF CONCEPTS, AND RELATED DIDACTIC MATERIAL FOR TEACHING GEOMETRY

Nicoletta Lanciano
Dept.of Mathematics, University of Rome “La Sapienza”

Abstract

Observation is made of the difficulties and processes which future teachers of mathematics encounter in the construction of some concepts in geometry during training. In this research, special attention is paid to the vocabulary the trainees employ and to the material instruments they select in order to construct aids for teaching mathematics in situations where the body, as well as thought, is active and where the senses have been activated in order to find out results about space.

Introduction

"A great advantage of geometry is precisely the fact that the senses can assist the intelligence and aid identification of the direction to take."

"An immobile being would never have been able to acquire the notion of space since, being unable to correct the effects of changes of external objects through its movements, it would have had no reason to distinguish them from changes in state"

In agreement with these two statements made by Henri Poincaré, we consider important to investigate how knowledge in geometry is built, what obstacles and what difficulties need to be dealt with when the senses are involved, when there is an appeal to intuition and when manual dexterity, movement and sight are involved in order to improve our knowledge of space and concepts in geometry. Research of this kind is possible in teaching situations in which the teacher foresees activities of such a type.

Moreover, we consider that perception is essentially multi-sensorial: the brain conditioned by the action selects and integrates a plurality of sensations. Therefore, in order for the brain to be able to select and integrate, it is necessary that it has at its disposal a multiplicity of phenomena and direct experiences. One needs to be able to make reference to and draw on actual experiences, not only imagination, in order to be able to construct concepts and establish relationships. This is the central thesis of the book by Berthoz (1997) entitled "Le sens du mouvement". "Therefore our *sense* of space, and I would add, the emergence of a mathematical concept of space, is far from being a question of the brain alone, but lies rather in the relationship between body and brain: the body too, from the vestibular system to the muscles of the eyes and arms and to the skin, participates in the representation of space and of movement and contributes to the constitution of the memory of them with its specific systems of reference." (Longo, 1997)

In relation to this we deem it opportune that concrete objects are also present in teacher training in geometry and that the future teacher must learn to plan and construct these, in order to be able as a teacher to use them as mediators in his or her actual teaching. The history of science is without doubt one of the sources to be drawn on in order to find pertinent concrete objects.

From a didactical point of view, this work also helps to deal with and question some "false ideas" and convictions held by students regarding mathematics (Furinghetti, 2002). These include:

- mathematics is a solitary activity made up of individuals in isolation

- students who have understood the mathematics studied will be able to solve a problem presented to them in five minute or less

- mathematics requires logic, not intuition

- mathematics is not creative

- mathematics is not a social process, that is to say one constructed thanks to the comparison of ideas

- problems have only one solution and only one process for resolving them.

The observed context

The observed context is made up of a Laboratory situation on a Didactic of Mathematics course in the teacher's formation. The students have a degree in scientific subjects and are doing a specialization course in order to teach science and mathematics at middle-school level (11-14 years).

On the course they worked on some mathematical subjects: on the one hand, topics like examples of didactic transposition and, on the other, aspects like more accurate epistemological reflection at adult level.

The time available for this work is 5 meetings of 4 hours each.

The didactic contract is explicit: teachers and students have already got to know each other during the lecture course and it is clear that in the Laboratory, dedicated to the preparation of a **didactical project** it is expected that each student will have to collect or produce concrete instruments, and instruments for reflection which will be useful for planning the teaching activities in class. It is also rendered explicit that during this work continuous reflection on what one is doing and thinking is expected: such awareness is favored by the requirement of keeping a 'logbook' in which to note one's story as an adult, difficulties met, discoveries made and elements of mathematics which have become clear during the work.

The students work in small groups: cooperation and exchange between peers are fundamental elements in this type of training.

In order to work effectively the "hands need thought" and "thought requires the hands" to sustain it and not only the hands: the eyes, the memory of an effort, the movement of one's body in space and the brain are activated together.

The **didactical projects** are linked to the "fields of experience" encountered by the students during the Didactic of Mathematics course and from which everyone has made a selection. This period is therefore largely taken up with the actual design and construction of aids, objects and models: such constructions sometimes present themselves as problem-solving situations in which apparently very simple subjects are dealt with, but these turn out to be full of implicit and hidden characteristics.

Concrete materials for the teaching of mathematics

Along with Bartolini Bussi (2001) we recognize that "artifacts are structured materials to be used in teaching, the very instruments of a sector of activity, but can also for example be ancient tools or at least instruments from the past" or ones elaborated in other cultural settings.

On the other hand “the transparency of any technology always exists with respect to some purpose and is intricately tied to the cultural practice and social organization within which the technology is meant to function: it cannot be viewed as a feature of the artifact in itself but as something that is achieved through specific forms of participation, in which technology fulfils a mediating function.” (Meira, 1995). Or, in other words, it is in the social practices, in the particular and actual conditions in which an instrument is used, conceived and constructed, that this offers its potential: these are not intrinsic and a priori, and neither are they always explicit and expected.

"This creates a problem for teaching” continues Bartolini Bussi: “how can a teacher plan and run an effective activity when this eliminates opacity of representation from the artifacts? In other words, how can the teacher guide all the pupils to carrying out reasoning, which is supported by the physical use of the instrument, while, at the same time, maintaining a distance from it and thus gaining the freedom of mental experiments? A response to this is to be sought in the epistemological, cognitive and didactical analysis of the cultural practices and social interactions either constructed or which it is possible to build in the classroom."

On the other hand, the way in which concrete materials are used and in which it is decided to use them is constantly related to the personal phenomenological experience. They constitute processes and are, as such, open to modification and supplement. We encourage taking into consideration the flexibility of all instruments, listening to and observing the interaction (teacher-teacher trainee) during which discussion of redesigning takes place. According to Rabardel (1995) for example, there are at least two possible interpretations for each technical instrument: on the one hand it was built according to specific knowledge which assures the achievement of certain objectives (here we refer **to cultural artifact)** and on the other it is used by the individual according to patterns of utilization of an individual nature (here we refer to **instrument**).

During the Laboratory sessions, the teacher’s intervention promotes and supports, through dialogue and bibliographical recommendations, the epistemological analysis of the artefacts, the cognitive analysis of the use of the instruments and the didactical analysis of the activities linked with them.

Some examples

Three artifacts are presented in what follows:

- an artifact from the history of Mathematics: the example of Leon Battista Alberti’s “diffinitor”, described in his book “*De statua"* (about 1430);

- an artifact designed in a particular context (precisely the teaching of Astronomy) and open to being redesigned and adapted to new situations;

- an object from the teaching of Mathematics itself, designed to visualise a construction of the ellipse.

A) Leon Battista Alberti's "diffinitor"

The "diffinitor" is an artifact described by Leon Battista Alberti in his work "De statua" which is useful for identifying a point in space using 3 coordinates (one angular and 2 linear). Anyone who wishes to construct it starting from a drawing (fig 1) immediately runs into some problems, some of them practical, and some others linked to the relationship between the observed or read image and the mental image made of that object. Only the picture of fig 1 was presented to the students: so students have to choose and to buy the materials to use to build their instruments.

In the logbook which follows, it is possible to read about how learning led to a modification of behavior. One can also see that this modification occurred because the meaning given to one's direct experience has been modified. From a logbook, course in the Didactic of Mathematics:

"..even the teacher learns by doing.. or: the story of the construction of L. B. Alberti's "diffinitor""

…"the need and desire to reproduce one of the instruments mentioned, in particular L. B. Alberti's "diffinitor", stemmed from having thought of using anthropometric measurements as the subject of my didactical project and about the research by various artists regarding the proportions of the body. I must say that the course reading did not enable me to understand how such an instrument worked and when the professor told us to create one, I objected that I "had not understood how it works and so I cannot create one.." In reality I was being asked to put into practice what we go around repeating "learn by doing..." Hence together with my group I went off in search of materials. We needed cardboard, scissors, glue, adhesive tape, paper fasteners, string and lead.

We had to create an instrument composed of a graduated circumference, a graduated rod and a plumb line, that too graduated. This instrument served L. B. Alberti to find the position of a point in three-dimensional space: he in fact, having a model in front of him, established the plastic position of the model by finding 3 coordinates in space, an angular dimension and 2 linear dimensions, and then transferring them on to the statue.

The first difficulty was to find the good materials, in particular the string and the lead: we adapted by taking a ball of string for parcels and some die. The prototype conceived was made completely from cardboard, both the circumference (the disc) *for which we used a paper plate, and the rod.. The second difficulty was to find the center of the disc and to graduate the circumference. We marked only the following fundamental angles on this: 0°, 30°, 45°, 60°, 90°, 180°, 270° and 360°. However, on taking the first measurements we realized that these angles were not sufficient. We overcame this problem through a photocopy of a goniometer glued on to the cardboard disc. The third difficulty came during the execution* *of the rod and above all in understanding how we would insert the plumb line. For the plumb line, the most difficult thing was to graduate it: we therefore decided that we could establish the point on the plumb line and for the reading we would compare the length of the string with the meter marked on the graduated rod. We started to take measurements: it was only on the second day that we were able to work out the use of the "diffinitor".*

<u>*...The "diffinitor" is prepared... first considerations:*</u>

a) It is true, by working on it, that it is possible to work out how it functions; the first measurements were not exact: the "diffinitor" serves for finding a point in space and not just any measurements.

b) …c) L. B. Alberti (XV cen.) introduced this manner of finding points in space and in doing so is like the Cartesian method of the 3 coordinates in space (x, y and z) (XVII cen.).

d) Lastly, but most importantly, in building the instrument we worked just like our pupils and we fully realized that while we were working we coordinated the manual activity by using both our thought and our eyes.

<u>*Why introduce the pupils to such an instrument:*</u>

1) To make them aware of the fact that the body has always been the subject of measurement right from the earliest times, and of how artists work in order to create sculptures.

2) To use an ancient artifact, one used by an artist in the 1400s.

3) By using such an instrument, and even more so when constructing the instrument itself, they coordinate their manual activities both with thought and with their eyes.

4) To use a different object for measuring, and hence a different way of measuring.

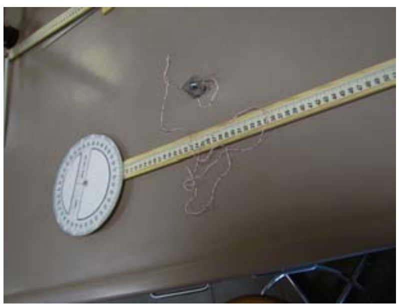


Fig. 1 Fig.2

Fig 1 the "diffinitor" in the book "De Statua"

Fig 2 a model of a "diffinitor" created by a student

Moreover, it is through constructing that the conceptual aspects emerge: in this case in particular, reflection leads to the need to have 3 coordinates in order to identify a point in space and to how these may not all be linear in type. In the meantime the more typically didactical aspects also emerge: for example if the long rod of the “diffinitor”, that on which the distance of a point from the center of the head can be read, is positioned under the graduated circle (fig 2). Indeed, in this position, its first part remains hidden and hence the numeration from 0 to r, where r is the radius of the circle, cannot be seen. This factor may seem inessential but is instead a delicate one from a teaching point of view because it is not always obvious to pupils where they should start from on a ruler when counting. In short, an instrument for teaching purposes is one thing and constructing one for practical use is quite another. It is possible to get around this, for example by marking the centimeters on the back of the instrument too.

B) The three-legged compass

Usually a compass is made up of 2 needles hinged together. In our experiment, the requirement is to construct a large compass with 3 long needles about 1 meter long (fig 3). The origin of this instrument is to be found in Astronomy, where 3 directions are needed to indicate the 3 points on the horizon where the sun rises or sets on the days of the Solstices and the Equinoxes, during one year. It can serve to sight and record angles in meso space (Berthelot & Salin, 1992; Lanciano, 1996): we can observe the angular distance between 3 different points and compare pairs of angles between them.

The instrument is presented to the students only in spoken form, only in words. It is in wood, big and solid enough to be used in the open meso and macro space.

However, an analysis of the use reveals some difficulties. Imagine that we sight 3 objects at different linear distances from the observer, for example a nearby tree, the dome of a church at 500 meters and a hill 2 km away, but at an angular distance which is equal in all cases. The question is to know whether the respective angles are perceived as equal or as different? To what extent does the amount of space interfere with the measurement of the angle in consideration? What knowledge of the angle does this instrument (which in terms of its size, is appropriate for use in an outdoor space) mobilize?

As far as the construction of the instrument is concerned, other problems arise. How is it possible to join 2 or 3 wooden legs and a goniometer all together, thus all joined to form the big compass? Can we screw the goniometer to the legs at a point other than where we have already screwed these legs together? Would this still allow the legs to rotate freely? For one point (the first screw) an infinite number of straight lines pass through but for 2 points only a single one would pass through, and, hence, it would no longer be possible to open the compass. During reflection on the instrument the screws have become "points "and the legs "straight lines". During the design it is necessary to carry out a mental experiment and hence it is not merely a question of manipulating an object.

The big compass to be used in meso space and also in order to look far away on the horizon in macro space, can be useful for avoiding certain typical errors in the comparison between angles which has to do with the length of the sides. In this case the sides are in fact all very long and different from each other: the distance between the observer and a dome or a terrace…

Examples of errors in construction can be seen in figure 4 and 5. In one case, the center of the goniometer does not coincide with the vertex of the angles indicated by the legs. In the other case, the lines traced on the rods are not parallel with the edges of these. Once again the vertex of the angles measured using the goniometer is different from the point in which the legs are screwed together: an inaccuracy in tracing the line is reflected in an error of the instrument. These may appear to be the errors of a child, and yet, they were observed with adults. The interaction between peers in the workgroups provided the necessary help for favoring not only correction, but also a deeper reflection on the instrument itself and on the meanings of the geometric concepts involved. The mediation of the instrument, and in this case of a system of gestures, leads, among other things, to work on the relativity of points of view and on the problems of visual parallax.

From a logbook (the text of the logbook is written in italics, whereas the comments on it are not; A and B designate the texts of the two students):

<u>*Material required:*</u> *A) 3 wooden sticks (length between 60 and 100 cm, breadth between 3 and 5 cm, thickness 0.5 and 1 cm) B) 3 90 cm wooden sticks*, the other 2 dimensions are not stated*; …*<u>*Execution:*</u> *1) trace the line that divides the stick (all 3 of them) in half with a felt-tip pen* (There is no mention to the fact that such a line must be parallel to the edges of the stick); *2) make a hole in the sticks with the drill; it should be assured that the hole is about 2 cm from the end of the stick and along the line drawn (as indicated in the figure!);3) use the screw to tie the 3 sticks together, without however tightening the screw too much because the sticks must be able to rotate around each other (it has to work like a fan); 4) stick the photocopy of the goniometer on to the card with glue: the center of the goniometer must be 2 cm from the edge of the card; 5) fix the goniometer to the compass in such a way that it appears most adapted to its compass: puncture the card and attach it to the screw; glue it to one of the 3 sticks;* (Here the student should also have mentioned the fact that the center of the angles marked on the goniometer must coincide with the point where the screw is). *When moving the 3 sticks must indicate the angles on the goniometer.* (The 3 central lines are the ones which must be considered).

<u>Use</u>: A) point the middle stick at a reference. Moving the other 2 sticks so as to delineate (delineate is not an exact term) the angle desired (it is not clear on what basis this is done). Read the size of the angle on the goniometer (with 3 sticks 2 angles are formed and not just one). B) point the central stick at a given point; open the other 2 sticks and point them at the 2 references. The measurements of the 2 angles formed can be read directly

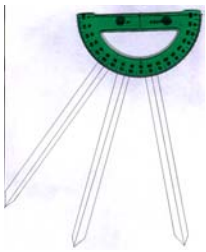
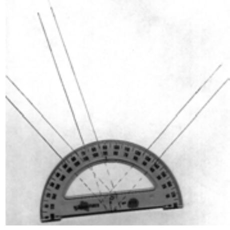
3	4	5
		

Fig 3 the three-legged compass: a successful construction

Fig 4 an error: the central lines are not well centered on the three needles

Fig 5 an error: the three needles are not well connected

The analysis of different logbooks reveals the need to try to get the trainers to reflect more about the possible difficulties linked, for example, with the concept of angle.

<u>C) The construction of the ellipse with the isoperimetric triangles with equal bases</u>

To obtain an ellipse with the isoperimetric triangles made of colored card (Barra, Castelnuovo, 2000) it is necessary to fix the perimeter of the triangles and then select an appropriate base and trace pairs of sides in such a way that their sum is constant. The strategy of using 2 rules empirically starting from the ends of the base and randomly seeking an appropriate inclination for the sides, proves to be a rather ineffective one: this randomness produces inaccurate results. Anyone who adopts such a strategy also demonstrates that they do not know how to use a compass, even when this is suggested. In short, the function of a compass is to trace the locus of points equidistant from a given point but to do so requires active hands, hands that try things out and “heads that think”: or rather everything that mobilizes that thinking with one’s hands too that is indispensable for carrying out a “didactical transposition "of knowledge. The compass is not the only instrument which can assist the drawing of the triangles requested: some string and 2 nails can also be used, employing the so-called "gardener’s method". The suggestion of using the compass, or the short rope, in this case creates a new situation in teacher-pupil interaction, one which is the richer and more capable of involving different meanings, the more profound and accurate the exploration of the instrument and of the knowledge incorporated in it is, and the more therefore a situation of semiotic mediation is created. The problem can also be solved with a wooden stick for each triangle, the stick being as long as the perimeter and opportunely cut into 3 parts: there are thus multiple solutions for the initial problem and these induce reflection on how in doing geometry one refers to some particularly simple, and as a consequence preferred, forms which are recognized implicitly without the need of an analysis, in this case triangles, and without resort to a definition. These forms possess certain properties which are easy to manipulate, for example in this case, the presence of 2 acute angles. On the other hand, not everyone accepts the same evidence: some very simple phenomena and for example those with great symmetry present themselves as evident: but along with Rouche (1999) we must ask ourselves “what evident elements are there for each one?” and “what evident elements are most common?” These evident elements are often supported by something encountered before, an experience. Some phenomena which did not appear evident become so following certain appropriate observations, constructions or manipulations. This consideration is the basis, moreover, for the search for the "limiting cases"(Barra, Castelnuovo, 2000), which are typically a cultural experience capable in many situations of rendering evident properties which were doubted or hidden.

Discussion

We have observed that when faced with some technical difficulties relating to the quality of the materials used or to their assembly in the construction of instruments, some students are ready to give up on the instrument being accurate, thus revealing themselves not to have a deep understanding of the concepts that the instrument brings with it. The emphasis placed on "doing" is in this case a great help in leading reflection to the root of the concepts.

There is confirmation, already in this phase of the research, of the usefulness of the laboratory in "getting involved " as adults reflecting together on their learning, on how they do things when faced with problems not already known, to gain experience; to feel at ease when using a material (a diagram, a speech, a written text, an object) in order to understand and observe others who use it in a different way from their own; to gain experience through an understanding of one’s difficulties.

The work, not only in the classroom but also in meso space (for example in a square or near a town monument), and even in greater spaces where an ability to conceptualize is even more necessary, leads the future teachers to reflection on the differences between this and the micro space of the book or the unchanging space of the classrooms (Lanciano, 1999), starting from one’s perception and one’s capacity to know, in this case linked essentially to geometry: such activities, in which looking and moving are equally important, constitute a setting which sets in motion previously dormant capacities. This also brings about a profound reflection on the teaching-learning of geometry and on knowledge of space.

REFERENCES

Barra M., Castelnuovo, E. (2000): *Matematica nella realtà*, Boringhieri, Torino

Bartolini Bussi M. (2001): Strumenti reali ed esperimenti mentali nella didattica della matematica, *Atti Castel San Pietro*, Pitagora ed

Berthelot R., Salin M-H. (1992): L’enseignement de l’espace et de la géométrie dans la scolarité obligatoire. Univ. De Bordeaux I

Furinghetti F. (2002): *Matematica come processo socioculturale,* IPRASE, Trento

Longo G. (1997): Géométrie, mouvement, Espace : Cognition et Mathématiques, à partir du livre Le sens du mouvement de A. Berthoz, Odile Jacob, 1997, *Intellectia* 2, 195-218

Lanciano N. (1996) : “L’analyse des conceptions et l’observation en classe: outils pour une définition des objectifs éducatifs et des stratégies pédagogiques pour l’enseignement de l’Astronomie à l’école élémentaire en Italie ”, *Thèse*, FPSE, Université de Genève

Lanciano N.et al. (1998): *Geometria in città*, Centro U. Morin, Paderno del Grappa

Lanciano N. (1999), “Il valore educativo della geometria”, Cooperazione Educativa n 4

Meira L. (1995) : Mediation by tools in the mathematics classroom, Proc.19 PME Conference, vol 1, 102-111

Rabardel P. (1995), *Les Hommes e les Technologies : Approche cognitive des instruments contemporains*, Armand Colin, Paris

Rouche N. (1999): La géométrie et la nature des choses, Proceedings I, *Histoire et épistémologie dans l’éducation mathématique*, Belgium, 43-63.