STUDYING GEOMETRIC FIGURES AT PRIMARY SCHOOL FROM SURFACES TO POINTS

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Abstract

On the one hand, difficulties in geometry in relation with figures are well known and yet studied from different theoretical perspectives. On the other hand, at primary school in France, geometry is generally less considered than numbers. The purpose of this research is to develop some teaching situations usable by any teacher and focussing on one aim: to learn to analyse figures, that is to say to see some lines, points, properties that do not appear at first glance but are important to construct or reproduce the figure. We identify some didactical variables of these situations and put them in relation with students' difficulties.

I. General frame, questions and methodology

Our research raised from several observed facts and one fundamental hypothesis about teaching geometry at an elementary level (primary school and beginning of secondary school).

1. Some observations and reflections about teaching geometry in French primary school.

- Many scholars (for instance see Mammama & Villani, 1998, Laborde & Capponi, 1994, Berthelot & Salin, 1998, 2000-2001) identified several strong difficulties related to the use of figures in solving geometry problems at secondary school concerning both constructions of figures and proofs of properties of geometric figures. On the one hand, students trust what they see on the drawing (figure drawn on paper) instead of reasoning on the geometrical figure connecting theoretical objects defined by some properties; on the other hand, they do not see what they should see. For instance, they cannot identify relevant parts of figures to use the theorems they need and know, they do not think of continuing lines that could easily reveal some interesting alignments or intersections.

- Similar exercises may be found in textbooks for the last year of primary school and the first year of secondary school but with quite different expectations from teachers at each level, especially concerning reproduction of figures.

- In the national assessment at the beginning of secondary school, apparently similar

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tasks concerning geometric figures produce quite different results.

- In primary schools, the teaching of geometry is often reduced to the teaching of cultural vocabulary concerning some objects of space, some plane surfaces and some basic objects or relations as points, straight lines, segments, parallel or perpendicular lines and to the use of usual instruments for drawing geometric figures.

- In primary schools, teachers usually regard geometry as less important than numbers. The importance of problem solving is considered in the numerical field for which there are many documents designed for teachers. On the contrary, the teaching of geometry lies quite often on "ostension" (showing): geometrical objects are shown, students have to recognise them, but they are rarely used to solve problems (Berthelot et Salin, 1994, Gobert 2001).

2. First analysis and fundamental hypotheses.

a) Some explanations of these facts may be retained at first:

- Specificity of geometric knowledge itself opposing two aspects: on the one hand a practical knowledge related to space mastery, activities of tracing and moving, visual perception; and, on the other hand, an axiomatic theory, preferential field for deductive reasoning, this aspect becoming more important in secondary school. No consensus really exists on what to teach in geometry at primary school, on the way to take into account, in elementary teaching, the relations between practical geometry and theoretical geometry.

- Teachers training: many primary school teachers themselves have had difficulties with geometry in their secondary studies and they are not very confident in this subject. It is established that a teacher is less safe when s/he favours problem solving in her/his teaching. The more initiative is left to students, the more the teacher has to manage with unexpected processes or reasoning. This requires a better mastery of knowledge.

b) The main hypothesis supporting our research is the following:

The relation of students to figures is a key point in their relation to geometry and it is necessary to take into account in teaching that students have something to learn to articulate in geometry the register of figures and the register of language (Duval, 1994, 1995). From nursery school to the first years of secondary school, students must move from a vision of figures as surfaces in a material form (templates) they can handle, cut up, or colour to a vision of figures as points and lines characterising it and allowing to construct it with instruments bearing themselves mathematical knowledge. At this elementary level, we do not oppose theoretical geometry and practical geometry on material because we think that theoretical objects may emerge from the solving of practical problems. Therefore primary school is for us a crucial moment for fundamental learning governing success in geometry problems at secondary school. Working on figures is one of these fundamental learning objects:

working on drawings to involve properties of geometrical objects representing these drawings.

(H) The main objective is to bring students, in the course of primary school and the first years of secondary school, round to change progressively their view on geometrical drawings and materials so that finally they conceive geometrical objects. For this, we think that it is possible to choose situations that "force" this view change and bring students to "perceive" geometrical properties. Jointly to this change, begins some emergence of both geometrical objects and relevant vocabulary.

Remark that our hypothesis does not concern the existence of levels as Van Hiele levels and on ways to change such a level for students, but it lies on two assumptions:

- at any moment (or at any level), the view of students on drawings (material figures) is depending on situations

- one can act on these situations to change the view of students, to help them to see properties that they do not see at a first glance, for instance to consider alignments, midpoints, ratios on a drawing, to move some parts…

Our theoretical framework is the Theory of Situations (Brousseau, 1997), as used for geometry by Berthelot et Salin (1995) and Argaud (1998).

c) More precise hypotheses for this research

We will now specify our main hypothesis (H) relatively to the reproduction of figures[2] in terms of four new hypotheses.

(H1) Two significant elements of this view change concern dimension change:

- Passing from the vision of a figure as a surface (via a template for instance) to a vision of it as lines (the edge of the template). For example, a situation encouraging this change of view is to reproduce a broken template as in figure 1. To restore the missing vertex, students have to continue straight lines.

- Passing from the vision of a surface as lines to a vision of it as characteristic points. A situation encouraging this change of view is, for instance, to reproduce a figure like the one in figure 2 in an empty frame at a different size. Identification of some alignments and intersections is needed to know where to stop the segments.

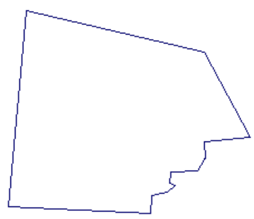

Figure 1

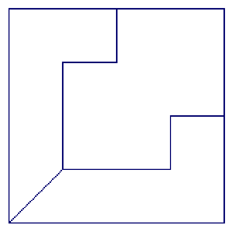

Figure 2

[2] From a theoretical point of view, we retain the distinction of Parzysz (1988,1991) and Laborde & Capponi (1994) between drawing and figure, but we respect the usual use of "figure" in a meaning of "drawing" to speak of usual activities in primary school. Reproduction of figures is a type of activity used from nursery school to secondary school.

(H2) Another significant element is to dissociate a geometrical object from the instrument that is usually used to produce it (for example circle/ compass; straight line/ ruler; right angle/ set square).

(H3) There are specific elements to take into account for specific figures. For example, passing from surfaces to lines is sufficient to construct a square with usual instruments, but for a circle you have to consider a specific point: the centre. The ruler is a universal template for straight lines but there is no universal template for circles or arcs.

(H4) The change of view goes with a change of language: language helps changing view and language evolves as view changes.

3. Questions and methodology

a) There are three main directions in our research :

- Develop a finer analysis of national assessments in order to support our observations

- Elaborate and experiment sequences of geometric situations involving our hypotheses

- Test the robustness of these situations when they are proposed to standard classes without researchers being in the classroom.

In this paper, we develop only the third point. The research is still in progress and this issue was studied only at first and second grades (age 6-8) and with one type of situations, called here "restoration of figures"[3]. Cognitive and mathematical activity of students is quite dependent on the choice of deleted parts to restore. Our objective is double. On the one hand, we aim to relate this choice with cognitive and mathematical task and with the ability of these situations to help students to change their view on figures and to draw geometrical objects from requirements of drawing. On the other hand, we aim to test the viability of this kind of situations in ordinary classes and their ability to help teachers to change their view on figures.

b) Our method includes several steps :

- We prepared one situation of restoration of figures implemented in a class of first grade and observed by researchers and some "pedagogic advisers"[4].

- The sequence was analysed by researchers and advisers. A grid for analysis of figures to restore was produced with them, identifying some didactical variables.

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[3] Students are provided with two sheets of paper. A figure is drawn upon one of them; on the other one, there is the same figure but some parts have been deleted. Students have to complete this second figure to restore a figure superposable to the first one. The figure is also available on transparencies so that they may check their reproduction after they have finished.

[4] Pedagogic advisers are teachers with no class; they help inspectors and take part in teachers' training, especially in service teachers' training.

- The advisers produced themselves another figure and implemented a situation of restoration of this figure in some classes and stored the students' productions.

- The development of the situation in classes and some productions were analysed during two or three meetings between advisers and researchers.

- Finally student's productions were analysed in a finer way by researchers.

II. Data gathering and results

In this research, there is a large interaction between methodology and results. The first example of this interaction is the elaboration of a grid to analyse situation of restoration of figures. The grid provides means to analyse figures but it is also a result of our research. Indeed, the theoretical hypotheses helped us to construct a first version of the grid but the effect of variables has to be checked by observation of students. Moreover, our purpose is to elaborate situations usable by "standard" teachers so we plan also to elaborate a version of the grid usable by teachers. After the presentation of this grid[5], we present the implementation in class and give some more information on the gathering of data before coming back on students' productions and discussing some perspectives.

1. The grid to analyse situations of restoration of figures[6]

a) Variables concerning complexity of figures

We consider figures composed of segments. Such a figure may be seen as composed of several polygonal surfaces placed side to side or one inside another. We suppose for this paper that we delete only segments or parts of segments and never a surface constituent of the figure[7]. Nevertheless, to restore a part of a segment missing, it is often necessary to consider other segments, especially when one end of the segment disappeared. Therefore the restoration of a line has to be regarded not separately but with its environment. Segments may be seen as edges of surfaces and the part to restore as a hidden or torn zone like in D on figure 3. The complexity of the restoration depends in particular on characteristics of the zone to restore. Restoration concerns parts of segments or whole segments. In the case where no whole segment disappeared, we will call "junction" a zone to restore surrounding a point (often a meeting point). Some variables concerning this junction may be characterised by an ordered pair of integers (n,p) = (number of segments arriving in the zone, minimal number of segments needed for restoration). It is clear that p is at most equal to n and that $p < n$ if two segments arriving in the zone are on a same straight line. We take an example used by the advisers (figure 3) to explain the grid.

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[5] This version is for the researchers after experimentation, this is why we consider it as a result, especially the increasing difficulty between (3,2) and (2,2) junctions.

[6] Of course, the grid is expressed, in terms of researchers' vision of the figure, which is different from students'.

[7] This case was studied too and might be discussed in oral presentation

For our figure, we have (2,1) for I and for J; (2,2) for A, B, C; (4,2) for D and G; (3,2) for E and H, (4; 3) for F, (4,4) for K. A first idea would be to measure complexity with n or p or n+p because the junction seems more complex when many segments arrive in the zone. Nevertheless, the observation of students showed that they have difficulty in determining where to stop drawing a line, especially in "corners" as in A, B, C, and K.

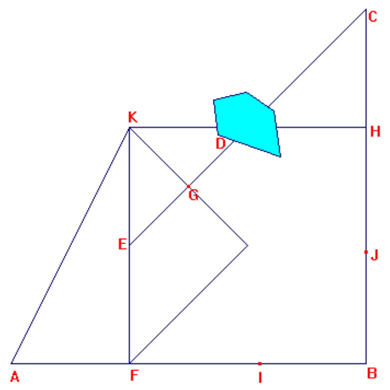

Figure 3 (Picasso)

To restore such a zone with one ruler, you have to continue one line beyond the wanted point, continue the other line as far as the first one and then rub out what is beyond. Children do not seem to easily use their rubber in this case. They try to guess where to stop, draw too short lines and continue in several lines. Then, it seems that the fact that n = p increases complexity of restoration. Moreover, the number of lines is not the only element to consider. In some cases indeed, you have to take into account an order to draw the lines, for example for a junction (3,2) as in E; in other cases, it is not necessary, for example for a junction (4,2) as in D. Perhaps it should be possible to measure something about complexity with p-(n-p) = 2p-n, but we did not try to have a general formula. We identified three basic junctions: line (2,1); knot (3,2); corner (2,2) with increasing difficulty. Other junctions are composed with these ones. For example a junction (4,2) is like two lines, a junction (4,3) is like two knots. We have n = 2p if there are only lines, and n = p if there are only corners.

In the case where a whole segment disappeared, it is necessary to take into account lengths or alignments with points out of the zone to restore. It gives another reason for difficulties. Another variable concerning both the zones to restore and the whole figure is the direction of segments to restore: horizontal and vertical lines are easier to perceive so an alignment of two segments in such directions is also easier to perceive. Other variables concern the whole figure, for example the direction for reading: from left to right and from top to bottom. A difficult junction may be more difficult if placed on the left and on the top of the figure than on the right and on the bottom. Another global variable is the fact that the figure represents something like a castle or it is purely abstract. The interpretation of the figure by students may give them some landmarks and, for example, avoid some omissions of segments because of their significance in the drawing.

b) Other variables

To implement a situation of restoration in class, it is important to take into account other variables, in particular:

- The material given to students to draw figures (one or more rulers, templates for some parts of figures...) and to check elements on them (templates, figure achieved on transparencies with more or less fine lines…

- Conditions of realisation, for instance presence of a model on the table of the students or model stuck on the blackboard.

- Appearance of the figure and hidden parts: only lines or draft of broken surfaces or colours…

- For templates, the fact that they have a symmetry axis or not.

- Management of the class, especially of explanations: for instance, is there a collective explanation during the restoration of the figure by the students?

2. Implementation in classes

The four pedagogic advisers prepared two figures: the one (Picasso) presented in figure 3, and another one representing a castle[8]. For each figure, they conceived partially deleted drawings of different complexity referring to a previous version of our grid. There were only junctions around a point, the case where one segment was entirely deleted appeared only for the last version in one class so that only few students had to deal with it. For example, for classes where figure 3 (of course without letters and so on) was given as a model, there were six versions with increasing difficulty according to a first version of the grid. In figure 4, you can see the first, the third and the fourth ones. Students were given the third version to start. The model was available on the students' table. When they thought they had finished, they might verify their productions with transparencies and restore again the same version if they were not satisfied. If they could not succeed after three trials, they were given the 1st version.

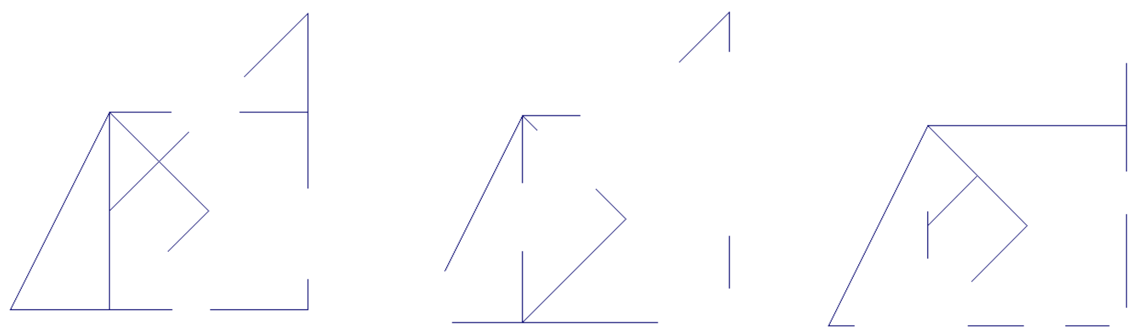

Figure 4

If they succeeded, they were given the 4th one and so on. Templates were available in two classes dealing with the castle. The advisers implemented themselves the sequence, each with one or two classes of grades 1 or 2 with one figure (figure 3 or castle). They gathered students' productions and gave an oral account of the

[8] In a first time, the figure 3 (Picasso) was used in two classes and the other one in three other classes. About two months later, figure 3 (Picasso) was given to classes that dealt with the cattle before and the castle to another class.

sequences. After a first collective (advisers and researchers) analysis of data, they implemented the sequence with the other figure in the same class or in other classes so that three classes or more treated each figure.

3. About the students' productions

- The first remark concerns the richness of productions and the interest of students. Most of them were quite demanding with themselves and started again the restoration several times to succeed at last.

- Analysis of productions helped us to elaborate the grid, particularly to identify the difficulty of junctions (2,2). Children hesitate to continue too long lines: we observed successive extensions and a freehand finishing. Another variable was identified: the length of segment left.

- Observation of students shows that they attend to one zone at a time and do not restore at once one segment with two deleted zones.

- The use of a ruler is unskilful for many children.

- Some students, particularly in 1st grade and in "weak" classes, try to close the figure as near as possible, linking ends of the nearest segments like in figure 5. They had the model on their table but they did not look at it.

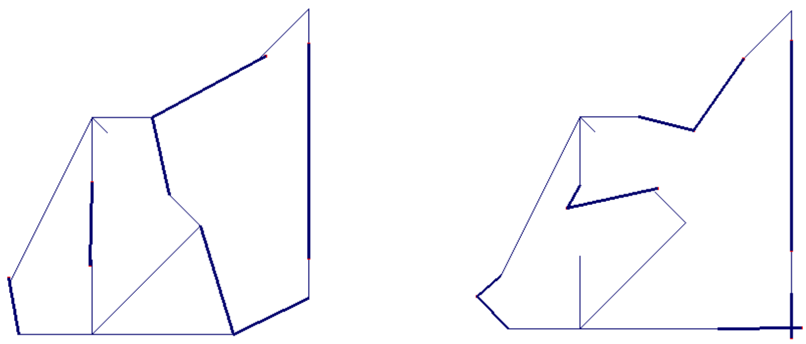

Figure 5

- Restorations of castles were more alike the model than for the other figure. For this one, the big oblique line was often forgotten or badly restored. This line is hardly interpretable as the edge of a surface, but we cannot know if the best result for the castles was caused by the possibility to interpret or by the availability of templates.

Conditions of observation are not sufficient to draw more precise conclusions about actual difficulties of students and on the persistence of these difficulties, particularly the difficulty of corners and the closing of the figure. A question is to know if these difficulties disappear for older students without specific learning or if they persist for some students. According to our problematic, one point is interesting: the fact that children attend one zone at a time. A mediation of the teacher seems necessary to change the students' view. It should be interesting to do a study on a longer time to see if this kind of situation allows a durable learning on this point: to look for alignments.

4. Perspectives for teacher training

First, we proposed the advisers to prepare a situation of restoration and to propose the teachers to implement it in their classes with an observation from the advisers. But none of them could propose to teachers the situation they elaborated. All of them managed themselves the situation in classes because they were not sure of what would happen and they did not feel able to explain to teachers what to do with this situation, especially concerning management of collective phases. Hence a first perspective for our research is to improve our grid for this type of situations to make it more reliable and easier to use for teachers. Moreover, it seems that transparencies failed to play the foreseen role to help students to understand the problem as well as to give them means for rectifying their production. The management of instructions is not so easy.

The effect of abstract or representative figure is interesting to evaluate not only from the perspective of students' learning but also from the perspective of teachers' teaching. It seems that teachers prefer representative figures because they can include them more easily in a story. Usually, in first grades, teachers have a theme they use for their teaching in all subjects. The use of a representative figure may lead to a better success but a less efficient learning.

In this paper we developed only one type of situations but we conceived others, some of them were experienced in 5th grade. Our project is to elaborate a progression concerning figures from first to sixth grades and to experiment it in ordinary classes.

An important issue in order to evaluate the situations is not only to know if they allow provoking the aimed learning but also if they are viable in ordinary classes. For this, we have to know better effective practices of teachers in geometry. In this sense, we have elaborated a questionnaire that will be proposed to teachers during this year.

In conclusion, our objective is to continue to study the possibility of taking into account in teaching some subjects concerning the analysis of geometric figures that are for the moment not taken into account in the actual teaching but let to the responsibility of students alone. We aim to develop graphic abilities for students inside "geometry 1" in the sense of Houdement and Kuzniak (see their paper in this group). But teachers must be in "geometry 2" to choose relevant figures and to choose what part to erase so that students develop graphic abilities in relation with geometrical thinking and supporting geometrical knowledge. For this we try to design didactic situations viable in ordinary classes and appropriate teacher training.

References

Argaud H.C. (1998) *Intérêts et limites de l'environnement Cabri-géomètre en géométrie plane pour l'enrichissement de l'espace des problèmes et dans la constitution des milieux pour la validation dans des situations d'apprentissage à l'école élémentaire autour des relations de parallélisme, de perpendicularité et d'égalité.* Thèse, Université Joseph Fourier, Grenoble.

Berthelot, R. & Salin, M.H (1994) L'enseignement de la géométrie à l'école primaire, *Grand N*, 53, 39-56.

Berthelot, R. & Salin, M.H (1995) Savoirs et connaissances dans l'enseignement de la géométrie, in Arsac, G., Gréa, J., Grenier, D. & Tiberghien, A. (éditeurs) *Différents types de savoirs et leur articulation*, (288 pages)187-204, La Pensée Sauvage Editions, Grenoble.

Berthelot, R. & Salin, M.H (1998) the rôle of pupils' spatial knowledge in the elementary teaching of geometry in Mammana, C. & Villani, V. (eds) (1998) *Perspective on the teaching of geometry for the 21st century,* pp. 71-78. Dordrecht, Kluwer Academic Publishers.

Berthelot, R. & Salin, M.H (2000-2001) L'enseignement de la géométrie au début du collège. Comment concevoir le passage de la géométrie de constat à la géométrie déductive ? *Petit x,* 56, 5-34

Brousseau, G. (1997) *Theory of didactical situations in mathematics. Didactique des mathématiques 1970-1990,* Kluwer Academic Publishers, Dordrecht, 336 pages.

Duval, R. (1994) Les différents fonctionnements d'une figure dans une démarche géométrique. *Repères-IREM*, 17, 121-138.

Duval, R. (1995) *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels,* chapitre IV Figures géométriques et discours mathématiques, pp. 173-207, Bern, Peter Lang.

Gobert, S. (2001) *Questions de didactique liées aux rapports entre la géométrie et l'espace sensible, dans le cadre de l'enseignement à l'école élémentaire,* thèse de doctorat, Université Paris 7.

Laborde, C. & Capponi, B. (1994) Cabri-géomètre constituant d'un milieu pour l'apprentissage de la notion de figure géométrique. *Recherches en didactique des mathématiques,* 14(1.2), 165-210.

Mammana, C. & Villani, V. (eds) (1998) Perspective on the teaching of geometry for the 21st century. ICMI Study, Dordrecht, Kluwer Academic Publishers, 364 pages.

Parzysz, B. (1988) Knowing vs Seeing. Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics,* 19 (1), 79-92.

Parzysz, B. (1991) Espace, géométrie et dessin. Une ingénierie didactique pour l'apprentissage, l'enseignement et l'utilisation de la perspective parallèle au lycée. *Recherches en Didactique des Mathématiques*, 11(2.3), 211-240.