ANALYSING PROCESSES OF SOLVING WORD PROBLEMS IN MATHEMATICS LESSONS INTERACTION: FRAMINGS AND REFERENCE CONTEXTS BETWEEN REAL WORLD AND MATHEMATICS

Ralph Schwarzkopf

University of Dortmund, Germany

Abstract: *On the one hand, solving word problems is an important learning goal of mathematics lessons. But, on the other hand, it is one of the most difficult demands for students. Whereas many investigations analyse the strategies to solve word problems in interviews and tests, empirical investigations that focus on the depending classroom interactions are rarely represented in mathematics education. The goal of the presented investigation is to analyse chances and difficulties of interaction processes in mathematics lessons of early grades, in which teacher and students are solving word problems.*

Solving word problems appears as an important learning–goal in the curricula that teachers try to realise in mathematics lessons. The goal of the presented research project is a better understanding of solving word problems within mathematics classroom interaction processes. This project has just started, and the presented thoughts are of preliminary character.

In the first section, aspects of two theoretical approaches cited in the project shall be cleared briefly. Two questions will be formulated within these approaches to differentiate the focus of the investigation. An analysis of two short episodes from a fourth grades class will illustrate these questions in the second section. The closing remarks will discuss some results of the analysis.

1. Theoretical Framework

The author follows the interpretative paradigm, citing mainly two approaches of qualitative research in mathematics education. On the one hand, theories of symbolic interactionism and ethnomethodology are assumed to analyse the organisation of meaning in interaction processes (e.g. Bauersfeld / Krummheuer / Voigt 1988, Voigt 1994, Yackel 2000). On the other hand, an epistemological perspective is cited for analysing the structure of the knowledge, developed in the classroom interaction (s. Steinbring, e.g. 1999, 2000). Due to space restrictions, both approaches are discussed very briefly.

Aspects of symbolic interactionism: Framings

According to this approach, ”(...) social interaction is a process that *forms* human conduct rather than simply a setting in which human conduct takes place” (Yackel 2001, p. 11). Hence, the participants construct their individual sense of the content of the interaction process by participation. They do this within their *framing of the situation* (Krummheuer 1992). Roughly speaking, the framing of the situation gives a context for the individual to interpret the interaction, and is responsible for his decisions about rational acting. Hence, within different framings the participants are acting in different ways concerning the relevance of facts, the meaning of assertions, the acceptance of statements, the rules of correct reasoning and many other aspects.

Regarding word problems as a demand of translation between real world and mathematics (cf. Müller 1995), two framings may be activated while solving word problems: On the one hand there are “real–world” framings, giving an “everyday–understanding” of the word problem. On the other hand there are “mathematical” framings, may be activated by the given question of the task or the context of the mathematics lesson. To solve a word problem, the student would have to relate knowledge, built within these two framings to each other.

But observing empirical data one can see that the interaction process is more complex. Students (and teachers) sometimes seem to understand a word problem neither mathematically nor in a real–world–sense. Nevertheless, they act in a rational way. One goal of the presented investigation is to reconstruct the rationality of such discussions, in other words: Which framings can be reconstructed in the interaction process on the way from giving a word-problem to the construction of a solution?

Aspects of epistemological theories: The epistemological triangle

This approach to mathematics classroom interaction focuses on the structure of the developed knowledge (Steinbring 1999). Briefly speaking, knowledge is divided into three functional components: *Symbols* are necessary to present the knowledge in some kind of codification, but they are not meaningful for themselves. Any understanding of symbols requires the construction of a *reference context* that builds the basis for an interpretation of the symbols. The reference context provides a possibility to operate with symbols in a meaningful way. Building relations between symbols and reference contexts again can not be done in an “intellectual vacuum”. It requires the appearance or the creation of an underlying *(mathematical) concept*, which provides the integration of the knowledge into theoretical structures.

According to this approach, learning is a circulating process of constructing relations between these three functional components of knowledge (fig. 1). Solving word problems can be described as creating concepts that extend real world knowledge by creating mathematical relations in the context of the task (cf. Steinbring 2001). The second question discussed in this paper can be formulated from an epistemological point of view as following: Of what kind are the relations between real–world knowledge and mathematics structures that students and teacher construct while discussing word problems?

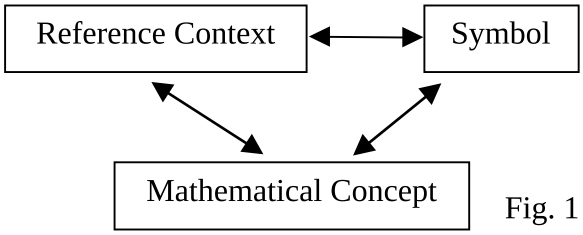

Fig. 1

2. An example from a fourth grade class

In the following, the questions presented above shall be illustrated by analysing a word problem–solving process. The data is taken from the regular mathematics lessons of a fourth grade class of primary school (10 year–old students in Germany). Within this lesson, the teacher confronts the students with the following task, taken from their mathematics textbook (Müller / Witmann 1997, p. 10):

On the motorway from Gießen to Dortmund:

On further signs there are the kilometres information:

Dortmund	151	143	135	132
Hagen	135	127	119	116
Siegen	56	48	40	37

Calculate in all cases the difference

A) Between Dortmund and Hagen

B) Between Dortmund and Siegen

Fig. 2

Following the intentions of the mathematics textbook–authors (cf. to this type of word problems Wittmann 1994), the children would first of all have to solve the given calculation tasks and find out that there are two results only. Afterwards, they have to explain this phenomenon by a reflection about the real–world sense of the calculation: The *differences* between the given numbers have to be interpreted as the *distances* between the assigned cities.

Within the presented lesson, the teacher initiates the following way of problem solving: At the beginning, the participants discuss some real–world aspects of the task. Afterwards, the teacher makes a sketch of the motorway and the given signs on the blackboard. The students then do the calculation and observe that they lead to two results only. At the end of the lesson, the teacher wants the children to explain this constancy by discovering the real–world meaning of the differences.

Due to space restrictions, the following analysis focuses on two short episodes of the problem solving process. Within the first episode, at the beginning of the lesson, the participants construct a somehow realistic background of the word problem. The second episode takes place at the end of the lesson: The participants argue for the constancy of the calculated results on the basis of the painted sketch.

In general, clearing the realistic background and creating a sketch are surely useful to solve word problems. But within the presented interaction process, both become problematical. The participants construct self–dynamical framings and somehow empirical, direct and inflexible links between the epistemological functions of knowledge, in other words: they develop an understanding of the word problem's context that does not (and can not) lead to a correct explanation of the calculation results.

First episode

The teacher demands the students to read the word problem. Afterwards, the children are asked to discuss their first impressions. This is the answer of Werner:

> W *They are on the motorway from Gießen to Dortmund. [...] And then they see this blue sign. And then, that there are still 157 kilometres left to Dortmund.*

Werner tells a short story about some people ("they") who drive on the motorway and read the given sign of the word task. These people use the sign in a way, that is typical for acting in real–world: They read the distance on the sign, to find out how many kilometres they will still have to drive until they reach Dortmund. Obviously, the demanded calculation of the word problem is not relevant within this story: Who would subtract in every day life the numbers given on signs on the motorway?

From an epistemological point of view, the whole word problem consists of several symbols, and some of them become meaningful for the interaction by Werner's answer. The student proposes a story, which tells us how to use the symbol "motorway–sign" given in the textbook: It is a direct supplier for information about a distance of a car to Dortmund. Hence, the story becomes a reference context for interpreting some of the given symbols. The relation between story and sign is basing on a concept of finding out the distance between a car and its destination (fig. 3).

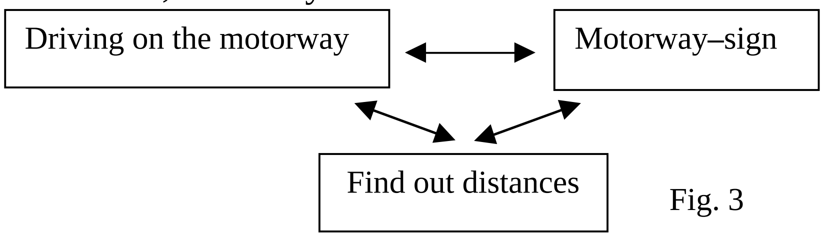

Fig. 3

A few seconds later, the teacher picks up the story of Werner as following:

> T *Okay. Werner found out, where the car drivers are. He already told us, but some of you were talking so I would like to hear it again. Where are they driving or if you are in the car, where are you driving? [...]*

By her reaction, the teacher shows herself satisfied with the story of Werner as an adequate reference context for interpreting the task. Furthermore, she wants the children to imagine that *they* are the drivers on the motorway. Hence, the context–giving story of Werner becomes an officially demanded framing by the reaction of the teacher. In the ongoing lesson, the teacher tries several times to activate this "story–bound framing".

<u>Second episode</u>

This episode deals with an explanation for the observed constant calculation results. In advance, a sketch is made on the blackboard and the children solved the calculations. It is already cleared that there are only two calculations–results and this observation is marked within the sketch by vertical lines in some of the signs (see fig. 4).

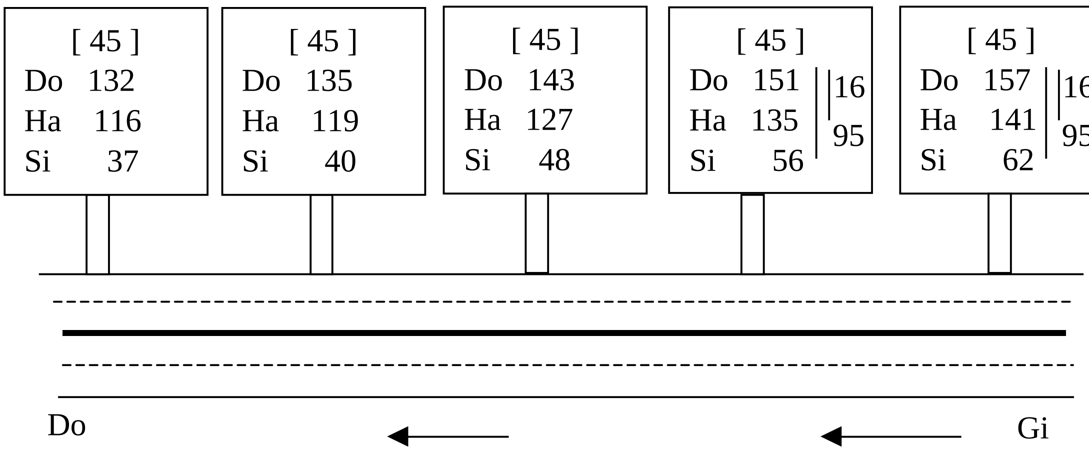

Fig. 4

The sketch shows a part of the motorway and all of the signs given in the word problem. Some arrows painted below the motorway illustrate the direction of the car from Gießen to Dortmund. This sketch is compatible with the story–bound–framing of the word problem: one can imagine cars passing the signs and the drivers reading them to inform themselves about the distance left to reach Dortmund. But regarding the sketch concerning the observed phenomenon, it is rarely helpful: The positions of the cities are not illustrated. Hence, the real–world meaning of the results, namely the distances between the cities, can not be found in the sketch. Nevertheless, the sketch becomes important for the ongoing lesson.

At the beginning of the following episode, the teacher demands explanations for the constant results:

> *T* *[...] Is there any reason for this. We are driving on the motorway and see the blue signs. It has to be like this. Just look: You are driving by car (points along the motorway on the blackboard) read the signs and if you calculate this then the difference between these cities (points at "Do" and "Ha" on the most right sign) noted on the signs is equal, why?*

On the one hand, she stresses the story–bound framing as an adequate point of view to produce arguments. Pointing at the sketch is compatible with the story–bound framing – one can imagine passing and reading the signs while driving on the motorway. On the other hand, the teacher stresses the calculations and their constant results. This seems to be strange within the demanded framing, because the story does not give any motivation for the drivers to calculate the differences.

An epistemological point of view leads to a better understanding of the teacher's intervention. For an explanation of the constant results, the children have to relate the "differences of the numbers" to the "distances between the cities". A sketch could be a helpful reference context for the children to build that conceptual bridge between the calculation results and their meaning in real–world.

But the cities are not integrated in the sketch and one can find only symbols for their positions in the painted signs. Hence, the sketch can not fulfil the function of a reference context, regarding the demanded reason. It becomes a symbol, which the children would have to interpret by creating a new reference context (fig. 5).

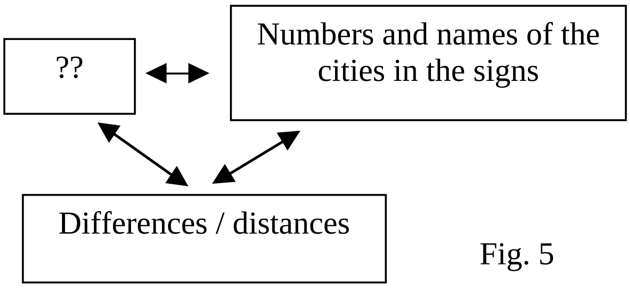

Fig. 5

However, the student Anja is able to construct a relation between the calculated results and the given sketch in the following way:

A *[...] Well I think the signs are all standing in the same distance to each other and if they are standing in this same distance then the numbers are of that kind that it is exactly equal.*

In her explanation, Anja uses some structures of the sketch – in fact, the signs painted on the blackboard have nearly the same distances. She fulfils the teacher's demand: Anja uses the sketch as help for building an alternative sense of the calculated results. According to this interpretation, one could name her underlying framing a "sketch–bound" one. The rationality within this framing is compatible with the children's social experiences with demands of the teacher: if the teacher stresses the importance of the sketch, the sketch must show the demanded reason.

From an epistemological point of view, Anja creates a concept to relate the results of the mathematical calculations to their meaning in the real–world: She interprets the mathematical differences as distances in the real–world. But in her answer, the sketch becomes very powerful: Anja refers to the only distances that are presented in the sketch, namely the distances between the signs. According to this interpretation, Anja uses the sketch as a reference context to interpret the symbolised calculation results as distances in reality (fig. 6). In her proposal, the sketch is taken for reality; hence it has to show all relevant structures.

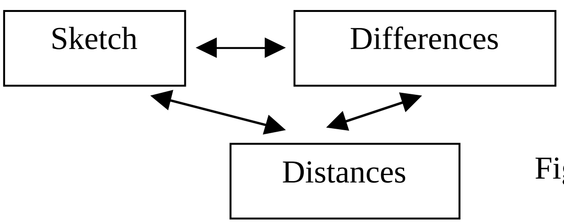

Fig. 6

The lesson ends with an adequate reasoning, basing on more interventions of the teacher. Due to space restrictions of this paper, these episodes can not be discussed.

3. Closing remarks

In the presented episodes, two framings, which lead to different horizons of understanding the word problem, could be reconstructed:

- The *story–bound–framing* leads to a realistic understanding of the given information, but not to an adequate interpretation of the calculation task.
- Within the *sketch–bound–framing* it is possible to interpret the mathematical differences as distance in real–world. But the sketch is taken for reality and the interpretation leads to distances between the signs.

Both framings illustrate aspects of the word problem – one could imagine driving on the sketched motorway, passing and reading the signs. But, for an adequate interpretation of the calculated results, passing the signs is less important than passing the three cities, in other words: The main part of the trip, namely the segment which should be related to the calculation–results, is neither part of the story nor of the sketch. Hence, within these framings, the context of the word problem is empirically extended, but structurally restricted.

The epistemological analysis leads to comparable results, concerning the relations between mathematics and real–world knowledge:

- The story leads to a direct, empirical understanding of the distances given on the motorway–signs: The numbers on the signs show the momentary distance of the car to the assigned city. Hence, the numbers become a *direct supplier* for information in real–world.
- Using the sketch as a reference context leads to a wrong interpretation of the calculated distances – the sketch is also used as a *direct supplier* for the real–world meaning of the differences.

Hence, the sketch in fact leads to a relation between real–world context and calculation results of the word problem. But this relation looks like a somehow direct, inflexible link: The mathematical result is identified with those empirical aspects of the story that can be seen in the sketch.

In this sense, the understanding of the word problem is not extended, but restricted within the constructed framings and reference contexts. The focus of the ongoing investigation is on a better understanding of this problem: How do teacher and students manage a balance between restricting the real life experience to mathematical relevant aspects on the one hand and extending real life by invisible mathematical structures on the other hand?

References:

Bauersfeld, H., Krummheuer, G., Voigt, J. (1988): Interactional theory of learning and teaching mathematics and related microethnographical studies. In H.-G. Steiner and A. Vermandel (Eds.): Foundations and methodology of the discipline mathematics education. Antwerp: University of Antwerp, 174-188.

Krummheuer, G. (1992): Lernen mit „Format“. Weinheim: Deutscher Studien Verlag.

Müller, G. (1995): Kinder rechnen mit der Umwelt. In: G. Müller, E.Ch. Wittmann (Eds.): Mit Kindern rechnen. Frankfurt a.M.: Arbeitskreis Grundschule e.V., 42–64.

Müller, G.N. / Wittmann, E.Ch. (1997): Das Zahlenbuch. Mathematik im 4. Schuljahr. Leipzig: Ernst Klett Schulbuchverlag.

Steinbring, H. (1999): Mathematical Interaction as an Autopoietic System - Social and Epistemological Interrelations. In: I. Schwank (Ed.), European Research in Mathematics Education. Proceedings of the First Conference of the European Society for Research in Mathematics Education Vol. 1 + 2. Forschungsinstitut für Mathematikdidaktik Osnabrück. Vol. 1

Steinbring, H. (2000): Epistemologische und sozial-interaktive Bedingungen der Konstruktion mathematischer Wissensstrukturen (im Unterricht der Grundschule). (Abschlußbericht zu einem DFG–Projekt, 3 Bände). Dortmund: Universität Dortmund, April 2000.

Steinbring, H. (2001): Der Sache mathematisch auf den Grund gehen - heißt Begriffe bilden. In: Ch. Selter, and G. Walther (Eds.) Mathematiklernen und gesunder Menschenverstand. Leipzig: Ernst Klett Grundschulverlag, 174 - 183.

Voigt, J. (1994): Negotiation of Mathematical Meaning and Learning Mathematics. In: Educational Studies in Mathematics. An International Journal. v. 26(2-3), 275-298.

Wittmann, E.Ch. (1994): Üben im Lernprozess. In: E.Ch. Wittmann, G.N. Müller: Handbuch produktiver Rechenübungen. Band 2: Vom halbschriftlichen zum schriftlichen Rechnen. Stuttgart: Klett, 180.

Yackel, E. (2000): Explanation, Justification and Argumentation in Mathematics Classrooms. In: van den Heuvel-Panhuizen, M. (Eds.), Proceedings of the 25th International Conference for the Psychology of Mathematics Education, Utrecht, The NetherlandsFreudenthal Institute, Faculty of Mathematics and Computer Science, vol. I, 9 – 24.