THE TRIANGLE AS A MATHEMATICAL OBJECT

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This work constitutes the first part of a research into the institutionalisation of the concept of the triangle in primary school. We first report on experiments into the personal meaning of the word 'triangle' and the gap between the personal and geometrical meanings. The cultural background revealed by this leads to an epistemological reflection on the nature of the mathematical object. On the basis of the experimental results and theoretical reflection, we put forward an activity for classification of shapes, which aims to emphasise children's conceptions and to help them to bring the concept progressively into focus. Results with implications for teaching are discussed and areas for future research are indicated.

Introduction

The triangle is one of the first geometrical shapes presented to children, who very early learn to recognise it; they distinguish triangles from circles and squares. In Italian primary schools, progress traditionally was as follows: the teacher started with polygons, named according to the number of sides; a three-sided shape was named "triangle". Nowadays many teachers prefer to start with teaching spatial awareness (for example, following tracks or trails) and with manipulation activities, recognition and classification of shapes. Subsequently the traditional classification regarding sides or angles is presented. Teachers do not normally find that children have difficulty with this. In both traditional and modern approaches the terminology is taken for granted but as research has show (Medici, Speranza, Vighi, 1986), this can lead to problems. The word "triangle" is used for operations and for recognition; sometimes a definition is also given, but frequently no check is made on whether the concept has been interiorised. Moreover, questions are not asked about the personal meaning of the concept, not even whether it differs (and if so, how far) from the institutional meaning (Godino, Batanero, 1994). In collaboration with some researcher-teachers[1], we decided to investigate the empirical-perceptual, figural

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(Fischbein, 1993) and lexical aspects. This was to understand how it would be possible to impart an understanding of the theoretical aspects in such a way as to construct a process of transition from conceptions and misconceptions to the institutionalisation of the concept itself.

Theoretical Framework

What is a triangle? Is a triangle an abstraction of a concrete objects? In nature there are very few examples of triangles. If we try to find, in the manner of Galilean philosophy, correspondence between objects in nature and mathematical shapes, it is difficult to place the triangle. According to Plato, the triangle is an idea, which exists independently of human thought, and according to Aristotle a triangle is an immanent shape upon real objects. "In Aristotle's view, in order to speak scientifically about a concept, one needs to have a "universal understanding" of it, that is to know its true nature." (Speranza, 1996). Euclid defines the rectilinear figures (Definition XIX) those bound by straight lines, trilateral shapes are bound by three straight lines. He classifies them according to sides or angles (Definitions XX and XXI).

Since "every mathematical concept is bound to function as a representation, given that there are no "objects, real things" to show in their stead or as their evocation" (D'Amore, 2000), it may be useful to refer to a variety of different "registers of semiotic representation" to approach a mathematical concept, or, as is said these days, a "mathematical object": "The coordination of many registers of semiotic representation appears fundamental for a conceptual learning of objects: we must not confuse the object with its representations and that it must be recognised in each of its possible representations" (Duval, 1993). So we decided that pupils should work on drawings, words and definitions. But, "… we must be able to deal with products of some process without bothering about the processes themselves" (Sfard, 1987). The questions on links between processes and objects are well known: concept formation takes place by means of a hierarchy which starts from action and leads through concrete processes, by encapsulating (Dubinsky in Tall, 1991), to objects onto which prime new processes (Sfard, 1991).

Methodology

In order to investigate what the word "triangle" evokes in the mind of the pupils, three different preliminary activities were prepared. The first concerned drawing, the second was based on a brainstorming experiment and the third was based on "definitions" given by the children. In this way these prepared the ground for the concluding activity on the classification of objects. The table on the next page shows the activities and their links with the registers of semiotic representation. The activities were carried out by pupils aged between 7 and 11 years. A large number of classes participated in the initial activities (20 classes, from 7 to 11 years old), and a smaller number of sample-classes were involved in the final activity (6 classes, from 9 to 10 years old). Four classes in particular were involved in the project over three years. We thus had the opportunity to follow and to study the progressive development of the concept in the minds of these children.

ACTIVITY	SEMIOTIC REPRESENTATION
Drawing Brainstorming	} ----------------- Geometrical Figures
Brainstorming Definition	} ------------- Natural Language

Activity 1: Drawing

Pupils working individually, using a sheet of blank paper and free hand were asked to carry out the following tasks:

- Draw a triangle
- Draw a different triangle
- Draw another different triangle.

The children were also asked to explain the choices they made in their drawing, and to explain the ‘differences’. It is important to clarify the use of the word ‘different’ in the task. In an early experiment the instructions were given as follows: Draw a triangle; draw another triangle; draw another triangle again. Almost all children (70 %) drew three congruent triangles, equilateral or isosceles, and thus carried out the task correctly. They confirmed our starting hypothesis that a ‘triangle’ for a child is equilateral, or sometime isosceles for simplicity of drawing, and has one side horizontal. Using the word ‘different’ in the instructions resolved this problem, although at the same time it raised new ones, such as what the word ‘different’ means for a child. The word led the children to draw ‘other’ triangles, making them look for real differentiation in their drawings. The ways in which the tasks dealt were, briefly: the first triangle drawn generally had one side horizontal; the small children first drew the outline and then coloured in the shape, while the older ones drew only the outline. Almost all the triangles drawn had acute angles. A number of children drew three distinct geometrical shapes, first a triangle, then a circle and then a rectangle. The adjective “different” clearly made them think of a change of shape, a strong differentiation. The most significant result however was that the triangle is essentially the same, equilateral and with one side horizontal. It is “the” triangle, some children call it the “normal triangle”. As a consequence, the second and third can only be the same as the first even with regard to dimensions: the pupils draw three triangles which are substantially congruent. But the need to respect the task made the children think about how they could make them different. They chose various interesting possibilities:

a) giving each triangle a different colour;

b) posing “the” triangle in different positions, “pointing downwards”, “with the point on the right”, or “rotated” or “crooked”. The idea thus emerged of geometrical transformation associated with the idea of movement, but the idea of invariant by isometric transformations was not grasped;

c) changing dimensions – drawing first one triangle, then one smaller and one bigger or alternatively three scaled triangles, (“one big, one medium-sized and one small”), or using both the previous criteria;

d) changing the shape, above all in the older classes, where classification on the basis of sides has not yet been dealt with in school. “The” triangle can become “wider” (“it has been lowered or widened”), “higher” (isosceles), “longer on one side and shorter on the other” (scalene), or with a different “stature”. The idea is thus to alter the shape as a whole, which can be done by changing its height. The children did not think of the traditional classifications of changing the lengths of the sides or the width of the angles to make the diversification. What is involved is both important and surprising: when teachers explain these classifications, pupils appear to understand them without difficulty, but, in this context, they do not use them spontaneously. Furthermore, of course, if they are actually used, misconceptions are often revealed. For example, there are those like Euclid, who consider the classifications as partitions: a triangle is exclusively right-angled or isosceles. In fact, in our experiments with older classes, who recognised traditional terminology, the names often did not correspond with adequate drawings. It was as though the gap between mental and conceptual images had not been overcome. Another significant aspect is that the need for diversification led a number of children to change the sides of the triangle. They thus obtained what they referred to as a “moved” triangle, with ”scalloped” sides or a triangle with curved sides or even, more surprising, a triangle where one “point” (in the sense of point of a knife, thin and sharp extremity) is rounded off (this provides a jumping off point for the subsequent work). Thus many children do not have the idea of side or rather they have, a very specific idea, certainly not Euclidean.

In conclusion, an examination of the protocols strongly reinforced the idea that there is a wide gap between the geometrical and "lived" triangle and that there is much work to be done in helping the transition from the personal to the institutional meaning. But how is it possible to build a bridge between the two?

Activity 2: Brainstorming

In order to further investigate the linguistic and figural aspects of the question, we carried out a brainstorming experiment. Each child was given a piece of paper with the word "Triangle" written in the middle, with the task: draw everything that the word suggested.

The activity was carried out in 6 classes, with children from 8 to 10 years old.

The following is a summary of the various types of drawings obtained in this way, running from the more to the less common. The drawings are related to:

1. triangles (equilateral or isosceles);
2. objects with triangular form (roof, beak, road signs, road-side triangle carried in a car, pyramid, pencil point, ice cream cone etc.) In fact, both in children's colouring albums and in pre-school teaching aids, the triangle is presented by showing three-dimensional figures where the two-dimensional representation is triangular in form. In this way a stereotype is created (Marchini, Rinaldi, 2002) which always appears in the various activities proposed and is clearly a well rooted idea even in pupils in the higher classes;
3. angles and "objects with angles" understood as "edges" (< and > symbols, angle of a table, etc.);
4. the word "three" or the number "3";
5. three points or objects with points (peak, nail, heart tooth, leaf with pointed edges, fir, etc.)

Activity 3: Children's "definitions".

Is it appropriate to give a definition of a triangle at primary school?

Some teachers opt for the shape aspect only and, from a collection of plane geometrical shapes, merely indicate all the triangles saying: "This, this and this are triangles". They are only concerned that their pupils are able to recognise triangles. Others give an over-detailed explanation such as: "A triangle is a two-dimensional shape (a polygon) with three sides and three angles". They thus introduce a mathematical object via a definition. It is however well known that the process of acquisition of new knowledge occurs in parallel with other mental processes. And up to what point is the definition interiorised by the pupils? To seek an answer to this question we gave the children the following task: write an explanation of what a triangle is, without drawings, for Ufomat, who comes from another planet. The Children were in the third and fourth year of primary school (approximately 8 or 9 years old) and worked in groups.

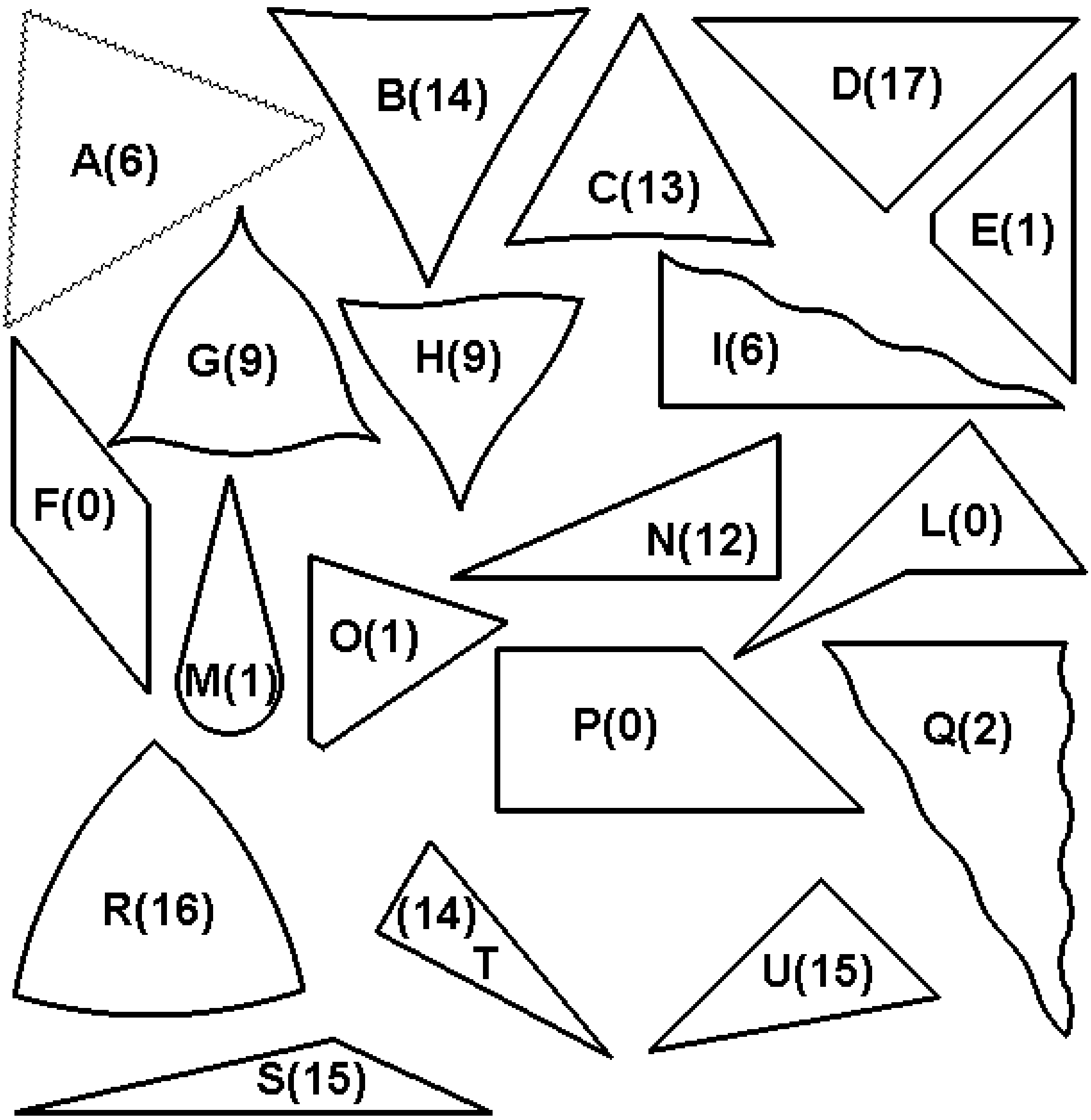
The analysis of the answers shows that children tend to first give a definition (either the teacher's or one of their own, for example "It is a shape with three peaks"), then a linguistic explanation (three angles) and finally they refer to resemblance with objects in triangular form. When pressed by their teacher, emphasizing that Ufomat must be able to understand on the basis of their explanation, they give operational explanations, such as: "Take three straight sticks lying flat and arrange then so that each pair of ends touches" or "Draw one horizontal line and two diagonal ones, all three joined together". Here another aspect emerges: the distinction between triangle understood as boundary and portion of a plane.

Final Activity: From the manipulation of shapes to the concept of a triangle

The three starting activities described above had the sole exploratory aim of establishing the ways in which a child conceives a triangle. Each activity could form the basis of further research. In my work they were used mainly to identify the features of the shapes to be used in the final activity. My teacher colleagues and I, for example, would never have thought of including curved or non-straight sides in the shapes. On the basis of these results, 19 geometrical shapes with particular features were identified and were to be classified[2]. Children were asked to look at the 19 shapes and their features, identify their main characteristics and decide whether they corresponded to the explicit or implicit definition of a triangle. We cut out copies of the shapes in paper or cardboard and divided the children into groups (17 groups of 4 children, from 9 to 10 years old). These children had not yet learnt about the geometry of the triangle from their teacher. Each group was given a kit of objects with the following task: stick each object onto a poster divided into two parts, one side for "triangles" and the other for "non-triangles" (this term was coined by the children themselves). They were also asked to justify their choices in writing. Each group presented their poster to the class, justifying the results of their work. The

[2] The nineteen shapes are illustrated on the next page. I am grateful to Igino Aschieri for the computer elaborations of the shapes.

pupils were thus involved in discussions firstly within each group and then with the class as a whole, in the presence of the teacher (Bartolini Bussi, 1989).



Note: The number on each shape (A-U) represents the number of groups which identified it as a triangle out of a total of 17 groups.

We make several observations on the figures and their classification. There are 5 triangles: D, N, S, T, U. The figure D was recognised by all children because it is rectangular and isosceles, for some pupils the others are not triangles for the following reasons: N because it is not isosceles, S because it does not have equal sides, T because it does not have equal sides or equal angles, U because it does not have equal sides. There are few or no doubts regarding some of the figures: F is a rhombus, P is a trapezium, L has 4 angles, O and E are lacking a point (some pupils do not notice the absence of the points and see three sides), M has only one point, Q appears to some children to be triangle with “2 corrugated sides”. The remaining shapes are the most interesting. The figure A has many points: the number of points appears more important than the figural aspect. For pupils it either has at least 6 or many points, or it “has no points where they should be at the intersection of the 3 equal sides”. B has 3 equal “sides” (curved), 3 angles, so is predominantly symmetrical and regular, C also has these features even if it has “one side not perfectly straight”, R is predominantly regular (“sides” slightly corrugated). B and R could be triangles in a non-Euclidean geometry. The classification of G, H, I is very difficult: they have three points, but strange sides! G “has 3 points even though the sides are wavy” or “has 3 equal sides” or “does not have straight sides”. Another important aspect is the distinction between “point” and “angle” made on the basis of concavity: “G has 2 angles and 1 point”, H “has 2 points and a angle”.

The criteria most used by children are the following:

- Equilateral triangle shape: A, B, C, R, G, H
- Corrugated or wavy sides: A, G, H, I, Q
- Curved sides: B, C, G, H, M, R
- Lacking a point: E, O.

Conclusions

An analysis of the posters shows that, apart from those made without a real classification criterion or with incoherent application of the chosen criterion, most groups carried out classifications “incorrect” in Euclidean sense, but nonetheless based on well-defined criteria. Often recourse was made to natural language and analogies “It is a triangle because it has the shape of a roof, a beak …” or “It is not a triangle because it is the shape of a drop”, etc. The most frequently used criterion was ‘points’, expressed in phrases such as ‘it’s a triangle because it has three points’. We would note that from a theoretical point of view we are not in Euclidean context, but rather in differential geometry for the children: the “points” of a triangle are not only its vertices, but can also be the *singular points* along the boundary. In fact as Piaget shows (Piaget, Inhelder, 1947) the child in recognising geometric shapes from four years upwards will often use touch and thus discover angles or corners as ‘something that pricks’. We saw that the children in our experiments also used this rather ‘primitive’ criterion, although they are somewhat older than those described by Piaget. We were working with children having more experience of spatial awareness and greater familiarity with geometrical shapes. They are in fact the same children who correctly drew and described triangles in the three starting activities described above. However in this context, although they used the classical definition, “it is a triangle because it has three sides", they revealed pre-conceptions with regard to the word "side". This did not refer to a segment, but to something joining two consecutive vertices in whatever way. Sides are sides even when they are "curved" or "wavy". So one of our conclusions is that children see a side only in its topological aspect. In a lesson after discussion of the group posters, the teacher suggested that a class poster should be drawn up as result of joint agreement on the objects to be considered triangles or not. The previous discussions had led to the elimination of some classification criteria proposed by the individual groups, so it was necessary to reach agreement on a single criterion which could be shared by all. In the majority of cases the older primary classes eventually reached the result of classifying as triangles only those shapes, which would be considered as Euclidean triangles. The children used expressions such as "It is a triangle because it has three straight sides and three sharp points". The classification of the 19 shapes was also carried out by younger children. Here we saw a wider variety of criteria; for example isosceles or not, or the choice of the figural aspect which the teacher accepted provisionally making the reservation that they would go back to the subject at a later date.

Further research into this field could usefully look at our last activity and use the Euclidean word "trilateral" which unlike the word triangle is probably not used by the children in everyday life. It could also be interesting to carry out the same experiment with older pupils and perhaps even with adults!

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