REAL AND VIRTUAL CALCULATOR: FROM MEASUREMENTS TO DEFINITE INTEGRAL

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In this paper I present a case-study, part of a teaching experiment in the secondary school (11th-12th grade), focused on the construction of the concept of integral in the environment of measurements, considered by the point of view of Tall's cognitive roots. Particularly, the study refers to the students' cognitive processes with the framework of embodied cognition and instrumental approach. The aim is to analyse the passage from finite sums to infinite ones, with the mediation of the technology.

INTRODUCTION

This article arises from different motivations: cognitive, instrumental and curricular: they are at the basis of a teaching experiment at secondary school level, based on an approach to definite integrals, starting from various activities of measurement. From the cognitive point of view, the problem consists in the analysis of possible continuities or discontinuities that the passage from finite sums to infinite sums may determine. In fact, the major difficulty for understanding the concept of definite integral is the existence of the limit of the finite sums, which approximate an area under a graph. From an epistemological perspective, this passage implicates a discontinuity, which may be overcome in a cognitive way. In order to do this, an aid can be offered by the use of technology, so the instrumental point of view is necessary, for studying the mediation of a tool in activities aimed at the construction of the meaning of definite integral. The last but not the least, it has to be considered the curricular point of view, for planning a teaching experiment and realising it in classroom activities. In Italy, the integrals are often taught by the teacher in a way that hides the constructive procedure of resolution. In fact, the definite integral is usually introduced as a set of primitives, obtained from derivation by an inverted process, using sometimes a very high level of complexity. Then, the definite integral is defined as the difference of a primitive function in the extremes of an interval. The result is interpreted as the measurement (with a sign) of the area under a graph, eventually with the presentation of the fundamental theorem. This theorem, even when proved, remains obscure for the majority of students (if not for all), instead of clarifying the passages. This arguments are concentrated - in the curriculum - at the end of the final year of school (13th grade), nearly at the conclusion of Calculus.

THEORETICAL FRAMEWORK

In the research of Mathematics Education, theoretical frameworks have been elaborated, to explain from various points of view (cognitive, epistemological), how students construct the mathematical concepts, that is learning in its deepest meaning. In this article I will refer to the frameworks introduced by Sfard (1991), namely the theory of process and object; by Dubinsky (1991), who studied the interiorisation of processes or the incapsulation of processes in objects; and finally by Gray & Tall (1994) for the notion of procept.

To these references I add Tall's cognitive roots, of which two in Calculus are: the local straightness and the area under a graph, respectively of derivation and integration (Tall, 1989; 2000; 2002). For Tall, the historical succession of the concepts of derivative and limit (derivative was founded before than the limit) is linked to an analogue cognitive succession. This approach is also used in recent French studies of Mathematics Education on Calculus, referring to derivative (Maschietto, 2001).

In their book "Where mathematics comes from", Lakoff and Nùñez (2000) have introduced the Basic Metaphor of Infinity, which arises when we conceptualise the result of a never-ending process. The two domains (source and target) of the metaphor are characterised by an ordinary iterative process with an indefinite number of iterations, an initial state and a resultant state, after each iteration. The crucial effect of the metaphor is to add to the target domain the completion of the process and its resultant state as a unique final state. This metaphor allows us to conceptualise "potential" infinity (which has neither end nor result), in terms of a process with a unique result.

Relying on the work of Bruner (1966), Tall (2002) proposes to classify the representation modalities in three distinct ways: *Embodied*, *Symbolic-proceptual*, *Formal-axiomatic*. The first is based on human perceptions and actions in a real-world context; the second combines the role of symbols in arithmetic, algebra and symbolic Calculus, based on the theory of these symbols acting dually as both processes and concepts; the third refers to a formal approach starting from selected axioms and making logical deductions to prove theorems. The word *Embodied* refers to the theory of embodied cognition, introduced by Johnson (1987), Lakoff & Núñez (2000), but it is used by Tall in a more specific meaning: "to refer to thought built fundamentally on sensory perception as opposed to symbolic operation and logical deduction." (Tall, 2002).

Other than the cognitive aspects, it is necessary to consider the way in which the technological tools act on the concepts of Calculus and the way by which those concepts can model the didactic transposition. This is called the process of instrumentation, and involves many studies, among them, for example the one of Artigue (2001), based on the research of Verillon & Rabardel (1995) about the ways by which an artefact becomes an instrument for a student: through the appropriation of schemes of use (instrumental genesis). In this approach, it is important to consider also the role of the teacher, introduced by Mariotti (2002): in fact the artefact, even if it incorporates a mathematical meaning and some schemes of use, it does not function automatically in the construction of this meaning. However the artefact - inserted between the students and the teacher - can be used by her/him to exploit the communication strategies and to guide the students towards the meaning.

In this theoretical context, I present a research project aimed at the social construction of the concept of definite integral as a measurement tool for the area (this is the teaching practice perspective). In the research perspective, the aims of the project are: the passage from intuition to conceptualisation, with the support of metaphors (see also Ferrara's paper in the Group 1 of this CERME); the mediation offered by technology for the construction of definite integral, in favouring cognitive continuity where there are epistemological discontinuities (see also Arzarello's paper, in this CERME Group).

Particularly, I discuss a case-study on the conceptualisation of the definite integral, as an exact measure, obtained by approximated measures, based on Tall's cognitive root "area under a graph"[1].

THE TEACHING EXPERIMENT

The starting point is the measurement activity, in order to avoid theoretical constructions, that can be sources of misconceptions or wrong images (Rasslan & Tall, 2002) and to approach the problem of measurements with an embodied perspective (Tall, 2002). The teaching experiment is composed by two parts: the first, made in the environment of paper and pencil, is aimed at the calculation of approximate measures of areas and perimeters, both in the geometric plane and in the Cartesian plane (Robutti & Sabena, in print). The second is characterised by the use of a technological environment (calculators TI89), to determine areas under graphs in the Cartesian plane. The choice here consists in the realisation of a group of activities before the regular lessons on Calculus. Therefore, the same group of 25 students did the first part at the end of the 11th grade and the second in the 12th grade. The students are medium achievers, in a traditional[2] course of secondary scientific school.

METHODOLOGY

The teaching experiment is planned inserting the activities during the mathematics lessons. Each activity (from 2 to 4 classes) is done by the students divided in small groups (3 to 4 people), and it is followed by a classroom discussion on the results of the groups, guided by the teacher, who gathers the solutions and institutionalises the knowledge (Brousseau, 1997). The collection of data is made by a video-camera and written notes, in order to analyse students' gestures and language.

THE ACTIVITY

In the 11th grade, the students carried out some activities in order to determine an approximated area of a shape using different grids, and to come to the idea that (a

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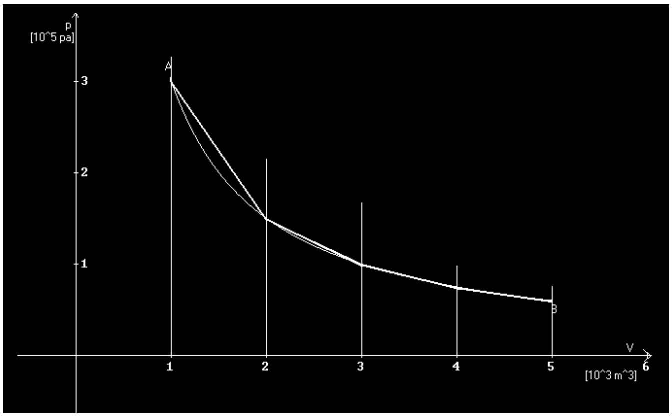
[1] As referred by the French *Commission de reflexion sur l'enseignement des mathematiques*, in its *Rapport d'etape sur le calcul* (2000), it is necessary to let live both the aspects of primitive and infinite sum (definite integral), through activities involving changes and rate of changes. It is fundamental to show the division of space in order to approximate variables with constants, for the passage to a limit. The *Rapport d'etape* mentions, as meaningful example, the area under a graph, saying that the approximation becomes a key-element for the conceptualisation.

[2] Traditional means here without a regular use of technological tools as computer, calculators, etc.

student said):*"The smaller the grids, the more they are precise"* (the number of squares which contain the shape and are contained in it), *"Because we used a measurement unity smaller ... ! You go towards* [he closes his index finger and his thumb]*"*. At a question posed by the teacher: *"Shall we stop here?"* the students answered: *"It is possible to go on to infinity"* and *"The uncertainty decreases more and more and the relative error ... goes towards zero"*.

At the beginning of the 12th grade, new activities are solved by the students, the first of which is the determination of the work made by a perfect gas during an isothermal transformation, represented by a hyperbola (figure 1) on the Cartesian plane (p,V)[3]. From the discussion about different procedures (obtained by the students' small groups) to determine the work (which corresponds to the area under the hyperbola) it arises the need of having an algorithmic formula, that can be implemented in a program of a symbolic calculator. Two algorithms are compared: one based on the approximation of the area using trapeziums (figure 2), and the other based on the rectangles below and above the graph of a function.



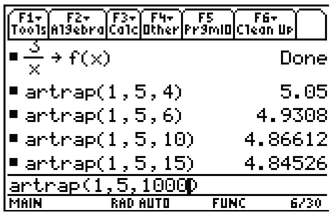


Figure1 *Figure 2*

The program of the trapeziums, written on the calculator, is used by the groups to determine an approximation of the area, for n=4 trapeziums. Then a discussion among all the students converges on the possibility to improve the approximation.

Teacher: *"If we want to calculate this area, what can we do with this program? Will we be happy to use it only once?"*

Fabio: *"No, no! We can use it with ... 6 trapeziums!"*

Andrea: *"But even with 10, so it is more ... it is less approximate!"*

Fabio: *"Even with 15, then!"*

Andrea: *"Even with 50!"*

Francesco: *"No, no, 1000!"*

Teacher: *"And the more trapeziums we use"*

[3] p and V respectively mean pressure and volume of a gas.

Andrea: *"And more precise is the result!"*

The idea that, while the number of the polygons increases, the result is more precise, is present in the students' minds, from the activities in paper and pencil of the previous year. Now, this idea is reinforced by the use of the program, which let the students divide the interval on the x-axis in a great number of parts. It is recognisable a transformational reasoning (Simon, 1996), as a dynamic interpretation of the increasing of precision in terms of numeric values of the area, depending on the increasing of the number of trapeziums. In fact, during the discussion reported above, this division into intervals, which become smaller and smaller, is a procedure made at a mental level (not at a physical one), and represents the first step towards the construction of a mathematical object.

In a small group discussion, a student says:

Erika: *"No, it must be smaller, because, look!"* She scrolls with her finger from top to bottom the column of uncertainties and then the first column (n).

The words and the gestures used by Erika reveal a dynamic interpretation of the table (which contains areas with the corresponding uncertainties), while the number n (of the subdivisions) changes. Erika (as other students do) scrolls the column from top to bottom, looking at the values of uncertainty and its changes. She points her attention to the global trend of numbers, observing the local changes from one value to the successive. After the students have used the program, the display - showed in figure 2 - has been projected, by the teacher, on a screen; after that a new discussion begins:

Teacher: *"The best we said was?"*

Andrea: *"The last!"*

Teacher: *"Why?"*

Andrea: *"Because it has more intervals and then ... "*

Stella: *"Because it gets nearer to the area"*

Teacher: *"But why is it so precise, if there are more intervals?"*

Andrea: *"Because ... with more intervals ... it is possible to give a better approximation of the curve with a line going to a more ... microscopic, and then ... nearer"*

Andrea's last phrase is interesting, because it reveals a passage from the global properties of functions to the local ones, as if Andrea could notice the local properties of a graph, after having observed the global ones, thanks to the sub-division of the interval on the x-axis. The student has the intuition that the more the intervals, the better is the approximation of a curve with segments, which are closer to the curve. The word "microscopic" reminds to the local approximation of curves with lines, that is the theoretical base of Calculus. The discussion continues with the next excerpts:

Teacher: *"The last is more precise: what does it mean saying more precise?"*

Andrea: *“That it gets nearer to the average value”*

Students: *“That it gets nearer to the real value”*; *“That it gets closer to the real value”*

The students come to the second step of conceptualisation: the idea that the last result of the program, which approximates the area, is more precise than the previous ones, because "it gets nearer to the real value". This second step is characterised by the consciousness that there exists a "real value" for the area, even if they don't have it, at the moment, because they have seen a succession of values approximating the area, but not "the end of the story".

Until now, students' attention has been focused on different procedures for approximating an area, among which some were implemented on the calculator. Now, their attention moves to the input/output processes of the implemented procedures (trapeziums and rectangles), and they are asked to compare the two processes, in order to understand which one is better for an approximate value of the area. During the discussion, two positions rapidly arise: one for the trapeziums, the other for the rectangles.

Fabio: *“Artrap is more rapid”*

Francesco: *“But it isn't more precise. Not generally, we don't know if it is, with another curve, because artrap isn't the curve, we don't know"*

Fabio: *“But it is the one which is nearer”*

Erika: *“But sure, in the interval there is the area, between ardif and arecc; it isn't more precise, in my opinion, because it is a unique value" ....*

Erika: *“**In the interval** there is surely the value we are looking for”*

Andrea: *“Yes, but you can't work with **an interval**, while you can work with **a number**"*

Francesco: *“Instead we know that of course artrap isn't **the** value, because it is approximate” ...*

Francesco: *“But, if we decide that an error of 0.1% is ok for us, we do ardif and arecc, so we are sure that the found value is the right one”*

The debate deals with the possibility of having **a value** for the area, using the trapeziums process (artrap is the name of the program on the calculator), with the impossibility, for this value, of being the real one. From the other side, the rectangles process gives **an interval**, that contains **the number** area, which can be given with a fixed approximation (ex. 0.1%), but it is impossible to use an interval in other calculations. In this phase, the students go on in the process of conceptualisation, passing through a third step: the comparison between different procedures, characterised by the naming (to use, for a process, a name coming from the program on the calculator: "artrap, ardif, arecc").

In the successive discussion, after having concluded that it is better the rectangles process, for its generality, the students are guided, by the teacher, to link it with another theoretical content, developed in the previous year.

Teacher: *"What do we remember thinking back to this situation?"*

Stella: *"The square root of 2"*

Teacher: *" The square root of 2. That is, when did we construct what?"*

Francesco: *"The contiguous classes"*

The students link the process of rectangles with the construction of a real number, as the square root of 2. But it isn't sufficient, because, if they understand the analogy between the measure of the area and a real number, they aren't able to bridge the gap from the approximation process to the exact value of the area. In order to do this, the teacher introduces an ideal calculator to help them:

Teacher: *"Now I am in an ideal calculator, which doesn't exist of course, and I imagine to do the calculation.*

Francesco: *"At the end we will have a root"*

Teacher: *"A root?"*

Francesco: *"No, a number ... What is their name?"*

Teacher: *"Real"*

The **ideal** calculator plays the important role of metaphor, that leads the students towards the measure of the area, thought as a real number. The program of rectangles on the **real** calculator, in fact, doesn't give this number (as soon experienced by **Francesco**: *"I put infinite instead of a number n, and the calculator answers undef* [undefined]*"*).

We can interpret this metaphor in the sense of Lakoff & Nùñez (2000), identifying the source domain with the **real** calculator (with rational numbers, operations, in particular sums), and the target domain with the **ideal** calculator (with real numbers as contiguous classes, operations without limitations, particularly infinite sums).

This metaphor is fundamental to do the passage from discrete to continuum, which marks the necessary step (the fourth and the final, in this activity) for giving a meaning to definite integral, not only as a process, but also as a concept. The metaphor itself guides the students to a conquest in cognitive terms: the understanding of a process (the approximation of an area under a curve by finite sums) which potentially goes on indefinitely, (the number of rectangles increasing) in terms of a concept. The concept itself is shown through a command (Integrate) on the calculator, that gives the definite integral, surely as a *black box* from the point of view of Calculus, but as a *transparent box*, from the point of view of mathematical meaning. The event described above can be recognised as a BMI: Basic Metaphor of Infinity (Lakoff & Nùñez, 2000, p. 159), in which it is possible to reach a final resultant state that follows every non-final state of an iterative process. A particular case of BMI is the Metaphor: Infinite Sums Are Limits of Infinite Sequences of Partial Sums (Lakoff & Nùñez, 2000; p. 197), that can be recognised in our situation.

The role of technology in this learning activity is of great importance, because it supports students in all the steps of conceptualisation. In the first and in the second steps, thanks to the program on the calculator, they come to the idea that the intervals on the x-axis can become smaller and smaller, and the last result which approximates the area is more precise than all the previous. In the third step, with the comparison between different procedures, they give a name to these procedures, using the same names of the programs on the calculator. The artefact calculator plays the role of an instrument for determining approximately an area, through the schemes used by the students (Verillon & Rabardel, 1995). In order to pass to the final step, the instrument is not sufficient and the role of the teacher becomes fundamental (as said by Mariotti, 2002), in the introduction of a metaphor based on the instrument utilised. The use of the metaphor of the ideal calculator refers to the same schemes activated by the students (the partial sums of areas) and extends them to infinity, in order to help pupils to do the passage towards the meaning of definite integral.

Therefore technology, in a real way and in a metaphorical way, plays here the role of a mean supporting the students in the construction of a concept from a process, in order to come to a procept (Gray & Tall, 1994).

CONCLUSIONS

This example gives a suggestion to Mathematics Education research: the possibility that the embodied cognition could furnish the tools (i.e. metaphors) in interpreting learning situations from a cognitive point of view. In this study we can observe that the metaphor introduced by the teacher is the launching pad for the students. In fact, it supports them in overcoming an epistemological discontinuity, that is the passage from finite to infinity, from discrete to continuum, marked by the definition of definite integral as the limit of finite sums.

It is important to point out the role of metaphors in teaching practice, not only in research. It can be a future research field, the study of Embodied Cognition with an educational perspective, to find what elements of the theory and in what ways they may play a fundamental role in linking research to practice.

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