LOCAL STRAIGHTNESS AND THEORETICAL-COMPUTATIONAL CONFLICTS: COMPUTATIONAL TOOLS ON THE DEVELOPMENT OF THE CONCEPT IMAGE OF DERIVATIVE AND LIMIT

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Recent literature has pointed out pedagogical obstacles associated with the use of computer environments on the learning of mathematics. In this paper, we focus on the pedagogical role of computers' limitations on the development of learners' concept images of derivative and limit. In particular, we intend to discuss how the approach to the concepts can be properly designed to prompt a positive conversion of those limitations to the enrichment of concept images. We use the notion of theoretical-computational conflict (Giraldo, 2001a) to support the discussion.

THE NOTION OF LOCAL STRAIGHTNESS

David Tall (1989) defines a *generic organizer* as a learning environment enabling learners to handle examples and non-examples of a mathematical concept. Generic organizers can be computer software providing quick responses to users' exploration. The design of a generic organizer must be based on a *cognitive root*, a central idea holding two fundamental features: make sense (at least potentially) for the learners and enable cognitive expansion. Generally speaking, a cognitive root does not correspond to the formal definition. In the case of the concept of derivative, the theoretical embedding – the concept of limit – is not familiar to students in elementary calculus courses. On the contrary, it ends up being deeply unfamiliar to human intuition, as its historical evolution testifies (see e.g. Cornu, 1991; Sierpinska, 1992). Therefore, the formal definition of derivative does not fit as a cognitive root, since the first condition above does not apply (even though the second one certainly does).

On the other hand, Tall (2000) claims that the notion of local straightness is suitable as a cognitive root to this concept. This notion is based on the fact that a curve graph looks straight if closely magnified. According to the author, it is a primitive human perception of the visual aspects of a graph and is deeply related to the way an individual looks along the graph and apprehends the changes in gradient. Thus, in an approach based on the notion of local straightness as a cognitive root, the gradient is presented as the slope of the line mingled with the curve. The associated generic organizer is a computer environment allowing the user to draw a graph, change graphic windows and observe the consequent changes in graph's appearance (Blokland, Giessen & Tall 2000). The local magnification process is instanced on figure 1, for a differentiable curve, namely $y = x^2$, which acquires the aspect of a straight line, and for a non-differentiable one, the blancmange function[1], which preserve its wrinkled aspect.

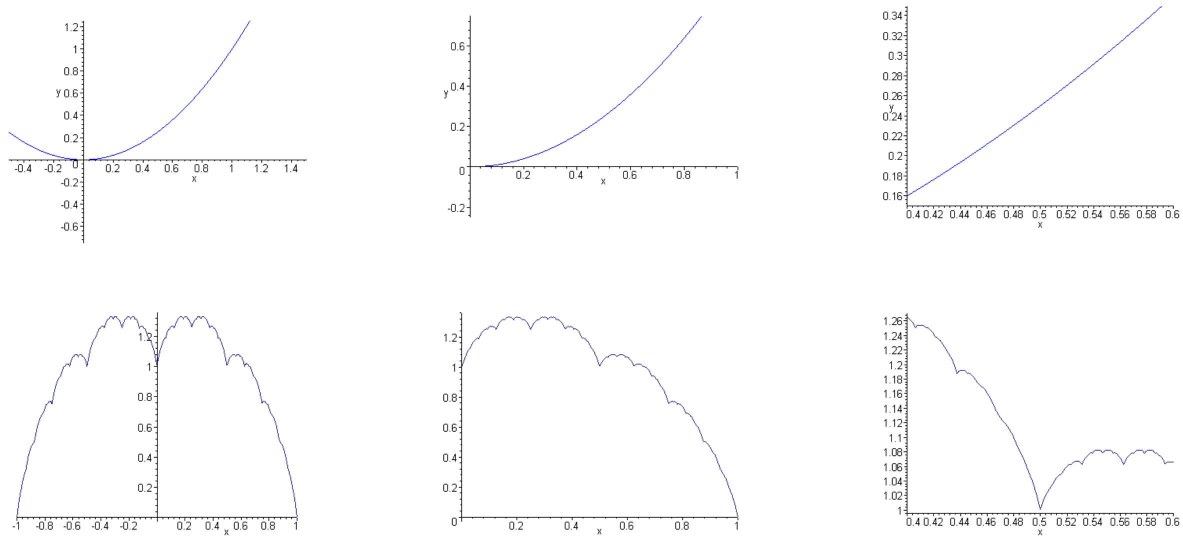

Figure 1: The local magnification process for differentiable and non-differentiable curves.

We have designed and tested a generic organizer (Giraldo, 2001b; Giraldo & Carvalho, 2002b), named *Best Line*, which allows learners to compare graphic and algebraic representations on the local magnification process. *Best Line* is a Maple routine with inputs: a function f, a point x_0 in f's domain, a slope a for a line passing through $(x_0, f(x_0))$ and a value for $h = \Delta x$; and outputs: the graphs of f and of the line $y = ah + f(x_0)$ in the interval $[x_0 + h, x_0 - h]$, a vertical segment linking the curve to the line (representing the difference $\rho(h) = f(x_0 + h) - (ah + f(x_0))$) and the numeric values of $\rho(h)$ and $\rho(h)/h$. The main idea is to compare the graphic and algebraic local behaviours of the curve $y = f(x)$ and the line $y = ah + f(x_0)$, both for $a = f'(x_0)$ and $a \neq f'(x_0)$. Figure 2 reproduces examples of computer screens generated by *Best Line* for $f(x) = x^2$ and $x_0 = 1$, with $a = 2 = f'(x_0)$ (above) and $a = 2.5 \neq f'(x_0)$ (below).

By displaying both the graphic and algebraic representations, we aim to prompt a broader view to the fact that, among all the straight lines passing through $(x_0, f(x_0))$, the tangent is the one which *best approximates* the curve, in the sense that not only the difference $\rho(h)$ tends to zero, but so does the ratio $\rho(h)/h$. The picture of the

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[1] The blancmange function is defined for $x \in [-1,1]$ by the sum of the series $b(x) = \sum_{n=1}^{+\infty} b_n(x)$, where $(b_n)_{n \in N}$ is a sequence of modular functions inductively defined as follows: $b_1 = |1 - |x||$; $b_{n+1} = \left| \frac{1}{2^n} - \left| b_n(x) - \frac{1}{2^n} \right| \right|$. Each b_n is non differentiable at the points in the form $x = \pm \frac{k}{2^{n-1}}$, for $k = 0, \ldots, 2^{n-1}$. The sum b is continuous, since it is the uniform limit of a series of continuous functions, but nowhere differentiable.

graphs provides a geometrical interpretation to the approximation. As the user zooms in, by changing the value of h, this value acts as reference unit to the picture. If the straight line displayed is not the tangent, the vertical segment is always visible. On the other hand, if it is the tangent, that segment quickly disappears from sight. In other words, in this case, $\rho(h)$ approaches to zero, *even when compared to the unit* h.

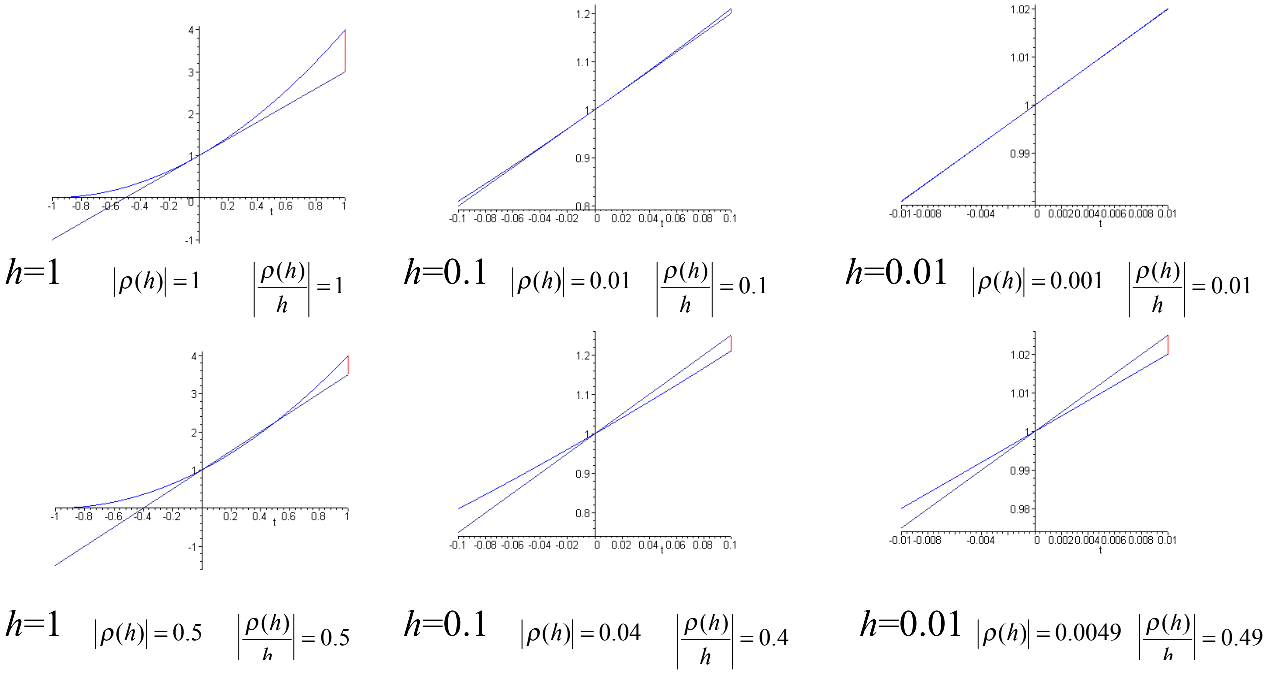

Figure 2: Screens from the generic organizer *Best Line*.

The approach described above has been tested in a calculus course in a Brazilian university. Among the students who took that course, we select the subject of the case study described in section 3. We instance below two excerpts (translated from Portuguese) from other participants' written questionnaires:

> Roughly speaking, we could compare it [the local magnification process] to the view of Earth we'd have if we observed it from the space, something like a sphere. But, if we come back and observe it from the surface again, it has details, which were undistinguishable before.
>
> Actually this function [the blancmange] really seems a mountain, a very irregular one. [...] It's like one'd take a big log and wanted to balance it on this very irregular mount. You cannot do that.

THEORETICAL-COMPUTATIONAL CONFLICTS

Research results show that misused computational environments can have negative effects to the learning of concepts. For instance, Hunter, Monaghan & Roper (1993) observed that students using software *Derive* did not need to substitute values to get a table and sketch functions' graphs. As a result, students did not develop the skill of evaluating functions by substitution. Even students who could perform the evaluation before the course seemed to have lost the skill afterwards.

In Brazil, Belfort & Guimarães (1998) observed teachers dealing with a dynamic geometry environment. They were asked to find empirically the rectangle with perimeter $40m$ and the greatest area. Due to floating point errors, the software would give approximate results. Therefore, three pairs of teachers obtained the maximum area of $100m^2$, but the values for the side AB were different. The authors report the teachers ended up in a deadlock, and were unable to figure out which would be the correct answer. However the investigation about the software 'mistake' led to the necessity of a theoretical solution. The authors conclude that it is possible to use software limitations as a tool for the development of deductive reasoning.

Hadas, Hershkowitz & Schwarz (2000) present a set of activities designed on a dynamic geometry environment to motivate the need to prove, by causing surprise or uncertainty from situations in which the possibility of a construction was against students' intuition. The number of deductive explanations increased considerably in situations involving uncertainty. The authors conclude proofs were brought into the realm of students' actual arguments, and they naturally engaged into the mathematical activity of proofing.

Doerr & Zangor (2000) report pre-calculus classroom observation on the use of graphing calculators. The authors claim that, contrary to previous concerns, the device did not become a source of mathematical authority. They remark that perspective was a consequence of the approach adopted by the teacher, particularly by her awareness to limitations of the calculator and her belief that conjectures are proved on the basis of mathematical reasoning.

Many authors agree that the effects of computers on mathematics learning do not depend on any inherent feature of the devices themselves. Rather, such effects are consequent from the way they are (mis)used. Tall (2000) affirms that the focus on certain aspects and the negligence of others may result in the atrophy of neglected ones. The experiment reported by Hunter, Monaghan and Roper, in particular, has uncovered a *narrowing effect*: *intrinsic characteristics of the computational representation led to limitations on the concept images developed by learners*. Generally speaking, many limitations of computational representations for mathematical concepts arise from the algorithms' finite structure. Figure 3 displays the process of local magnification of the curve $y = x^2$, around the point $x_0 = 1$, performed by Maple. Since the curve is differentiable, it should look like a straight line. Rather, due to floating point errors and/or limitations of the underlying algorithm, for very small values of graphic windows ranges (on orders lower than 10^{-6}) the software draws a polygonal.

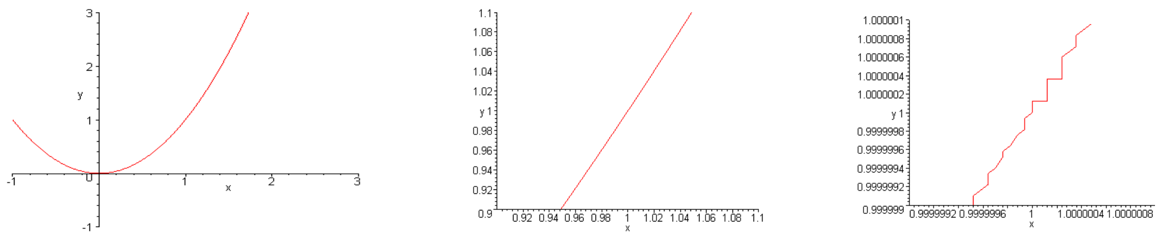

Figure 3: A theoretical-computational conflict.

To study more carefully those situations, we defined (Giraldo, 2001a) **theoretical-computational conflict** to be *any situation in which a computational representation is apparently contradictory to the associated theoretical formulation* (see also Giraldo & Carvalho, 2002a; Giraldo, Carvalho & Tall, 2002).

In our own interpretation, the narrowing effect observed on Hunter, Monaghan and Roper's experiment was not due to the occurrence of theoretical-computational conflicts, but, on the contrary, to their absence. Overuse of computational environments – specially when not confronted to other forms of representation – may contribute to the conception that limitations of the representation are characteristics of the mathematical concept itself, leading to the development of narrowed concept images. Sierpinska (1992) remarks that the awareness to the limitations of each of the form of representations and to the fact that they represent the same concept are fundamental conditions for the understanding of functions.

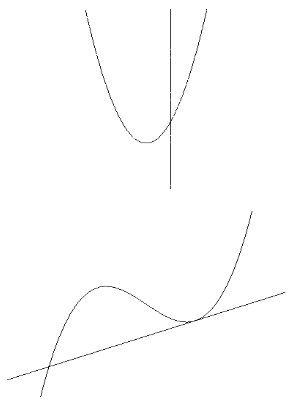

Figure 4: A line which cuts the curve only once, but is not tangent; and a line which cuts the curve twice, but is tangent.

For example, the representation of derivatives as slopes of tangent lines, as usually presented in classroom, comprises some limitations. Tall (1989) observes that of notion of tangent line in students' concept images is strongly linked to geometry problems about the construction of tangent to circles. The approach to those problems focuses on global geometric relationship between of the curve and the line, particularly, on the number of points of intersections. Thus, the notion of being tangent – to 'touch' in one single point – figures in opposition to notion of being secant – to 'cut' in two points; which do not corresponds to the concept of tangent, in the sense of Infinitesimal Calculus (figure 4). It is likely to expect that learners build narrow concept images having those ideas as main references. In fact, Vinner (1983), for example, observed that many students believe that a tangent line can only 'touch' the curve, but not 'cross' it. When asked to sketch the tangent line to the curve $y = x^3$ on the point $(0,0)$, some of those students drew a ray with origin in the point, not crossing the curve, and others a tiny segment close to the point $(0,0)$, but with slope different from 0.

However, into a richer context, where awareness to the limitations is encouraged, this same representation can take on a *converted* pedagogical role: it can underline that, *to study the differentiability of a curve, it does not matter what happens far from the tangency point*. In the same way, if theoretical-computational conflicts are emphasized instead of avoided, the limitations of computational representation can assume a *converted* role: they can work for the enrichment, rather than the narrowing of concept images. In particular, by emphasizing the conflicts arising from the finite structure of computational algorithms, the infinite nature of the central concepts of Infinitesimal Calculus can be highlighted.

A CASE STUDY

The experiment reported in this section is part of a wider study, in which six participants, selected from the course described on the first section of this paper, were observed in personal interviews dealing with theoretical-computational conflict situations (using software Maple). The interviews were tape recorded and fully transcribed. Global results are currently being analyzed. We will focus on the results of one of the participants, Antônio (pseudonym). We will summarize his responses (translated from Portuguese) to four interviews, concerning the concept of derivative.

Interview 1: Participants were given a few general questions concerning their conceptions about functions, continuity and differentiability.

Antônio was asked how could he decide whether a function is differentiable, given the algebraic expression. He stated that a function would be differentiable if he could apply known formulae to evaluate derivatives. He was then asked how he could decide about the differentiability if, instead of the expression, the graph of the function on a computer screen is given. He stated that he would zoom the graph in to have a more careful view, but it would be impossible to be sure, as computers are not flawless.

Interview 2: Participants were asked to gradually zoom in the graph of the function $y = x^2$ around the point (1,1), and simultaneously explain what they were observing. They would obtain screens similar to the ones shown on figure 3.

Antônio declared he would see something similar to the tangent straight line, as he zoomed in on the graph. When the software started to display a polygonal, he claimed that the computer was wrong, as this was not the expected result. After thinking for a while, he explained the computer's error:

> It's because the computer hasn't got idea what it's doing. It's kind of messing up the points. […] As the computer sketches the graph by linking the points and these points are results of approximations, so it links without thinking. It links the points, and whatever it gets will be the graph for it, as it doesn't know what goes on.

Interview 3: Participants were asked to zoom in the graph of the blancmange function around a fixed point, and explain what they were observing.

Antônio started by explaining the construction of the blancmange function. He showed good comprehension of the process:

> […] You are taking a number and multiplying it by $1/2$, taking that one and multiplying by $1/2$, by $1/2$. So, it's a geometric progression with rate $1/2$. […] Then, it's the sum of a geometric progression. The sum of a geometric progression is a limit, then it converges to a point. […] Then each point there is a geometric progression, it's the limit of a convergent geometric progression. It's there. […] It's well defined.

Starting the local magnification, he explained that, as the curve was not differentiable, the graph would become more wrinkled. However, to sketch the graph, the algorithm used a finite truncation of the series which defines the blancmange. As a result, it did not look more wrinkled, as Antônio expected, but quickly acquired a straight aspect. Antônio showed great surprise at that point, and asked the reason of the unforseen result. After listening to our explanation, he commented:

> Oh, I see. You could sum a few more steps, but not until infinity.

After thinking for a while, he proceeded, with increasing excitement:

> But infinity it [the computer] can't make. [...] Hey! I think nothing could make! [...] It can't add until infinite! There will be ever an infinity missing. And nothing can represent the infinity, as a whole, but we can show that it goes to that place, that it tends to that. [...] It's impossible to represent it, not on the computer, not on a sheet of paper, and not in anything else! The computer only represents things that a human being knows.

Interview 4: While dealing with the *Best Line* routine, participants were asked to explain their impressions.

Antônio declared:

> […] So, this guy here [points to the vertical segment on the screen] will have to decrease faster than *h*, because it *h* went faster, this guy would always be there. […] In the case of the derivative, this guy runs faster. We see it's a special property. [...] We've found a way to express the definition, but it's more [..] it's less hidden what it means. […] One can feel what's going on. Actually, we've written the definition, but this way is much clearer than the usual way, the limit and the straight line, and so on.

DISCUSSION

Since the first interview, Antônio clearly expressed his preference for algebraic representation. He states that the criteria for deciding about the differentiability of a function must be based on formulae. Moreover, he appears to be aware of the limitations of computational algorithms. Such mental attitude gave him means to quickly grasp the cause of the unexpected result on interview 2. In this sense, the theoretical-computational conflict was almost immediately solved by the student. On the other hand, in interview 3 a theoretical-computational conflict played a central role on Antônio's reasoning (see figure 5). In fact, Antônio's enthusiasm suggests the conflict actually triggered a new idea for him: *it is not possible to represent the concept of infinite by any physical means*. Moreover, he points out the reason for the impossibility: *infinity can never be attained*. The conflict led Antônio to grasp not only the limitations of the computational representation, but of other forms as well; and to figure out a conceptual distinction between finite and infinite.

Antônio's mental attitude towards conflict situations contributed to the results reported. The outcomes of the four interviews summarized above suggest that the conflict have acted as positive factor for the enrichment of Antônio's concept image of derivative and related notions. Nevertheless, other participants show quite different behaviors. In some cases, the conflicts do prompt students to engage into a rich reasoning. In others, the conflicts are barely noticed by students, as they are quickly solved. But some students seem not to cope with conflict situations at all. The global results of the investigation in which this experiment is comprised are currently being analyzed. One of our aims is to understand more clearly in which situations conflicts do have a positive role for the enrichment of learners' concept images, in particular, in which sense and in which extent learners' previous attitudes and background determine that role.

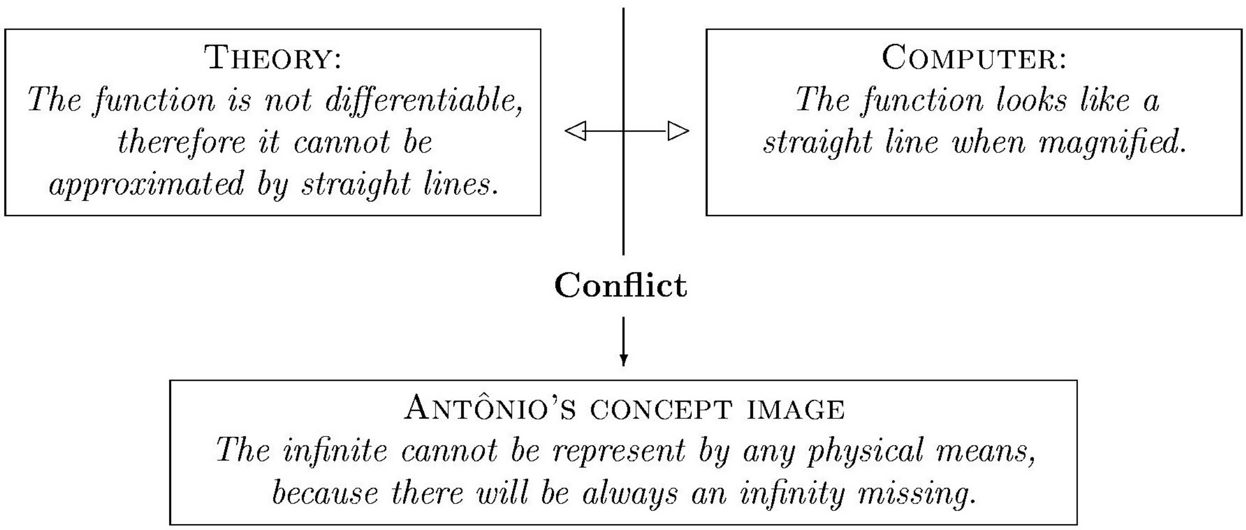

Figure 5: A theoretical-computational conflict acting in Antônio's reasoning.

PERSPECTIVES

Undergraduate teaching of mathematics often follows a model of *purely formal approach*, in which contents are presented into the same structure and order as the theoretical formulation. Several pedagogical obstacles have been pointed out as associated to that model. Cornu (1991) remarks that many expressions used in mathematical definitions have different meanings from current language. This is the case of the fundamental concepts of infinitesimal calculus, as ‘limit’ and ‘continuity’. Once the mathematical definition is formulated, the defined concept acquires the status of object itself, independent of the language employed. Thus, despite the fact that definitions are grounded on current language, their logical handling demands the abstraction of language. The main ideas necessary for the building of further theoretical developments often do not come out from formal definitions, but from related intuitive ideas (see e.g. Cornu, 1991; Tall & Vinner, 1981). Vinner (1991) stresses that the processes by which mathematical theories are formulated hardly correspond to their final organization. That being so, on introducing a given mathematical concept, we often appeal to representation forms different from the formal definition, and limited in relation to it – this is the case of the computational representations of Calculus concepts, the focus of this work. On the other hand, a model of approach supported by a single representation form is often associated to pedagogical obstacles from a different nature. As we have observed, if that is the case, a narrowing effect on concept images is likely to take place.

We read in the classic *What is Mathematics*:

> Whatever our philosophical standpoint may be, for all purposes of scientific observation an object exhausts itself in the totality of possible relations to the perceiving subject or instrument. (Courant & Robbins, 1941, p. xvii)

The aim of this work is to put forward an alternative model of approach, not purely grounded on formalism nor purely on imprecise representation forms. It is not meant to undervalue of the formalism, in relation to the imprecise. On the contrary, through the emphasis of limitations and differences, we intend to prompt the development of rich concept images, as well to stress the central role of the formal conceptualization on the construction of a mathematical theory.

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