WORKING GROUP 13
Applications and modelling

CONTENTS

INTRODUCTION TO THE WORKING GROUP "APPLICATIONS AND MODELLING"

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The discussions of the working group at CERME 4 were strongly influenced by different approaches towards applications and modelling, presented by various speakers which created a basis for a constructively critical and argumentative discussion. These discussions demonstrated that there does not exist a homogeneous understanding of modelling and its epistemological backgrounds within the international discussion on applications and modelling.

However, this is not a new situation at all. Nearly twenty years ago, Kaiser-Messmer (1986, pp. 83) showed in her analyses that within the applications and modelling discussion of that time various perspectives could be distinguished, internationally and nationally in Germany or German-speaking countries as well. These are the two main perspectives that emerged from the discussion that time:

- A **pragmatic perspective,** focussing on utilitarian or pragmatic goals, the ability of learners to apply mathematics to solve practical problems. Henry Pollak (see for example 1969) can be regarded as a prototype of this perspective.
- A **scientific-humanistic perspective** which is oriented more towards mathematics as a science and humanistic ideals of education with focus on the ability of learners to create relations between mathematics and reality. The 'early' Hans Freudenthal (see for example 1973) might be viewed as a prototype of this approach. Freudenthal changed his position at the end of his life, as he tended to take pragmatic aims more into consideration (see for example 1981).

Although these were the main streams of the discussion on applications and modelling further differentiations become obvious, especially on a national level. For a better understanding of the current approaches, the distinctions made by the scientific-humanistic perspective are helpful. Hans-Georg Steiner (1968) put **epistemological goals** into the foreground and emphasised the development of mathematical theory as an integrated part of the processes of mathematising. However, early attempts such as that of the French-speaking André Revuz (1971) are also important. He starts out from the triple situation-model-theory which means that models are constructed by starting from a situation which then leads to the development of a mathematical theory.

[1] The author wishes to express her thanks to Werner Blum for valuable contributions and discussions in the development of the new classification system.

Furthermore, an **emancipatory perspective** in the discussion can be identified, which is developing into socio-cultural attempts of mathematics teaching (for current approaches see for example Gellert, Jablonka, Keitel 2001).

A third stream, named **integrative perspective,** demands that applications and modelling should become subject to different levels of aims, that is to serve scientific, mathematical and pragmatic purposes but in a harmonious relation to each other. This perspective is not limited to specific aims and gets its strength from a wide range of aims and arguments (see for example Blum, Niss 1991).

The various perspectives of the discussion as reconstructed by Kaiser-Messmer vary strongly due to their aims concerning application and modelling. The appropriate references suggest various dimensions of aims. Kaiser (1995, p. 69f) distinguishes the following goals:

- Pedagogical goals: imparting abilities that enable students to understand central aspects of our world in a better way;
- Psychological goals: fostering and enhancement of the motivation and attitude of learners towards mathematics and mathematics teaching;
- Subject-related goals: structuring of learning processes, introduction of new mathematical concepts and methods including their illustration;
- Science-related goals: imparting a realistic image of mathematics as science, giving insight into the overlapping of mathematical and extra-mathematical considerations of the historical development of mathematics.

Comparable dimensions of aims are stated by Blum (1996, p. 21f) although he identified and described the nuances differently, and by Maaß (2004) as well.

Meanwhile, the current discussion on applications and modelling has developed further and become more differentiated. New perspectives can be identified which, as it became obvious from detailed analyses, emerged from the above described traditions or partly can be regarded as their continuations.

In the following, a classification system for present approaches of applications and modelling will be suggested by reverting to the previous differentiations summarized above but taking the current developments of the discussion on applications and modelling into consideration. This suggestion is based on recent analyses using literature mainly generated by ICMI and ICTMA activities and additional publications (see for example the reference list in the discussion document of the ICMI Study on Applications and modelling in mathematics education (Blum et al. 2002, p. 279f)).

This classification distinguishes various perspectives within the discussion according to their central aims in connection with applications and modelling and describes in short words the backgrounds these perspectives are based on as well as their connection to the initial perspectives. This ensures both a continuity for the present discussion as well as accumulates current perspectives coherently into the existing literature

Name of the approach	**Central aims**	**Relations to earlier approaches**	**Background**
Realistic or applied modelling	Pragmatic-utilitarian goals, i.e.: solving real world problems, understanding of the real world, promotion of modelling competencies	Pragmatic approach of Pollak	Anglo-Saxon pragmatism and applied mathematics
Contextual modelling	Subject-related and psychological goals, i.e. solving word problems		American problem solving debate as well as everyday school practice and psychological lab experiments
Educational modelling; differentiated in a) **didactical** modelling and b) **conceptual** modelling	Pedagogical and subject-related goals: a) Structuring of learning processes and its promotion b) Concept introduction and development	Integrative approaches (Blum, Niss) and further developments of the scientific-humanistic approach	Didactical theories and learning theories
Cognitive modelling	Psychological goals: a) analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes b) promotion of mathematical thinking processes by using models as mental images or even physical pictures or by		Cognitive psychology

	emphasising modelling as mental process such as abstraction or generalisation		
Epistemological or theoretical modelling	Theory-oriented goals, i.e. promotion of theory development	Scientific-humanistic approach of “early” Freudenthal	Roman epistemology

Table 1: Classification of current perspectives on modelling

When analysing the papers discussed during the sessions of the working group applications and modelling at CERME 4, one finds out that the apparent uniform terminology and its usage masks a great variety of approaches. It is remarkable that now, after a longer period of time, attempts from Roman language speaking countries were brought into the discussion on applications and modelling which start out from a more theory-related background. Partly they refer to the anthropological theory of didactics and to the approach of mathematical praxeologies of Chevallard emerging from anthropological theory, or they refer to approaches like that of Brousseau concerning ‚contract didactique'. In contrast to the approaches of realistic modelling, approaches such as those presented by Garcia Garcia & Ruiz Higueras at CERME 4, give less importance to the reality aspect in the examples they deal with. Both, extra-mathematical and mathematical topics may be dealt with, while the latter is then described as "intra-mathematical modelling". If the approach of praxeology becomes the main orientation, this leads to the fact that every mathematical activity is identified as modelling activity for which modelling is not limited to mathematising of non-mathematics issues. As a consequence these approaches show a strong connection to the science-oriented approaches of Steiner and Revuz for which mathematising and modelling is taken as part of theory development. However, these approaches are also rooted in the tradition of the scientific-humanistic perspective mainly shaped by the early Freudenthal. In his earlier work, Freudenthal (1973) understands mathematisation as local structuring of mathematical and non-mathematical fields by means of mathematical tools for which the direction from reality to mathematics is highly important. Freudenthal distinguishes local and global mathematisation, and for global mathematisation the process of mathematising is regarded as part of the development of mathematical theory. These approaches continue with a distinction developed by Treffers (1987): horizontal mathematising, meaning the way from reality to mathematics, and the vertical mathematising, meaning working inside mathematics. Freudenthal (like his successors) consistently uses the term mathematising. According to Freudenthal mathematical models are only found at the lowest level of mathematising when a mathematical model is constructed for an extra-mathematical situation.

Likewise, analyses show that the approaches from the pragmatic perspective were sharpened further until they became the approach of realistic modelling. For these kinds of approaches, authentic examples from industry and science play an important role. Modelling processes are carried out as a whole and not as partial processes, like applied mathematicians would do in practice. In summary, it can be stated that a characteristic of approaches described by Haines & Crouch or Kaiser is one in which modelling is understood as activity to solve authentic problems and not the development of mathematical theory. The described empirical studies even point out that newly learned knowledge cannot be applied directly in modelling processes, only with some delay. This fact has already been pointed out in earlier reports based on anecdotal knowledge (e.g. Burghes & Huntley 1982). In general, the presented empirical studies aimed at fostering modelling competencies demonstrate well underlying complexities which makes it difficult to achieve progress.

Besides these quasi polarising approaches, the realistic modelling and the epistemological modelling, there exists a continuation of integrative approaches within the educational modelling which puts the structuring of learning processes and fostering the understanding of concepts into the centre. However, the approach of educational modelling may also be interpreted as continuation of the scientific-humanistic approaches in its version formulated by Freudenthal in his late years and the continuation done by Treffers (1987) or respectively by De Lange (1987) for whom real-world examples and their interrelations with mathematics become a central element for the structuring of teaching and learning mathematics.

Within the discussion on applications and modelling, the approach of cognitive modelling, which exams modelling processes under a cognitive perspective, is new. Of course, the analysis of thinking processes by means of the approach of modelling is not new and is found in many theories of learning or cognitive psychology (see for example Skemp 1987). However, the analysis of modelling processes with a cognitive focus must be regarded as a new perspective as only recently a few studies were carried out, among others the study of Blum & Leiss which was presented at CERME 4.

The approach of solving word problems named contextual modelling, has a long tradition, especially in the American realm, but with the model eliciting perspective introduced by Sriraman at CERME 4 and referring to studies of Lesh & Doerr (2003), an explicitly theory based perspective has been established which is clearly going far beyond problem solving at school.

This perspective traces its lineage to the modern descendents of Piaget and Vygotsky, but also to American Pragmatists. The philosophy of this perspective (Lesh & Sriraman, 2005a, 2005b) is based on the premise that:

- conceptual systems are human construct, and that they also are fundamentally social in nature (Dewey and Mead);
- the meanings of these constructs tend to be distributed across a variety of representational media (ranging from spoken language, written language, to diagrams and graphs, to concrete models, to experience-based metaphors (Pierce);
- knowledge is organised around experience at least as much as around abstractions - and that the ways of thinking which are needed to make sense of realistically complex decision making situations nearly always must integrate ideas from more than a single discipline, or textbook topic area, or grand theory (Dewey);

- the "worlds of experience" that humans need to understand and explain are not static. They are, in large part, products of human creativity. So, they are continually changing - and so are the knowledge needs of the humans who created them (James).

In the contribution by Sriraman, an abstract task was used to decipher student understanding of modelling constructs developed within the models and modelling perspective. In particular the researcher was interested in knowing whether post graduate students could objectively apply the definition of a model eliciting activity to a problem that blatantly violated the design principles for model eliciting activities. Interestingly enough, Sriraman reports that the subjective experience of solving the problem caused considerable conflict for several students and prevented them from objectively applying the definition.

The papers discussed in the working group which two of them are not contained in the proceedings and therefore put into brackets, are classified according to the perspectives described in table 1.

Approach	**Classification of the papers and – if mentioned – referred theoretical protagonist**
Realistic or applied modelling	Haines/Crouch [Kaiser (Pollak)]
Contextual modelling	Sriraman (Lesh & Doerr)
Educational modelling; differentiated in a) **didactical** modelling and b) **conceptual** modelling	Vos (Freudenthal) Lingefjärd Henning/Keune (Niss) [Blomhoj (Niss)]
Cognitive modelling	Blum/Leiss
Epistemological or theoretical modelling	Garcia/Ruiz (Chevallard) Dorier (Brousseau)

Table 2: Classification of the papers presented at CERME 4

The classification of a paper to one category does not mean that the overall position of the researcher belongs to this category. It is possible and in a few cases even known that the overall approach of a person emphasises different aspects of modelling. Among others, Blum emphasises educational as well as cognitive modelling approaches in his recent publications. Furthermore it has to be pointed out that these classifications are not based on objectifiable and operational criteria but on the analyses of texts by means of a more hermeneutic understanding of text.

To summarise, these analyses demonstrate on the one hand that currently significant further developments are taking place within the discussion on applications and modelling, while on the other hand it became clear that these new approaches still go along with existing traditions and that they have developed further earlier approaches or fall back on them. However, the frequent usage of concepts from the modelling discussion should not be mistaken about the fact that the underlying assumptions and positions of the various modelling approaches differ widely. A precise clarification of concepts is necessary in order to sharpen the discussion and to contribute for a better mutual understanding. Thus, this suggestion for a description of the current discussion on applications and modelling is meant to be a first step into this direction.

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“FILLING UP“ – THE PROBLEM OF INDEPENDENCE-PRESERVING TEACHER INTERVENTIONS IN LESSONS WITH DEMANDING MODELLING TASKS

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Abstract:*In section 1, we will describe the starting point and the context of our research, the projects SINUS and DISUM. In section 2, we will present and analyse a typical example of a demanding mathematical modelling task, and report on how this task was used in the DISUM project. In section 3, the core part of thispaper, we will concentrate on some selected aspects of how teachers have dealt with this modelling task. Finally, in section 4, we will reflect upon these lessons and draw some conclusions.*

Keywords: Modelling, teacher intervention, empirical research.

① THE PROJECTS SINUS AND DISUM

Soon after the release of the unsatisfactory TIMSS results in 1997, the German government established a reform project that aimed at improving the quality of mathematics (and science) teaching: *“Steigerung der Effizienz des mathematisch-naturwissenschaftlichen Unterrichts”* (“Increasing the efficiency of math and science teaching”, abbreviation: *SINUS;* see Prenzel/Baptist 2001). It ran from 1998 to 2003. The participants were schools, 180 altogether, organised into 30 so-called “model projects” with 6 schools each, distributed all over Germany. The grades involved were 5-10 (that is, 10-16-year-olds). One of these 30 “model projects” was directed by us (Blum et al. 2000). The SINUS programme was, globally speaking, successful and was therefore considerably extended. The goal is to involve 2000 schools by 2007. The central aim of SINUS was, and still is, to teach mathematics so as to fulfil certain *criteria for “quality teaching”*. These criteria – both theoretically well-founded and empirically well-supported – are in short (Blum 2001, Helmke 2003):

I. Demanding orchestration of the teaching of mathematical subject matter

1 Aiming at *competencies* and providing *manifold opportunities* for learners to acquire competencies (opportunities for modelling, arguing, etc.; see Niss 2003).
2 Creating manifold *connections,* vertical ones (within mathematics) and horizontal ones (with the real world outside mathematics).

II. Cognitive activation of learners

3 Stimulating permanently *cognitive activities* of students, including *meta-cognitive* activities (that is a conscious use of strategies and reflections upon one's own activities; see, e. g., Schoenfeld 1992).
4 Fostering students' *self-regulation* and *independence* as much as possible, and reacting to individual students *adaptively,* based on a firm diagnosis.

In addition to these more subject-related criteria, there are criteria concerning general "classroom management":

III. Effective and learner-oriented classroom management

5 Varying teaching *methods* flexibly, and fostering *communication* and *cooperation* among students.
6 Fostering a *learner-friendly* classroom environment where learning and assessing are recognisably separated and *mistakes* are seen as good *learning opportunities*.
7 *Structuring* lessons *clearly* and using *time effectively*, among other things by preventing disturbances.
8 Using *media* (such as calculators and computers) appropriately.

In all aspects, the *teacher* has a crucial role to play. We can speak, in the words of Weinert (1997), of *"learner-centred and teacher-directed"* teaching.

In order to achieve this central aim of SINUS, two *guiding principles* were followed:

- The *"new culture of tasks":* Changing mathematics teaching requires the selection of appropriate tasks and their implementation in the classroom according to the quality criteria.
- The *"new culture of cooperation":* Changing mathematics teaching must be brought forward by the whole mathematics staff of a school, and more generally, all institutions (schools, universities) have to collaborate.

SINUS was, and still is, an ambitious programme. Quality teaching is not easy to accomplish, and classroom observations showed numerous *shortcomings.* Some of these shortcomings are definitely not due to a lack of practical realisation of existing knowledge by the SINUS teachers, but rather to

- a *lack of knowledge* of the actual procedures and difficulties of students when solving cognitively demanding tasks both in individual work and in pair or group work,
- a *lack of knowledge* of possible and appropriate ways for teachers to act when diagnosing students' solution processes and when intervening in case of students' difficulties, or in other critical situations.

"Appropriate" means "oriented towards the quality criteria", for example finding a proper balance between maintaining students' independence and self-regulation as much as possible *and* helping students to progress – an absolutely non-trivial problem for theory and practice!

These research questions were the starting point for the *DISUM* project (in 2002), an interdisciplinary project between mathematics education and pedagogy at the University of Kassel. DISUM means *"Didaktische Interventionsformen für einen selbständigkeitsorientierten aufgabengesteuerten Unterricht am Beispiel Mathematik"* („Didactical intervention modes for mathematics teaching oriented towards self-regulation and directed by tasks"; see Blum/Leiß 2003 and Leiß/Blum/Messner 2005). The subject of DISUM are modelling problems, mainly in grade 9. The project aims at dealing with these questions in a more systematic and carefully directed way than would have been possible in SINUS (that was established – and funded – as a practice-oriented classroom reform project). Accordingly, the *components* of DISUM are:

1. Detailed cognitive and subject matter *analyses* of modelling tasks (constructing the "task space" according to Newell/Simon 1972).
2. A detailed study and theory-guided description of actual *problem solving processes* of students in laboratory situations (pairs of students, sometimes with, sometimes without a teacher; method: videography and individual stimulated recall).
3. A detailed study and theory-guided description of actual *diagnoses* and *interventions* from teachers in these laboratory situations.
4. A detailed study of regular *lessons* with such modelling tasks, taught by experienced SINUS teachers, and a theory-guided description of these lessons, especially by means of our quality criteria.

For a considerable number of modelling tasks, these investigations have already been carried out. What will be done during the next two years is, in addition:

5. The construction of manageable and promising tools for
   a) the training of students in strategies for solving modelling problems,
   b) the training of teachers in "well-aimed coaching" of modelling problems.
6. A detailed study into the influence of
   a) students' use of strategies
   b) teachers' well-aimed coaching on mathematical achievement, in particular on modelling competencies of learners.
7. The implementation of the results into teacher education.

② THE "FILLING UP" TASK

One of the modelling tasks used in the DISUM project is the following:

Filling up
Mister Stone lives in Trier which is close to the border of Luxemburg. To fill up his VW Golf he drives to Luxemburg where immediately behind the border, 20 km away from Trier, there is a petrol station. There you have to pay 0.85 Euro for one litre of petrol whereas in Trier you have to pay 1.1 Euro.
Is it worthwhile for Mister Stone to drive to Luxemburg?

A global cognitive analysis yields the following ideal-typical solution, oriented to wards the well-known modelling cycle (fig.1):

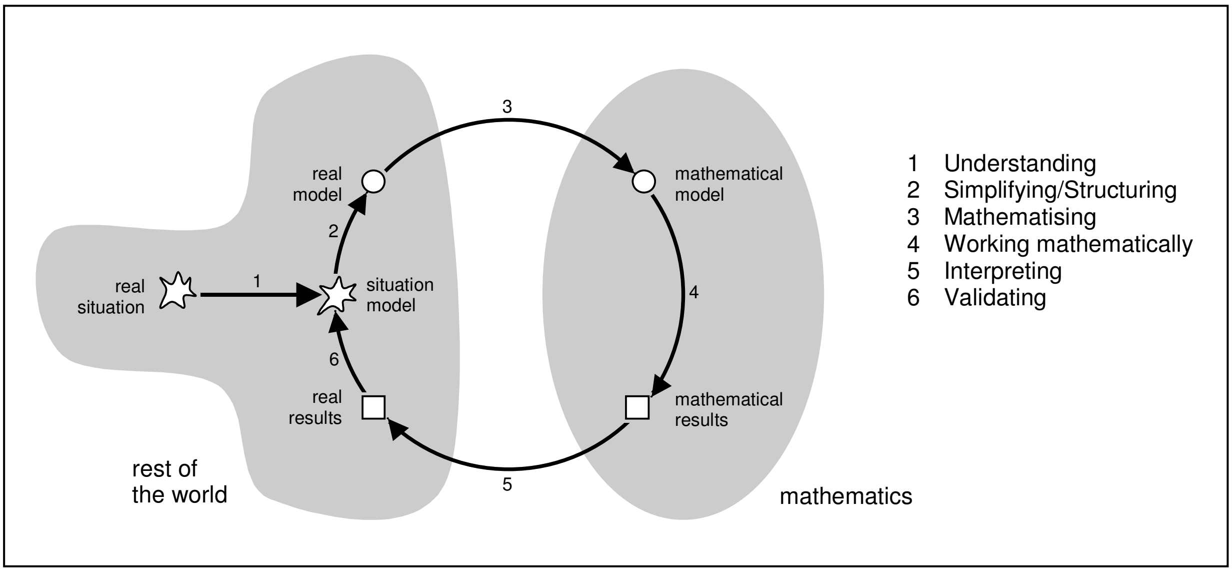
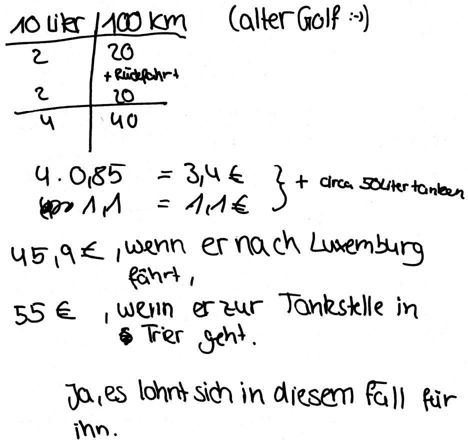


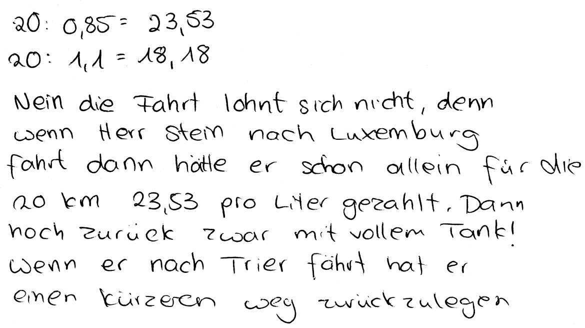
Fig. 1

First, the problem situation has to be understood by the problem solver, that is a *situation model* has to be constructed. Then the situation has to be simplified, structured and made more precise, leading to a *real model* of the situation. In particular, the problem solver has to define what "worthwhile" should mean. In the standard model, this means only "minimising the direct costs of filling up and driving". Mathematisation transforms the real model into a *mathematical model.* Working mathematically (calculating, solving equations, etc.) yields *mathematical results,* which are interpreted in the real world as *real results,* ending in a recommendation for Mr. Stone what to do. A validation of these results may show that it is necessary to go round the loop a second time, for instance in order to take into account more factors such as time or pollution. Dependent on which factors have been taken, the recommendations for Mister Stone might be quite different.

We have used the "Filling Up" task in lab sessions and in regular lessons as well as in various tests (in the SINUS project). Fig. 2 shows two *typical solutions* from students:



Standard model: comparing (only) the costs of driving and filling up



Traditional "solution": extract all numbers from the text and calculate these somehow, no matter what it may mean

Fig. 2

③ TEACHING "FILLING UP"

Our investigations have yielded a lot of interesting insights into students' problem solving processes and teachers' actions. Among the *results,* especially on the teachers' side, are the following:

- re-newed empirical evidence of the *indispensableness* of the well-known *modelling cycle* (see above), both as a metacognitive tool for students, and as an instrument for the teacher for diagnosis and well-aimed intervention
- a classification of various kinds of teacher *interventions*: related to content, to organisation and interaction, to motivation, and meta-level
- a distinction between working independently, with support from the teacher, on the one hand, and working totally on one's own, on the other hand; lack of support very often causes motivational, social, or cognitive problems and leads to failure
- a further development of existing *learning strategy models* (see, e.g., Kramarsky/ Mevarech/Arami 2002)*;* our model is comprised of five components: goals, volition, organisation, strategy, evaluation, and is actually doable by teachers
- insight into the importance of the teacher's broad *knowledge of the task space* as a solid basis for diagnosis, including the prediction of cognitive barriers, and for intervention. To put it less positively: the big influence of the teacher's idiosyncratic interpretation of the task space on his or her actions and, as a result, on the solution processes of the students.

For the rest of section 3, we will concentrate on the last-mentioned problem, in order to make just one aspect in the complex field of learning and teaching with modelling tasks a bit more concrete.

Several experienced SINUS teachers (in all kinds of schools and strands) have dealt with the "Filling up" task in their classrooms. In most classes, the lesson followed the same *pattern:*

1. Presentation and short discussion of the task
2. Dealing with the task individually
3. Solving the task in small groups
4. Presentation of the students' solutions
5. Reflection on the solutions

This pattern is different from the usual lesson script in Germany. However, this is still only a description of the surface structure of these lessons. Now, we will look a bit deeper.

All teachers had to solve the "Filling Up" task in advance by themselves. Let us take two teachers as an example. Teacher 1, Mrs. K., used the standard model which takes into account only the direct costs of filling up and driving. Teacher 2, Mr. R., considered more variables, such as time, and emphasised how important it is not to restrict oneself to the mere costs of filling up and driving.

Let us look at two excerpts from a 9th grade lesson taught by Mrs. K.

Excerpt 1:

S1:	What does "worthwhile" mean?
T:	"Worthwile" means whether it financially makes sense for him to drive across the border to fill up. Yes? Is the question okay?
S1:	Is it cheaper after all?
T:	Exactly. Whether it is profitable for him to drive cross the border or whether he should fill up in Trier instead. Exactly.

The student (a French exchange student) asks right in the beginning of the lesson what "being worthwhile" means. The teacher responds "whether it financially makes sense" and "whether it is cheaper", thus leading the students to the standard model. Later on in that lesson:

Excerpt 2:

S2:	You could also ask if maybe his workplace is past the gas station in Luxemburg because then he'd have to go that way, anyway.
T:	Yeah, okay, we still have to be realistic. If we take too many assumptions into account it'll get too tricky.

The student's question might easily lead to the consideration of time as an important factor. However, the teacher discourages the approach by speaking of "too many assumptions".

Let us now observe two excerpts from a 10th grade lesson taught by Mr. R.

Excerpt 3:

T:	So, what aspect have you incorporated into the 10 Euros?
S1:	The same as up here except with …
S2:	How much gas he gets.
T:	Just the fill up? Have you considered what driving costs apart from that? Or, yeah, he has to drive there and back. You have to estimate something.
S3:	How are we supposed to calculate that?
S1:	Yeah, mileage too and stuff.
T:	Yes, and time?
S1:	Looses value after all, and stuff.
T:	Exactly.

The teacher, in many respects a "non-invasive" type (in the sense of Barth et al., 2001), forces the students, rather early in their solution process, to take into account time and loss of value of the car as well.

Excerpt 4:

T:	Well, have you taken into account, if a car has 20000 km more on it it is worth less, after all, and …
S1:	Yeah, cause I had that …
T:	For that, he has to buy more tires and more oil and more of all kinds of things. Did you include that, too?
S2:	For how many kilometers do you necessarily need …
S1:	No. Whether he drives around 40 km in the city the whole time or goes there to fill up and comes back.
S2:	I even think it's almost better if you don't drive in the city but just drive straight through without stopping.
S1:	Yeah, cause in the city there's a lot of stop and go, there you have to, yeah, there it consumes more.
T:	It is certainly less than if he drives around in the city, but he doesn't drive around just for fun! He drives, after all, only if he has to drive. He is certainly not a fun driver, and if he's not driving around in the city he just drives to Luxemburg.
S1:	So if he, for example …
T:	Well, otherwise he would not drive these 2000 km, or in 10 years 20000 km.
S1:	Well, that balances out …
T:	Yeah, well, you have to …
S1:	Even if he drives around in the city? He drives there, but then he drives less in the city, they balance each other out a little.
T:	You have to consider that as well. That's what I'm aiming at, that you consider that as well.

The students argue that it might even be financially advantageous to drive out of town, but the teacher obviously cannot accept – probably caused by his ecological attitude – that Mister Stone is a fun driver, and emphasises that the loss of value caused by the additional mileage on the car is an important variable.

These examples show how strongly the *teacher's own conceptualisation of the task – resulting, among other things, from the teacher's preknowledge and beliefs – influences his or her type of intervention.* All SINUS teachers are familiar with the criteria for quality teaching, and their everyday lessons stand out very positively from typical German lessons. The intention of all participating teachers was, according to criterion 4, to foster students' self-regulation and independence as much as possible, in the sense of Montessori's "Help me to do it by myself", and to intervene in a minimal manner. In the end, however, it was even possible to assign students from various classes to their special class just by looking at the kind of solution they had accomplished.

We close this discussion with another excerpt from Mr. R.'s lesson which, in our view, shows a balance between preserving students' independence and supporting them actively.

Excerpt 5:

S1:	We don't know a lot of the data for the Golf, that's why we can't come up with an answer.
L:	What are you missing?
S2:	What it consumes and stuff.
L:	Consumes – your parents have a car?
S2:	Yeah.
S3:	A Clio.
L:	What does it consume?
S3:	Not much.
L:	What does "not much" mean?
S3:	So and so many litres per 100 km, but I don't know how many.
S4:	I think around 8 or 10, or? Could that be, roughly?
L:	That very well might be, yes. And what do your parents have?
S2:	Yeah, I have no idea how much.
S4:	We have an Escort, it's pretty much like a Golf, isn't it?
L:	That's definitely a good idea. So we'll take the consumption of an Escort.
S4:	Okay. Let's estimate, I don't know exactly, 9?
S3:	9.
L:	That's sounds pretty good, yes.
S4:	9 litres, okay.

The problem is – unavoidable in the solution of « Filling up » - that the students have to make assumptions about the gas consumption of the car. With the question as to how much their parents' car consumes, the teacher expresses two things: first, that the students are on the right track, and second, that the missing data have to be estimated, preferably by using everyday knowledge, not merely at random. At first glance, this intervention might appear rather common. However, it may be regarded as a minimal, independency-preserving intervention, as an effective compromise between saying nothing (leaving the students alone) and simply providing them with the missing data.

Of course, it might be argued that this could have been done in less time, for instance by only asking about parents' cars and not intervening any further.

④ EVALUATION

In section 3, we concentrated on only *one* aspect. There are many *other* aspects to be observed in these lessons. Looking at these lessons with "quality glasses" reveals that in all cases – in contrast to the large majority of everyday lessons in our country –:

- the teaching was oriented towards competencies, and the students had opportunities to model, to argue, to communicate,
- mental activities were stimulated,
- for the most part, the students could work independently,
- the atmosphere was tolerant towards mistakes and free of assessment.

However, some *problems* were also visible, in particular

- the difficulty of finding a proper balance between the students' independence and the teachers' intervention, influenced by his or her preknowledge and beliefs (as discussed in section 3),
- the lack of validation and of substantial reflection on the solution processes. There were indeed multiple solutions, and these were compared with each other (this alone shows that the observed lessons were far above average), but there were no discussion on the question of which initial data influenced the results, and in what way, and how accurate a result can actually be taking into account the rough assumptions made about tank volume and gas consumption of the car.

Only such functional analyses would yield a real understanding and would contribute to – in the words of Reußer (1998) – "the extraction of relevant conceptual-schematic and processual-strategic characteristics of a problem solution in an abstracting way". We refer to the discussion of that problem in Blum (2005).

So, there is certainly a potential for improvement even in the lessons of these experienced "best-practice teachers". More generally, the criteria for quality mathematics teaching have to be a central part of pre-service and in-service teacher education. The video documents produced in DISUM can certainly be used for the purpose of teacher education. This is already being done and will be done more extensively in the future (We refer again to Blum 2005 for a description of the teacher education programme COSINUS which, up to now, has already reached more than 70 % of all mathematics or science teachers in the state of Hessen, on a voluntary basis). We shall devote the final phase of DISUM, 2007 – 08, exclusively to an implementation of our materials and results into teacher education.

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AN INTRODUCTION TO MATHEMATICAL MODELLING AN EXPERIMENT WITH STUDENTS IN ECONOMICS

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Abstract: *The research presented here took place in the first year of French university with students graduating in economics and management. Our investigations show that these students do not greatly dislike mathematics, even if they often admit that they have difficulties. Moreover, they generally do not have any opinion about the utility of mathematics for economics. We have experimented with a teaching device, designed to change students' relation to mathematics and show them how to use mathematics in order to solve a problem in which mathematics does not appear at first. This situation is used as a paradigm for mathematical modelling.*
After a description of the context, we present this sequence with a brief analysis. Finally, we describe a didactical analysis using Brousseau's schema (completed by Margolinas) of the vertical structure of the 'milieu'.
Key words: modelling, mathematics applied to economics, proportionality, algebra, milieu.

INTRODUCTION

As teachers in charge of the mathematical instruction for students majoring in economical science and management in their first year of French university, my colleague and I have been concerned with the application of mathematics and the use of mathematical modelling in these fields.

Our teaching (Dorier and Duc-Jacquet 1996) includes several applications (mostly of calculus, series and linear algebra) to economics and management. In reference to the classification made by Blum and Niss (1991, 60-61), of the different approaches of teaching mathematics, including applications and modelling, our approach could be characterised as a 'mixing approach', in which "elements of applications and modelling are invoked to assist the introduction of mathematical concepts" (see Dorier to appear, for examples in English).
Blum and Niss (ibid., 53-54) have listed some of the obstacles to the integration of applications and modelling in mathematical teaching. They divide these obstacles in three categories, depending whether they refer to instruction, the learner or the teacher. In our teaching, we had the liberty to restrict the amount of mathematical concepts to be taught over the year, in order to have sufficient time to work on applications and modelling; this is a way of overcoming the obstacle from the point of view of instruction. To overcome the obstacle from the teacher's point of view, we worked with economists in order to investigate the economical contexts to which we could apply mathematics. The main obstacle came from the learner's point of view. Indeed, according to Blum and Niss:

> "Problem solving, modelling and applications to other disciplines make the mathematics lessons unquestionably more demanding and less predictable for learners than traditional mathematics lessons. Mathematical routine tasks such as calculations are more popular with many students because they are much easier to grasp and can often be solved merely by following certain recipes, which makes it easier for students to obtain good marks in tests and examinations."(ibid; 54).

At first, we thought that our students had a negative opinion about mathematics (due to their experience at secondary school) and would be glad to approach mathematics through applications and modelling in reference to their main subject, i.e. economics. However, we distributed a questionnaire to first-year students, about their perception of mathematics. We collected the answers to this questionnaire over five years. The main results show that our students are not simply weak students who dislike abstract mathematics and who are starved of practical application. On the contrary, they may like mathematics, even if it is complicated and even if they are not very successful, and last but not least, they do not care much about applications. What they like about mathematics is a certain form of security. For the vast majority, doing mathematics means finding the right recipe to guess the answer. If they see themselves as weak in mathematics, they often complain that they cannot find the right key to a problem, that they are not gifted. In a way, mathematics does appear to be a mystical subject, reserved to a circle of gifted people, who can magically find the right way to the solution.

Such representations are an obstacle to the use of applications and modelling in mathematics. Blum and Niss (1991) list five specific arguments for inclusion of applications and modelling in the instruction of mathematics. In relation to the state of our students' perceptions, two of them seem essential for our project, namely the 'picture of mathematics' argument and the 'critical competence' argument.

Indeed, the authors claimed that it is "an important task of mathematical education to establish with students a rich and comprehensive picture of mathematics in all its facets, as a science, as a field of activity in society and culture." They are also in favour of developing a "critical competence [which] aim is to enable students to 'see and judge' independently, to recognize, understand, analyse and assess representative examples of actual uses of mathematics, including (suggested) solutions to socially significant problems" (ibid., 43).

These two arguments seem essential for students who are in their last years of mathematical training and will have to use mathematics in an extra-mathematical professional context.

We have tried to apply these goals to the whole of our teaching. However, considering the main characteristics of our students, it seemed essential to initiate right from the first lecture a radical change in their perceptions of mathematics. Indeed, entering university is an important change in a student's life. Students expect some changes and it is an opportunity for the teacher to establish a new relationship with them. This is why we have decided to experiment with a teaching situation during the first one or two lecture of the year, in order to initiate a radical change in the students' expectations of mathematics, proper to make the use of applications and modelling more efficient in the rest of the year.

This project is an attempt to address issue 2 raised by Blum et al. (2001) in the discussion document of ICMI Study on applications and modelling in mathematical education: "What does research have to tell us about the significance of authenticity to students' acquisition and development of modelling competences" (op. cit., 160).

The aim of this paper is to present this situation and its experimentation with some theoretical elements for its analysis.

PRESENTATION OF THE SITUATION

This situation must have the following characteristics:

- The mathematics at stake must be elementary
- The initial problem must be easy to understand and posed in a totally extra-mathematical context.
- The answer should not be guessed to easily and yet be reachable with elementary mathematical competence.
- Different mathematical as well as not strictly mathematical models can be applied, giving partial or global, right or wrong answers.
- The situation must raise some issues concerning hypotheses to be made in order to make a real model of the initial real problem situation (Blum and Niss 1991, 38)

We chose the following problem, which is quite well known and has been experimented on in different situations:

> ***Wine and water problem**:*
> *Two identical glasses are filled with the same quantity of wine and water respectively.*
> *With a spoon, one takes some wine from the first glass and pours it into the glass of water and mixes it with the spoon.*
> *Then, with the same spoon, one takes exactly the same quantity as before from the glass containing the mixture of wine and water and pours it into the glass of wine, then mixes it.*
> *Which has the most? The wine in the glass of water or the water in the glass of wine?*

The situation was experimented on for five years by two different teachers, during the first lectures of the year with students entering university (between 3 and 4 hours, in two class slots). Four of these ten experiments have been tape-recorded. Apart from the teacher, one or two researchers were present and noted their observations[1]. The tapes have been transcribed and the students' written answers have been collected.

The mathematical lecture is given, in a lecture room with 150-250 students. In the experiment we used Legrand's (1988 and 2001) theoretical framework of *scientific debate in mathematics courses*. The scenario follows the following general schema:

[1] We would like to thank Annie Bessot for her participation in this research.

The teacher gives the text of the problem to the students, without extra comments.

First stage: give your opinion (*10 min. of research*) Students are asked to reflect on it individually or in small groups for about 10 minutes, to formulate a first opinion. The teacher is silent and does not circulate in the amphitheatre.

First vote: The teacher asks for a vote and writes the results on the blackboard according to four types of answer: “more wine” / “more water” / “other” / “?”.

The “other” and “?” answers are important, even if they may cover different types of arguments, they do not need to be discussed at this stage.

Students are asked to note the result of the vote and their own answer. The different kinds of answers are not discussed yet.

Second stage: convince the others *(about 20-30 min. of research)*

Now students are asked to write a letter to a friend far away in order to convince her/him of their opinion. The task is different, it is not only necessary to have an idea but they also have to find written arguments in order to convince someone else.

During this phase of research, the teacher circulates among the students, but should not say much and lets the students work on their own.

Second vote: Again students are asked to give their answer through a vote and the teacher marks the results on the blackboard.

The debate: This is the longest and most essential part of the situation. It may last over two hours. The task of the teacher is not easy, s/he distributes the round of speech. At several stages, s/he has to decide which type of answer s/he wants to put forward. For instance, right at the beginning of the debate, s/he will have to decide if s/he asks someone who thinks that there is ‘more wine’, or ‘more water’ or having vote for “?” or “other” to talk first. This will influence the rest of the debate. At first, for instance, it is better to ask someone who thinks that there is ‘more wine’ to talk first, in order to discuss the qualitative approach (see below). Throughout the debate, the teacher also has to make sure that everybody hears and follows, s/he writes the most important arguments on the blackboard using the exact terms of the students, summarises when necessary and institutionalises results when s/he judges that the discussion has come to a general agreement.

ISSUES ON MODELLING

We will now present a brief analysis of the problem before we come in the next section to a deeper didactic analysis of the sequence, especially regarding the question of mathematical modelling.

At first, one may not see the necessity of mathematics to solve the problem. Indeed, the problem does not raise any mathematical question. In this sense, the formulation used in the statement of the problem is (deliberately) deprived of any mathematical annotation (like, glass A and glass B, or such). Therefore, the problem can be approached on a purely qualitative basis. In this case, the most common first answer is something like: “There is more wine in the water than water in the wine, because the first spoon is full of wine, while the second spoon is not filled with pure water”. Of course, this argument can be rejected by the fact that some wine is brought back with the second spoon. In all the experiments, this type of argument appeared in the first stage of the debate (this is why it is essential that the teacher asks those who have voted for ‘more wine’ to talk first). Students may have some quite animated discussions, but they always realise that this type of argument come to a ‘dead end’ and that the qualitative treatment has to be overcome.

In response to this impossibility to solve the question on a purely qualitative basis, students may propose different types of answers, which come from different models of the problem. We give a list here, which is quite exhaustive. All these may not have appeared in all the experiments during the debate and appeared in a different order/different orders, but most of them can be found in all experiments in the students’ papers.

- *Numerical models*. These are specific cases in which the quantities of liquid in the glass and in the spoon are specified by numerical values. They lead to calculations, in which the main mathematical tool is the notion of proportion. Note that these models can be more or less general, for instance the quantity of liquid in the spoon can be expressed either numerically or as a proportion (or percentage) of the quantity of liquid in the glass. These examples, if correctly computed, lead to the correct answer, i.e. both quantities, of water in the wine and wine in the water, are equal. Nevertheless, the difficulties inherent to calculations in this context may lead to a wrong answer. In this sense, some choices of quantities can be more troublesome than others. For instance, the choice of *1000ml* for the glass and *10ml* for the spoon can easily lead to the idea that the second spoon contains *9ml* of water and *1ml* of wine (this is a typical mistake with proportion). Moreover, students convinced that there is more wine than water can unconsciously distort their calculations in order to prove what they are convinced of.

- *Extreme cases*. One can imagine that the spoon is as big as the glass (i.e. the first spoon empties the glass of wine). It is then easy to see that, at the end, each glass contains half water and half wine. On the other hand, one can imagine that the spoon is empty, in which case, the contents of the glasses remain identical. These two extreme cases are unrealistic and lead to the right conclusion without much calculation. A student offering such an argument is very likely to have a good understanding of the power of modelling for the situation.

- *Graphical models*. One can draw glasses and represent the liquids in it at the different stages, by cutting the content in the glass in different proportions. This leads to a more or less sophisticated graphical proof. A formal version of a graphical model appeared in several of our experimentations. The students had replaced the liquid by balls of two different colours in such a way, that calculations were easy to make.

• *Models using letters*. These can be mixed with numerical models, or even graphical models. Using letters to designate objects is often seen as a mathematical ability. Here, the use of letters is efficient, if it is applied to the unknown initial quantities of liquid in the glasses and in the spoon (the parameters of the situation), the second can be expressed either independently or as a proportion of the first. Below, we give a succinct proof, using Q as the quantity of water and wine in the glasses at the beginning and q as the quantity of liquid transferred with the spoon. The following table shows the quantity of each liquid in each glass at the three stages of the situation:

	Water in glass *A*	Wine in glass *A*	Water in glass *B*	Wine in glass *B*
Stage 0	0	Q	Q	0
Stage 1	0	$Q-q$	Q	q
Stage 2	$Qq/(Q+q)$	$Q^2/(Q+q)$	$Q^2/(Q+q)$	$Qq/(Q+q)$

The most complicated calculations occur in the second stage, when one has to find the quantities of water and wine in the second spoon. It is an interesting use of algebra and proportion, but we will not focus on this, in this text.

Besides answers of this type, with all the mistakes that can interfere, there are some possible models using letters, which are not pertinent. For instance, a student remaining in the qualitative approach can propose a solution, in which s/he decides to call x the wine and y the water, the rest of her/his argument being totally qualitative. There are also some mixed models in which the letters are used with both qualitative and quantitative value. Here is an example of what we have seen during one of our experiments, and is quite representative of a type of argument appearing in all experiments at some stage of the debate:

> "x is the water and y is the wine… so we have situation 1, where we have Δ equal, … the spoon, the level of the spoon, well a supplement. Then, we have a situation 1, where we pour the spoon of wine into the water. So it makes x plus Δy… Then we have a second situation, where we take a spoon of this mixture that we pour into the wine. It makes Δ, open a bracket, x plus Δy,…, plus y. then we develop…what is inside the brackets, it makes Δx plus Δ second y, … Δ square y… plus y. Then we see that we have the same quantity Δ… in the second situation, in the second glass, than the first situation, since there is Δy, a spoon of y, of wine, and Δx in the second situation, which is a spoon of… of water. Therefore, the supplement of wine in the water and the supplement of water in the wine, it is the same thing."

These different models proposed by the students have to be discussed during the debate, regarding the accuracy, the validity of the mathematical treatment and their degree of generality. These are all essential questions regarding modelling.

There is also another type of discussion, which usually only appears at a certain point in the discussion, at least after the purely qualitative approach has been rejected. This concerns questions regarding the hypotheses to be made about the real problem situation in order to make a real model, such as: "How can we be sure that there is exactly the same quantity of liquid in the two glasses?, "And in the two spoons?", "Are wine and water totally mixable?", "Since there is water in wine, how do we measure the quantity of water and wine in a mixture?".

We see that, besides the question of the use of algebra to solve the problem, we have there an enriching situation, proper to making important questions about mathematical modelling appear at an elementary level. In this sense, not only is this situation proper to initiate the change in students' perceptions (what the students think of) of mathematics, but it can also be used during the year, as a reference for several questions about modelling. The framework of the scientific debate is proper to generate a sufficiently rich discussion in order to make most different types of models and questions appear in the discussion. The teacher leads the debate in the sense that s/he is responsible for the validation and institutionalisation of the main results appearing during the debate.

Before we come to a more didactic analysis, we have to say that there is a very elegant and short solution of the 'wine-water problem' that does not necessitate calculation and use of proportion (as shown above). Moreover, this solution works even if there is not the same quantity of the two liquids at the beginning, and even if the two liquids are not, like wine and water, perfectly mixable. In other words, the same result holds if one starts with a glass containing any quantity of water and a glass containing any quantity of a liquid like oil for instance. Indeed, in the second spoon, both liquids can be present, so there is less water, but the small amount of water (let us call this quantity q') is exactly the same quantity of wine (or oil) brought back into the glass of wine (or oil), which means that in the glass of wine (or oil) there is q-q' of water and in the glass of water there is also q-q' of wine (or oil)!

This solution can arise among the students, but only at the end of the sequence. However, it can be explained to the students by the teacher (only at the end), if it has not arisen before.

DIDACTIC ANALYSIS IN TERMS OF 'MILIEU'

As we have seen, there is no mathematics visible in this situation at the beginning. However, students are in a mathematics lecture, therefore they know that they have to use mathematics to answer the question. The context of debate among students prevents any introduction of mathematics by the teacher. Thus students have the entire responsibility for building their mathematical strategy. In their individual research, they have to put forward their ideas regarding the situation and, during the debate, challenge their colleague's arguments. This is typical of a situation in which the learner is confronted with an 'antagonistic milieu' with which he interacts and has to acknowledge feedback from it. This is the basis for a didactic situation, in the sense developed by Brousseau (1986, 1997). Brousseau (1990) proposes a theoretical framework in order to analyse the different roles of the learners and the teacher in relation to the different levels of knowledge involved in a situation. In his theory of 'situations didactiques', teaching situations are described and classified according to the exchanges between students, the teacher and the *milieu*[2]. In this model, a learner solving a problem holds various positions, so does the teacher. Each different position corresponds to a different situation, with a different milieu, different knowledge and different postures for the learner and eventually the teacher. For each position, the triplet learner-teacher-milieu constitutes a type of situation, which is a certain level of analysis of the teaching situation in question. These different levels are theoretical models of the interactions between the students, the teacher and the knowledge in a certain milieu. Moreover, in the model, the different levels of situation and milieu fit into each other, like Russian dolls. Indeed, the milieu of level $n+1$ is constituted by the situation of level n. This is called the *vertical structure of the milieu*.

Originally Brousseau used this model in order to analyse the learner's work and created only four levels now known as the *sub-didactic* levels. While she was interested in interpreting not only the work of the learner but also the role of the teacher, Margolinas (1995 and to appear) introduced four new levels known as the *over-didactic* levels, which offer a kind of symmetrical analysis for the teacher. In the lower three sub-didactic levels, the learner: discovers the problem, makes real or mental experiments, searches in her/his previous mathematical knowledge what can help her/him, interacts with her/his friends, etc. These types of actions are 'below' what the learner does intentionally, in order to respond to the didactic injunction given to her/him in the didactic situation. In the three upper over-didactic levels, the teacher: thinks about her/his teaching in a general approach, in accordance with official guidelines, but also her/his representation of teaching and learning, s/he designs her/his teaching project, etc., before s/he implements it into the class. The lower over –didactic level (obtained by a descending analysis) and the upper sub-didactic level (obtained by an ascending analysis) define the didactic situation and must coincide for the situation to function correctly. Margolinas' analyses have pointed out some interesting didactic phenomenon due to the non-coincidence of the two didactic situations.

It is important to understand that, in this model, there is no notion of chronology. The student enters the situation by actions, but s/he may be in a position of acting on material while s/he tries already to answer the didactic injunction. All these levels are susceptible of interplays at any time during the teaching sequence. The model describes postures that overlap during that time, and are impossible to isolate in reality. It does not give account of chronological actions, since a student can be in two or more different postures at the same time and interact with different levels of the milieu, therefore being in different situations.

We will now give a brief description only of the four sub-didactical levels regarding our wine-water problem. We will give for each level, a description of the milieu (M),

[2] The term of milieu is generic in Brousseau's theory, it not only refers to something materialistic, it can include elements of knowledge, but also other students. It is essential to understand that a milieu is a theoretical object built by the researcher in order to analyse teaching situations.

the positions of the student and the teacher (St and T), the knowledge in question (K) and the situation (S). Note that in the sub-didactical levels, the teacher only appears in the two upper levels.

<u>*Ascending description of the sub-didactic levels*</u>

In bold characters are the names of the different elements in Brousseau's model

Level –3 (Objective situation)

M-3 (material): glasses, spoon, quantities of liquids, mixtures, transfer of liquids.

St-3 (objective): (imagines) actions of transferring parts of a liquid, pure or mixed.

K-3: knowledge about mixtures of wine and water. Additivity of quantities on a qualitative basis ("if one adds, it raises", "if one takes away, it diminishes", "in a mixture, there is less of each liquid than the whole", etc.)

S-3: (imaginary) manipulations of liquids' transfer.

Level –2 (Situation of action)

M-2 (objective)= S-3

St-2 (acting): quantifies and/or designs one or several models.

K-2: identification of the parameters of the situation: initial quantities in each glass, volume of liquid transferred by the spoon. Exploration of a model and its (implicit or explicit) hypotheses: equality of the quantities transferred at each step, perfect miscibility of wine and water, equality of the proportions of each liquid in the glass and in the spoon, etc. Elaboration of arithmetical or algebraic relations, congruent to the operations of transfer: what do we need to determine at each step?

S-2: designing of models (numerical, graphical, extreme cases, with letters, mixed).

Level –1 (Learning situation)

M-1 (action) = S-2

St-1 (learner): quantifies the decanting operations.

T-1: (observer) Observes students' capacity in using their mathematical knowledge in order to quantify (arithmetical and algebraic tools).

K-1: calculations, arithmetical and algebraic rules, proportion, percentage, meaning of the hypotheses in the model, etc.

S-1: solving the problem in the model(s).

Level 0 (Didactic situation)

M0 (learning) = S-1

St0 (student): writes her/his solution.

T0 (teacher): gives the problem to the students.

K0: use of models: results in a model give ideas on the initial question, pertinence of quantitative models.

S0: debate on the results of the different models.

Here we cannot develop in detail how to use this model to analyse the situation, but we will now give the most important results of this analysis. Note that this type of description is also important in order to help a teacher who would like to lead such a debate in her/his class.

Passing from the objective situation to the situation of action (levels –3 and –2) is essential regarding modelling, since it necessitates the recognition of the ineffectiveness of the qualitative approach. In their individual research, some students may not be able to overcome this stage. Here, the debate is crucial in order to make all students go beyond this stage, with minimal didactic injunction. The diversity of opinions in the class and the discussion of arguments among pairs is an essential part of the debate.

Level –1 presents some mathematical difficulties. Since this situation is not specifically designed in order to work on arithmetics or algebra, if the debate among students is not sufficient, the teacher may have to intervene in a more didactic way on this matter.

The teacher wishes to institutionalise not only the results on the wine-water problem, but also a more general result that can be qualified as a meta-level) about the use of mathematics in such a problem. In the over didactical level 1 (the **situation of project** in Margolinas' model) the teacher is in a posture (P1) of designer of a project (here introducing modelling) and the student (St1) is in a reflexive position about what he is learning. In other words, P1's project is to make students access explicitly to the level of St1, which normally remains unconscious in a situation (it belongs to the over-didactic levels). This will necessitate a negotiation from the teacher in the last phase of the institutionalisation. This is also the key in order to use this situation as a paradigm for all modelling situations to be studied in the future by students in mathematics lectures. The facts that the mathematics at stake in this situation are quite elementary and that the problem is easy to solve, make this meta-level accessible. The teacher must give an explicit discourse at the end of the situation, but also needs to refer regularly to this situation during the year. In our experiments, students had very positive reactions to this situation. The discussions were enriching and consistent. Parts of the debates were regularly evoked during the year on different occasions involving modelling. Students showed a significant change in their perceptions of mathematics.

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MATHEMATICAL PRAXEOLOGIES OF INCREASING COMPLEXITY: VARIATION SYSTEMS MODELLING IN SECONDARY EDUCATION

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Abstract: *We present part of a research developed in the framework of Anthropological Theory of didactics (ATD) on the study of "proportional relationship" and "functional relationships" in Secondary education in Spain. Firstly, and after dealing with some researches about mathematical education on "modelling and applications", we reformulate these problems on ATD. Secondly, and based on a curriculum and textbooks analysis, we describe the scope of "proportional relationship" and "functional relationships" in the Spanish Secondary education. Finally, and as a conclusion we suggest a possible educational process based on increasing complexity mathematical praxeologies, that, we think, will allow rebuilding "functional relationships" from their own raisons d'être: the study of variability.*

Keywords: Epistemological approach, modelling, praxeology, proportionality

1. INTRODUCING ANTHROPOLOGICAL THEORY OF DIDACTICS

In the works of Chevallard (1999), Chevallard, Bosch and Gascón (1997), Gascón (1998), Espinoza (1998), Bosch and Gascón (2004), it is shown the way that researches in Didactics in Mathematics have evolved in recent years and how the Anthropological Theory of didactics (from now on, ATD) has emerged, considering the incapacity of other theories to explain some aspects of educational phenomena. This new modelling also allows the emergence of new educational problems, which could not be set out in other theoretical frameworks.

1.1. The mathematical activity: mathematical praxeologies

One of the ATD basic axioms is that "toute activité humaine régulièrement accomplie peut être résume sous un modèle *unique*, qui résume ici le mot de *praxéologie*". (Chevallard, 1999, 223). Two levels can be distinguished:

– The level of *praxis* or "know how", which includes some *kind of problems* which are studied as well as the required *techniques* to solve them.

– The level of *logos* or "knowledge", of the "discourses" that describe, explain and justify the used techniques. This is called *technology* and the formal argument, which justifies such *technology*, is *theory*.

Mathematics, as a human activity, can be modelled in terms of praxeologies, called *mathematical praxeologies* or *mathematical organizations* (from now on, MO). In order to have the most precise tools to analyze the institutional didactical processes, Chevallard (1999, p. 226) classifies mathematical praxeologies as: *punctual, local, regional* and *global.*

In a simplified way, we can say that what is learn and taught in a educational institution are mathematical praxeologies.

1.2. The process of study: didactic praxeologies

Mathematical praxeologies do not emerge suddenly. They do not have a definite form. Otherwise, they are the result of a complex and ongoing activity, where there exist some invariable relationships in its operative dynamics, which can be modelled. There appear two aspects very close to the mathematical activity:

- The process of mathematical construction; the *process of study* and,
- The result of this construction; the *mathematical praxeology*.

Chevallard (1999, p. 237) places this *process of study* in a determinate space characterised by six educational stages[1]: (1) first encounter, (2) exploration of the type of tasks, (3) construction of the technological-theoretical environment, (4) work on technique, (5) institutionalization and (6) evaluation.

Once again, this process of study, as a human activity, can be modelled in terms of praxeologies, which are now called *didactical praxeologies* (Chevallard, 1999, p. 244). As every *praxeology*, didactical praxeologies include a set of problematic educational tasks, educational techniques (to tackle these tasks) and educational technologies and theories (to describe and explain these techniques).

There appears a new conception of didactics of mathematics, where *didactics* identifies everything which can be related to *study* and *aid to study*: “Didactics of mathematics is the science of study and aid to study mathematics. Its aim is to describe and characterize the study processes (or didactic processes) in order to provide explanations and solid responses to the difficulties which people (students, teachers, parents, professionals, etc) studying or helping others to study mathematics face” (Chevallard, Bosch y Gascón, 1997, p. 60).

2. MODELLING AS A MATHEMATICAL ACTIVITY

Researchers of didactics in mathematics have a growing interest since the middle eighties, on the role that modelling processes can play in teaching and learning mathematics in all levels of the educational system.

When formulating educational problems, modelling related problems are often linked to mathematical application problems and solving application problems (both integrated in more general problems of Problem Solving). Two different research trends can be distinguished in this framework. Obviously there are relations between them.

[1] The idea *educational stage* is defined not in a chronological or linear sense, but in the sense of dimension of the mathematical activity.

One trend is centred in the research of the role that modelling and applications can play as a learning tool. For instance, survey works carried out in the frame of Realistic Mathematic Education (Gravemeijer, 1994; Gravemeijer and Doorman, 1999).

The other trend is based on mathematical competencies and skills which all students should acquire (modelling and applications among them). Consequently, the underlying educational problems are focused on the study of the features of teaching and learning processes of modelling and applications, as well as on its way of integration in the curriculum of mathematics. (Blum, 1991; Niss y Jensen, 2002).

The main feature of most researches is that the sense given to the ideas of *modelling* and *modelling processes* is very close to the sense assigned in the mathematical institution. So, modelling educational problems is often linked to the mathematical applications problems to reality, or to other subjects, in accordance with the mathematical interpretation of the term *modelling*.

We want to remark the fact that, in the epistemological model of mathematics underlying in this research domain, the idea of *modelling* is not a problem, as it is not a problem, for instance, in the research of *biology* or *economics*. The built "patterns" of the *modelling processes* are very close to those suggested by mathematics itself. They are seldom modified or extended from the considered experimental facts.

3. THE MATHEMATICAL ACTIVITY AS A MODELLING ACTIVITY

One of the main axioms of ATD is that "most of the mathematical activity can be identified (…) with *a mathematical modelling activity*" (Chevallard, Bosch and Gascón, 1997, p. 51). This does not mean that *modelling* is just one more aspect of mathematics, but mathematical activity is in itself a *modelling activity*.

First, this statement is meaningful if the idea of *modelling* is not limited only to "mathematization" of non-mathematical issues. Second, this axiom will only be meaningful if a precise meaning is given to the *modelling activity* subject from the own ATD.

ATD proposes that the mathematical activity can be identified as an integrated and articulated process of successive extensions of MOs, which makes up a modelling process. This new view acquires full sense when considering *intra-mathematical modelling* as an essential and inseparable aspect of mathematics. If so, the researcher's interest is not focused in the relationship between mathematics and "real world", or other subjects, nor in the way could students establish this relationship. The interest is focused in the analysis and description of conditions and restrictions which allow the development of *study processes*. These processes start from relevant problems of the *raison d'être* of the knowledge which are preferred to promote and can create a mathematical activity characterized by the construction MOs of increasing complexity in a learning environment.

4. THE PROPORTIONALITY RELATIONSHIP IN SECONDARY EDUCATION

In Spanish Secondary Education, the *proportionality relationship* between magnitudes is a paradigmatic case of *'thematic confinement'* (Chevallard, 2001). The influence of upper levels of didactic codetermination on school mathematics causes its atomization and fragmentation in different *areas* and *sectors*.

The "classical problems" of *proportionality* (direct and inverse) are integrated in the "Numbers and algebra" *area* and the "Magnitudes" *sector*. However, the study of proportional functional dependencies is considered to belong to the "Functions and graphs" *area*.

From the analysis of the curriculum and different textbooks, we notice an inadequate articulation between both *sectors*, causing, among others, such educational phenomena as the isolation of the proportionality relationship in the possible relationships between magnitudes.

Students rebuild, at least, two isolated MOs about *proportionality*:

– The first, whose *raison d'être* is solving classical arithmetic problems on proportionality.

– The second is a more general problem in functional relationships between magnitudes.

In the first case, the *proportionality* relationship is essentially static: given three specific measurements (two of the same magnitude and the third one being different) the problem lies in calculating the missing measurement. In the second case, the *proportionality* relationship is essentially dynamic. School tasks focus on representing linear functions and studying their properties (intersections with coordinate axis, slope of a straight line).

5. MODELLING VARIATION SYSTEMS: A PROCESS OF REBUILDING MATHEMATICAL PRAXEOLOGIES OF INCREASING COMPLEXITY IN SECONDARY EDUCATION

As a **conclusion**, we suggest a didactic praxeology which allows rebuilding *relatively complete* local MOs (Fonseca, 2004) on the study of situations where two magnitudes vary depending one on the other univocally (functional dependency) in the third and fourth year of secondary education (14-16).

5.1. MOs' raisons d'être. *General issues*

When the origin of any domain of mathematics is analyzed, one of the essential issues to consider is the *raisons d'être* which caused its creation and development and its presence in educational systems. The identification of these *raisons d'être* will allow the formulation of *main issues* to create a relatively complete didactic process. This could be described as a study process characterized by rebuilding a set of increasing complexity MOs and will involve the development and performance of a *mathematical modelling activity.*

In relation with "functions", where it is not possible to formulate a unique issue precisely, the origin of the possible formulation lies on the *study of variations*; in the study of situations where two or more magnitudes vary, depending ones on the others, and situations where relative questions arise such as the way of classifying that variation.

In the Spanish Secondary Education, "functions" are created, first, as a means to represent and describe situations of variation, that pretend to have a "real" nature, whose existence and type of variation are given in advance. This way, the Spanish curriculum (Royal Decree 3473/2000) establishes, for the second academic year, the following criteria of evaluation:

> 12. Representing and interpreting Cartesian graphs and points of simple functional relationships, which are based on a direct proportionality and are given by value charts, and exchanging information between charts and graphs.
>
> 13. Obtaining practical information from simple graphs (continuous stroke) for the resolution of problems related to natural phenomena and everyday life.

Afterwards, these functions become independent of their role as *models* of specific situations (exogenous problems), and focus on the study of the characteristics of their graphical representation and their algebraic expression (endogenous problems). Also, the Spanish curriculum establishes the following evaluation criteria for the forth academic year:

> 9. Interpreting and representing in a graphical way all constant, linear, similar or quadratic functions from their characteristic elements (slope of the line, points of intersection with the axis, vertex and axis of symmetry of the parabola). As well as interpreting and representing the simple exponential functions and simple functions of inverse proportionality, using significant charts of values, where they can be also assisted by a scientific calculator.

When approaching the study of "functions" at schools, there is no focus on the specific type of variation. The type of variation that describes a function is studied in High Schools, when introducing the idea of derivative function and the idea of derivative of a function at a point.

We propose to include the study of the nature of the variation that is described by each functional relationship. This way, we will be able to construct a local MO that is *relatively complete* for the Secondary Education, and that will allow us its amplification to a *regional* MO, based on the study of systems of variation amongst magnitudes. In general, these issues will be similar to the following one:

$\boldsymbol{Q_{var}}$**:** How can we describe the type of variation between two or more magnitudes?

This issue is restricted to the case of *functional dependencies*, i.e. situations in which the quantities included in one or more magnitudes depend one-to-one on the quantities of another magnitude (independent variable).

5.2. An educational process: a "savings plan"

Let us consider, at least theoretically, a hypothetic situation in which we can distinguish between two or more magnitudes, among which a relationship is induced or justified by a *technological* component.

This situation, which is very general, can be placed in an economic and commercial environment: the planning of a "savings plan". This will be carried out by using a *technological* "discourse" chosen for didactic purposes. This basis will determine a first construction of the system we want to model, which must be unknown to the student. The election of this basis will be the first variable specification and limitation, and will set the relationship between variables: The magnitudes included will be time (V_1) and money (V_2).

The complete construction of the system will be done according to the following restrictions: V_1 and V_2 have a one-to-one relationship, and the set of quantities of V_1 is a discrete one, with its elements evenly separated.

The issue of the proposed environment may be generally formulated, in the following way:

Q_S : How can a specific "savings plan" be planned (SPl_i)?

This question is critical in several ways:

- The fact that Q_S is very general forces us to take new decisions about the possible type of variation between V_1 and V_2 (second stage of system construction).
- It can lead to a mathematical activity using basic technical elements, like those of arithmetic.
- The construction of different solutions, i.e. different savings plans (punctual praxeologies SPl_i). These will act as models of the original systems, and will be the source of new issues.

The system will not be constructed, and it will be the student's responsibility to create it. For this, he/she will have to:

1. Chose a first stage *Ci*, that will be provisional (situation parameter).
2. Decide how the following stages will be generated, i.e. the type of variation defining the system. There is not a single way of undertaking this task, but decisions must be taken regarding the two system variables. We will focus on decisions about the type of variation, expressed as a recurrence of first order: *if I deliver a C_n quantity in a "n" payment, in the "n+1" payment I will render a C_{n+1} quantity, that will be related to C_n in the same way as C_n was related to C_{n-1}.*

3. The student will also have to *simulate the system*, by means of constructing a set of stages that is comprehensive enough to study the evolution of the system.

In this context, the Q_S question will be specified in the following task:

> Π: We intend to plan a journey with enough time, as an end-of-year trip. For this, we must decide a savings plan that allows us to raise enough money. Though we do not know the specific amount of money needed, we can estimate it, as well as decide deadlines, amounts to pay, and so on. Obviously, the issue is not deciding now how much money must be rendered and how will it be done, but rather to start working on it, intending to foresee the end of the academic year and the needs by that time.

In general, the types of variation may be sorted out in three main categories: "equitable" (the same quantity is given in each payment), "cumulative with an increasing fee" (the fee is greater in each payment than in the previous one) and "cumulative with a decreasing fee" (the fee is smaller in each payment than in the previous one).

The different types of variation will be specified in each category. The following could be an example of a "cumulative with an increasing fee" type of variation: In the first payment, a *C* amount is delivered and, in the following payments, the same amount given in the last payment plus the original *C* is supplied. This way, in payment 1 we would give *C*; in payment 2 we would give $C+C$ and in payment 3 we would give $2C+C$, and so on. This is an *equitable* condition of the "variation of the variation".

In these cases, the system simulation requires the selection of particular values for the original parameters (the first fee C_0 and the *C* amount), as well as the generation of stages by means of basic arithmetic combinations. In the previous example:

τ_C^1

x (months)	0	1	2	3	4	5	
y (euros)	C_0	y_1	y_2	y_3	y_4	y_5	

$+C$ $\quad + (C+C)$ $\quad + (2C+C)$ ....

A punctual MO is constructed for each type of variation, and the MO is defined by an arithmetic method that allows the generation of system stages, the resolution of comparisons between different "savings plan" and the advance of the total amount that will have been raised by an *n* payment.

In order to progress in the study process, it is necessary to analyse the reach and legitimacy of these arithmetic methods. The student will need to develop new methods when facing a set of tasks for which the previous methods are inadequate, as they are too difficult or seem not to be applicable in this context.

There are at least two types of task that cause this development:

– Control and anticipation tasks: They will require the determination of the parameters that define the system, in order to obtain the desired final savings.

– Comparison tasks: These are related to the previous ones, as it is necessary to define certain original parameter values of two or more “savings plans”, in order to make one of them equal or exceed the other.

For instance, a “cumulative with an increasing fee” type of variation will require the following sort of control tasks:

$T_{control}$: *Each group must create a savings plan of the “cumulative with an increasing fee” type that would generate each of the following final amounts (* $C_f^1, C_f^2, C_{f,...}^3$ *). For this,* C_0*,* C *and the number of fees must be previously selected in a proper way.*

There are three parameters (C_0, C and the number of fees). Every time two of them are assigned a value, the calculation of the third one is a problem task, called *control task*:

– $T_{control}^{I}$: Once C_f, , C_0 and C are fixed, how many payments would be necessary?
– $T_{control}^{II}$: Once C_f, , C_0 and a $\hat{n}$ number of payments have been fixed, what is the value of C?
– $T_{control}^{III}$: Once C_f, C and a $\hat{n}$ number of payments have been fixed, what is the value of the original fee, , C_0?

The limitations of the arithmetic methods of stage simulations are stated by the savings plans tasks of comparison and of control and anticipation. This is done by creating the need of calculating the fee delivered in each payment, whether they have the same type of variation or not. This way, a new problem arises:

T : How can we obtain an algebraic expression that allows us to calculate at any moment the amount saved, according to the original parameters?

The resolution of this task can be very complicated. But in the restricted case of recursive “savings plans” that is being considered, the work developed on this recurrence leads to a general method:

τ_{rec}: Once different stages of a system are created, the method consists of developing a recursive process, in which each term is written in accordance with the previous one, until we return to the original parameters.

This is not a repetitive method, and depends on the type of variation under consideration. For the prior case of cumulative with an increasing fee plan:

$$\tau_{rec}: \quad y_0 = C_0 \ ; \ y_1 = C_0 + C \ ; \ y_2 = y_1 + 2C = C_0 + C + 2C \ ; \ \ldots \ ; \ y_n = y_{n-1} + nC = C_0 + \sum_{k=1}^{n} k \cdot C$$

Given the fact that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, we can assume that $y_n = C_0 + \frac{n(n+1)}{2}C = \frac{C}{2}n^2 + \frac{C}{2}n + C_0$.

This algebraic expression supposes an extension of the arithmetic method of stage simulation τ_C^1 and will lead to the development of the preceding punctual praxeology. This development will be demonstrated when resolving the previous tasks, and will lead to new problems.

For example, now it is possible to create new methods to solve $T_{control}^{II}$:

$$\boxed{\tau_{a\lg} \quad : \quad C_f = \frac{C}{2}n^2 + \frac{C}{2}n + Ci \longrightarrow C = \frac{2(C_f - Ci)}{\hat{n}^2 + \hat{n}}}$$

Due to the obvious limitations of the space given, we have only been able to outline an educational process that will help to build "functional relationships" through a study process at the secondary schools. This way, students will be able to construct mathematical praxeologies of increasing complexity. This process has already been implemented with students of the 4º year of the Spanish compulsory secondary education (15-16 years) and is currently being experimented with students of the same level in High Schools (17 years) and of the first stage of tertiary education (teachers training).

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GETTING TO GRIPS WITH REAL WORLD CONTEXTS: DEVELOPING RESEARCH IN MATHEMATICAL MODELLING

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Abstract: *We are concerned with applying mathematics in real situations and understanding real world- mathematical world transitions. We report on modelling and applications using exemplar models with conclusions on novice-expert behaviour.*
Key Words: Mathematical Modelling; novice-expert behaviours.

1. RECOGNISING MODELLING SKILLS

Society's view that mathematics is useful is reflected in the presence of applications and modelling in the mathematics curriculum in schools, colleges and universities. Our research has been concerned with how pupils' and students' achievement in applications and modelling can be recognised and understanding how students develop into good mathematical modellers. Mathematical modelling usually operates under teaching and learning paradigms that embrace either holistic or dissected approaches. Concentrating on the latter we developed multiple-choice questions (MCQs), providing readily understandable contexts for the student, that focus on stages within a modelling cycle (Haines & Crouch, 2001) with a view to charting students' progress per se and to provide a snapshot of the development of their understanding of mathematical modelling processes. These early MCQs were constructed in six analogue pairs for use as research tools in pre-and post-test format as a measure of student achievement; they were later extended to provide indications of the success of the curriculum and its delivery (Izard et al., 2003). We also began work on developing a rating scale for mathematical modelling with wide applicability. Further insights into mathematical modelling behaviour were obtained by students answering various MCQs, and then completing a reflective questionnaire on how they arrived at their preferred answers, followed by an in-depth interview with a tutor. We were then able to classify processes involved in problem solving and to report on difficulties faced by students in moving freely between the real world and the mathematical world (Crouch & Haines, 2004). Our classification is broad but effective, the three categories being a: where the relationship between the mathematical world and the real world input to the model is taken into account; b: where there is limited evidence of this being so and c: where there is no evidence at all or where the problem has been simply looked at in real world terms taking no account of either the model or the mathematics. Our results, from differing perspectives, provide strong evidence that student learning in the transition from the real world to the mathematical model, is hampered by lack of knowledge and experience of abstraction. This behaviour is not so marked when moving from the mathematical model to the real world, indeed in this case the higher process level a is more likely to be used.

2. USING EXEMPLAR MODELS: A MODEL OF A KIDNEY MACHINE

Presenting exemplar models is a classical approach to teaching applications of mathematics within mathematics and in other disciplines. It is usually teacher led in contrast to the holistic and dissected approaches mentioned above. It often involves the presentation, discussion and analysis of several applications in particular fields addressing a basic applied mathematics problem, which is to model a physical system mathematically so that it sheds light on the mechanical working of the system in the real world. For example, recent modelling courses at City University have included models: kidney machine; aggregation of amoebae; road traffic flows; dimensional analysis of physical phenomena; n-stage rockets. In this approach students acquire a strong understanding of particular models and learn to compare and contrast developments in complex models. More than other approaches, this needs a strong focus on aspects of modelling from the teacher because modelling itself is not the driver, though there are firmer opportunities to link knowledge due to the prospect of strong engagement and motivation. To fix ideas and to provide context for this discussion we describe a simple model for kidney dialysis.

Some health problems are readily understood by students, and kidney failure, leading to kidney transplants and kidney dialysis, is one example. If we concentrate on dialysis either as a long-term treatment or as an interim procedure prior to transplant, then the kidney machine is a fruitful bioengineering problem. It is well suited to a modelling curriculum where exemplar models are presented. The model outlined here, reported by Burley (1975), is used extensively with university students. Contextual information provided for students includes knowledge that the kidney is a body organ that filters out waste material such as urea, creatinine, excess salts etc. from the blood. Waste products in the blood pass through the porous walls to the insides of the active units in the kidney. If this process malfunctions, waste products build up in the blood to toxic levels resulting in kidney failure and, without intervention, death. Following kidney failure, waste products may be removed artificially through a dialyser or kidney machine. In the 50 years these machines have been in use, their design has improved efficiency and reduced costs.

A two-compartment model (Fig.1) has blood from the body and the cleaning fluid, the dialysate, in adjacent compartments. They are separated by a thin membrane, which allows waste products in the blood to permeate through to the dialysate. The flow through the membrane is by diffusion from high concentrations of waste products (in the blood) to low concentrations, the diffusive effects are improved by making the blood and the dialysate flow in opposite directions on either side of the membrane. The dialysate is constructed to suit the needs of the patient.

The rate of removal of waste products depends primarily on four parameters: the flow rate of the blood, the flow rate of the dialysate, the size of the dialyser and the permeability of the membrane. Amongst the assumptions are: that all properties depend only upon x, the distance along the dialyser; all properties are independent of time, a *quasi-static* assumption and the amount of material passing through the membrane is proportional to the concentration difference, a *quantitative* assumption about the permeability of the membrane.

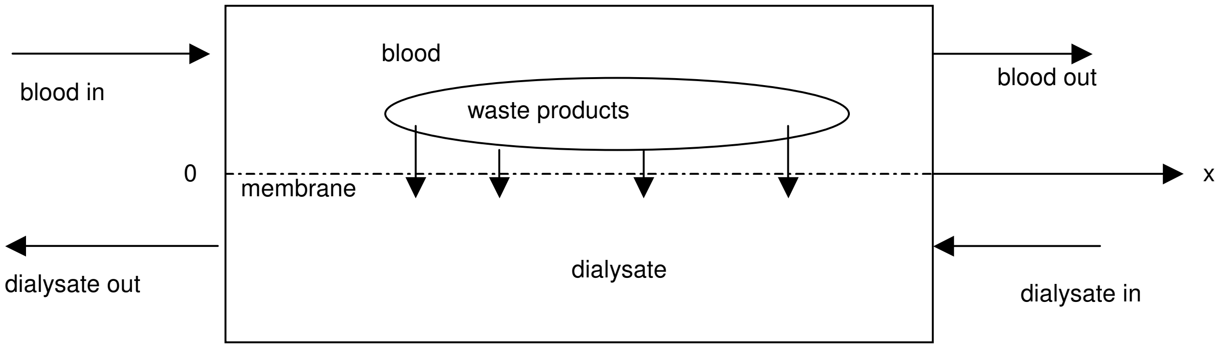

Figure 1. Schematic representation of a kidney machine

This leads to a model described by two coupled ordinary differential equations: $q_B \frac{du}{dx} = k(v - u)$ and $-q_D \frac{dv}{dx} = k(u - v)$ in which q_B and q_D are the flow rates of the blood and of the dialysate respectively, u(x) and v(x) are the concentrations of the waste products in the blood and the dialysate respectively and k is the permeability of the membrane. These equations may be solved by various methods and under different boundary conditions, simple ones being $u(0)=u_0$ and $v(L)=0$ referring to a base level u_0 for the concentration of waste products in the blood and clean dialysate. The clearance Cl of waste products is a measure of the efficiency of the machine, defined as a ratio of the difference in concentrations u on entry and exit to the machine to the concentration u on entry. The model of clearance, a function of key parameters, and its interpretation proves interesting.

3. MULTIPLE CHOICE QUESTIONS: KIDNEY MACHINE MODEL

We have constructed MCQs, focusing on stages of modelling, for use where mathematics and applications is taught through exemplar models. We now give each of the five MCQs, we comment on its structure and its associated partial credit assessment for each distractor. The MCQs were given to 51 final year mathematics undergraduate students in 2004. They were told that in each case they should indicate their preferred answer of the given options A, B, C, D and E and that credit is attached to more than one of these. Student responses show that in most cases the preferred answer was in fact that which gained most credit; for the reader, credit is indicated beside each option. However, the students were also asked to write a brief statement justifying their preferred answer. This statement, attracting credit as part of the coursework, gives insights into modelling processes and understanding.

Question 1 In a simple model of a kidney machine, the boundary conditions are $u(0)=u_0$ and $v(L)=½\, u_0$. Which of the following statements are true?

A. The dialysate is clean on entry to the kidney machine [0]

B. The waste products in the blood are at twice the level of those in the dialysate on entry to the kidney machine [2]

C. The blood is clear of waste products [1]

D. The waste products in the dialysate are at twice the level of those in the blood on entry to the kidney machine [0]

E. The waste products in the blood remain at the level u_0 [1]

Commentary: In order to understand this question, the mathematical expressions $u(0)=u_0$ and $v(L)=½\ u_0$ must be interpreted. In the modelling cycle, the activity falls within interpreting mathematics and concerns the transition from a model to a real world context. The distractors themselves are therefore phrased in real world terms. Option B, gives the correct mathematical interpretation of the boundary conditions and therefore attracts 2 marks. C could be true in the special case $u_0=0$ and for E this would be so if no waste products permeate through the membrane so in each case 1 mark might be awarded depending upon the justifying statement. Options A and D are untrue and attract zero credit.

Some justifying student responses:

Student M1: In his justifying his answer (A), M1 writes 'All the waste products in the blood can diffuse through the membrane from the blood and pass on into the dialysate, the dialysate *has* to be clean on entry to the kidney machine'. M1 has not connected with the model and the given boundary conditions. He has answered the question from the real world context only, for it is clear to him that for any kidney machine to be efficient it must use clean dialysate, never mind the given model.

Student M2: M2 relates 'I think B is the answer because I think at the beginning, there is no waste products in the machine. After the procedure, only half of the waste product from the blood flow to the dialyst. In the normal situation it must be $u_0 = v(L)$'. By choosing the correct answer B, M2 has interpreted the boundary conditions but in justifying his choice he focusses on the real world itself rather than the impact on the real world of the given boundary conditions. He has also not understood the steady state requirements of the model. He lacks understanding of the real world situation and how this is reflected in the given model. He suggests that the concentration of waste products in the dialysate is usually the same as that of the blood on entry to the machine. If this were so then the clearance would be zero.

Student M3: This student chooses A justifying that choice by 'The dialysate *should* be clean on entry to the kidney machine'. He makes no connection with the model.

Question 2: In a simple model of a kidney machine which of the following pairs of parameters would usually be adjusted to increase its efficiency (clearance).

A. The flow rate of the blood; the size of the dialyser [0]

B. The permeability of the membrane; the flow rate of the blood [1]

C. The flow rate of the dialysate; the flow rate of the blood [2]

D. The permeability of the membrane; the size of the dialyser [0]

E. The size of the dialyser; the flow rate of the dialysate [0]

Commentary: This question requires a practical understanding of the basis on which the model is constructed and how the parameters may be changed in the real world. In modelling terms, this question is on the boundary between the real world and the mathematical model either at the beginning of the cycle or at the end. The distractors are all focussed on the practicalities of these connections. Once the kidney machine has been constructed (made), it is very difficult to change the size (length) of the dialyser therefore options A, D and E do not attract credit. Option C is the preferred answer attracting 2 marks, since the flow rates of the blood and of the dialysate are easy to adjust. The permeability of the membrane is also difficult to alter, but it could be done by changing the membrane itself, therefore option B attracts 1 mark.

Some justifying student responses:

Student M2: In choosing option B, M2 says ‘ The flow rate of the blood is quite important but the permeability is more important. It is because if there is good permeability, the waste product can diffuse to dialyst faster. With suitable speed of flowing and the good permeability, the efficiency will be the best’. This is a good answer, but in focussing on permeability he has not understood that the flow rates are much easier to adjust in practical terms. He does make strong links between the real world and the model.

Student F1: She chooses the correct answer but supports it by quoting

‘We know that $\mathrm{Cl} = q_B \left(\dfrac{1-e^{-\alpha L}}{1-\dfrac{q_B}{q_D} e^{-\alpha L}} \right)$ where only q_B and q_D can be changed’. F1 does not recognise that α is a function of the permeability and of the flow rates of the blood and of the dialysate. She does not discuss the length L, she restricts her attention to the model and does not link with practical aspects to change parameters.

Question 3: Suppose that the permeability of the membrane is given by $k(x) = \begin{cases} x & 0 \le x \le \frac{1}{2}L \\ L-x & \frac{1}{2}L < x < L \end{cases}$ Which of these statements best describe this situation?

A. More waste products are removed from the blood at each end of the dialyser than at intermediate points [0]

B. Waste products are removed from the blood at a constant rate throughout the dialyser [0]

C. The removal of waste products from the blood increases towards the centre of the dialyser [2]

D. The removal of waste products from the blood varies through the length of the dialyser [1]

E. The removal of waste products from the blood increases as x tends to L [0]

Commentary: The permeability of the membrane, k(x), is a key parameter for the removal of waste products from the blood. This question introduces into the model a membrane for which the permeability varies along the length L of the dialyser. Understanding of the problem is increased if a graph of k(x) is included in the solution and the resulting graph is then interpreted in the practical situation. Although the question is firmly located in the mathematical world it requires continual referencing back from a sub-model to the real world situation. Since the permeability varies along the length L of the dialyser, option B attracts no credit. Options A and E indicate that k(x) is greater at x=0 and/or x=L than at intermediate points which is the opposite of the case so these two do not attract credit. Option D makes a general

statement on variability through the length L of the dialyser. This is consistent with the given model and so attracts one mark. C is the preferred option gaining full credit.
Some justifying student responses:
Student F2: F2 chooses the correct option C, but in supporting it, she makes no reference to the given model of permeability, preferring (wrongly) to rely on arguments based on her own interpretation of how the dialyser works. She writes: '…because since we are dealing with permeability of the membrane, at the start there is 0 permeability and towards the middle it increases, thus the removal of waste products from the blood increases as the blood flows towards the centre of the dialyser'. F2 focusses on her real world view, not linking it with the given model.
Student M5: Having chosen the correct option C, he justifies it thus: '$k(x) = \begin{cases} x & 0 \le x \le \frac{1}{2}L \\ L - x & \frac{1}{2}L < x < L \end{cases}$ Fick's Law states that the amount of material passing through the membrane is proportional to the concentration difference. Towards the centre the membrane is more permeable so more waste products are removed. (Diffusion through the membrane is from high to low concentrations)'. It is difficult to see whether M5 has interpreted the model of permeability correctly, without a sketch graph of k(x). The links between k(x), his own statements and the physical situation are far from clear. Stating Fick's Law is not relevant for this problem.
Question 4: A model of a kidney machine is described by the differential equations:
$q_B \frac{du}{dx} = k(v - u)$, $q_D \frac{dv}{dx} = k(v - u)$ with $u(0)=u_0$ and $v(L)=½\, u_0$
Which of the following statements best describes how they may be solved?

A. They can always be solved using a matrix method involving eigenvalues $0, k\left(\frac{1}{q_D} - \frac{1}{q_B}\right)$ [1]

B. They cannot be solved by substituting w = u – v, when the permeability k is either constant or linearly dependant on x [0]

C. They can be solved using a matrix method, when k is constant, involving eigenvalues $0, k\left(\frac{1}{q_D} - \frac{1}{q_B}\right)$ [2]

D. They can usually be solved by substituting w = u – v, if the permeability k varies with x [1]

E. They can always be solved using a matrix method involving eigenvalues $0, k\left(\frac{1}{q_B} - \frac{1}{q_D}\right)$ [0]

Commentary: Firmly in the mathematical world, this question requires a good understanding of the conditions under which a substitution method and/or a matrix method can be used and checks the accuracy of its application. The matrix method is an elegant and efficient method but it depends upon the transition matrix being independent of x, thus the preferred option C gains 2 marks. The equations cannot always be solved by a matrix method, option A attracts partial credit noting its veracity if e.g. k is a constant. Substitution works in easy cases where e.g. if k is either constant or linearly dependent upon x; so option D attracts partial credit but option B gains no credit. Option E has eigenvalues that are wrong, so gains no credit.

Some justifying student responses:

Student M6: Gaining partial credit with option A, M6 has concentrated on a learned method without understanding its implications. He writes: ‘ The matrix method can be used for any k, no matter k is constant or depend on x. It can be done by using eigenvalues but when we do the substitution method, we assume k is constant, not depend on x. Then we integrate with respect to x, [treating k is constant], so, substitution method we must use w=v-u’. He has not understood restrictions of the matrix method nor mastered the conceptual requirements of the substitution method.

Student M7: M7 uses a process of elimination to arrive at his preferred answer A. He says: ‘A,C can solve the differential equation. E is wrong eigenvalues whereas in C, k is constant i.e. k can be. Therefore by process of elimination A is the true answer’. He has not realised that for option A to be true it requires k constant. He has not discussed option D and so his elimination is incomplete. The mathematical solution is important in modelling and this student has not mastered a key element.

Question 5: In a simple model of a kidney machine, in which the permeability of the membrane is not constant, the Clearance Cl, of the waste products from the blood is given by $\mathrm{Cl} = q\left(\dfrac{1-e^{-L^2/6q}}{1-\frac{1}{3}e^{-L^2/6q}}\right)$ in which q is the speed of the blood in the machine and L is the length of the machine. Which one of the following statements is true?

A. There is a finite optimum length for the kidney machine that ensures maximum clearance Cl [1]

B. For a very small kidney machine, the clearance Cl corresponds exactly with the expected behaviour [2]

C. The maximum clearance, Cl, achievable by this machine is 3q/2 [0]

D. $\dfrac{\partial C_\ell}{\partial L} \leq 0$, for q constant, always [0]

E. the kidney machine should be of length $L = \sqrt{6q}$ [0]

Commentary: The transition from a mathematical model to the real world problem is difficult for students. This question is at that interface, requiring an understanding of the main dependent variable, the clearance Cl. In the models considered, the Cl is a monotonic increasing function of the length L and the blood and dialysate flow rates. Option D is therefore untrue and attracts zero credit. There is no optimum length L for maximum Cl, but the machine should be as long as possible. Option A would therefore attract zero credit but might obtain partial credit if justified in practical terms. Option B is the preferred answer, corresponding exactly with reality for, if the machine has no length then the clearance would be zero. Options C and E arise from false algebraic manipulations of the given formula and do not attract credit.

Some justifying student responses:

Student M3: M3 is guided by the real world situation to the detriment of the question and the model embedded in it. In choosing option A he says: 'The kidney machine is built to ensure maximum clearance Cl, by using an optimum X length which varies person to person. The bigger person the bigger kidney machine required'.

Student F4: F4 wrongly determines that option D is correct by an incomplete statement saying: 'x is monotonic increasing because gradient is not negative therefore $\frac{dC}{dL} > 0, \ldots$', oblivious to the fact that option D has the opposite inequality. She remains in a mathematical world in coming to her conclusion.

4. REAL WORLD - MATHEMATICAL WORLD TRANSITIONS: SOME PRELIMINARY RESULTS

Our previous research has considered students' mathematical modelling skills in terms of the expert-novice continuum (Crouch & Haines 2003, 2004). Novice first-year students have difficulty keeping the demands of the real-world and the model in mind at once. Novices tend to spend less time analyzing the problem statement (Schoenfeld, 1987), have difficulty distinguishing relevant aspects from the irrelevant, and think they have understood the problem sufficiently when they have not. Novices immediately tend to start generating equations without recognizing particular underlying abstract problem-types or being able to access relevant concepts and procedures (Glaser & Chi 1988).

In terms of the expert-novice continuum, we could argue that these final year students' modelling skills could be expected to fall into the 'intermediate' or 'subexpert' range, being no longer novices but not yet fully-fledged experts. Simon (1980) suggests that for complex tasks, it usually takes 10 years or more to become an expert. Patel & Ramoni (1997, pp87-93) classify final-year medical students as in the intermediate or subexpert range of expertise for the skill of medical diagnosis. Experts' store in memory of problem categories is extensively cross-referenced and experts are very efficient at ruling out wrong turnings in the problem-solving process early on. Intermediates also have acquired an extensive amount of domain knowledge, but it is not yet well enough organized to facilitate quite such effective problem-solving as experts, though much better than novices. It is therefore interesting to note that some of the students doing our MCQs, even when getting full or partial credit for their answers are showing by their explanations that their modelling skills may not be consistently working at optimal level, even though there appears in general to be considerable movement in that direction.

A preliminary analysis suggests that the majority of students are indeed getting full or partial credit for their answers to the MCQs, as we might hope for from final year students. However, some students still appear to be demonstrating difficulty in taking both the model and the real-world context into sufficient account and holding the balance between them, as experts would be expected to be able to do. Student M1 (MCQ1) and M3 (MCQs1,5) consider the real world situation without relating it to the model, whereas students F1 (MCQ2) and F4 (MCQ5) appear to engage only with the abstract mathematical model without reference to the real world. Goldstone &

Sakamoto (2003, p417,p443), reviewing research, suggest that there seems to be a mental competition between viewing an object in terms of its concrete existence in the real-world and viewing it in terms of its abstract symbolic representation. Whilst abstraction needs a perceptual grounding in the real world, too much attention to concreteness appears to interfere with the process of abstraction. This could imply that it requires considerable experience to keep both the real world and the model consistently in mind at once, without one or the other getting out of balance.

Experts can integrate abstract concepts with concrete detail to form a representation of the problem that is neither too general nor too concrete (Zeitz 1997, p48), and this representation successfully mediates between the concrete and the abstract (Goldstone & Sakamoto 2003, p417). In physics, for example, an expert's problem representation can be seen as a bridge between the detailed concrete physical situation and the abstract mathematics needed to find a solution (Zeitz 1997, p50). At this intermediate level, some of our students appear to be indicating that there are still some problems in consistently developing this bridge.

Some students' explanations indicate that they have not got a sufficiently accurate or relevantly detailed grasp of the real-world problem context. Students M2 (MCQs1,2) and F2 (MCQ3) have formed their own slightly inaccurate or insufficiently detailed view of the real-world context. Zeitz (1997) states that, in general, experts can focus on the detailed physical aspects of the problem situation that are relevant to the problem category. Lesgold et al. (1988) give an example of this from radiography, where experts looking at an X-Ray photograph can focus on specific relevant physical detail. People at an intermediate level of skill can still have difficulty deciding which of the information in the real-world situation is of high relevance (Patel & Ramoni, 1997, p56).

Other students appear to have some difficulty successfully deploying the relevant mathematical concepts and procedures (MCQ4, students M6 & M7). Lee & Anderson (2001) have demonstrated that overall skill at a complex air-traffic control task improves consistently with improvement in the sub-tasks involved. So students' overall modelling expertise may not improve until their expertise on mathematical sub-tasks improves, by being able to activate and utilize appropriate mathematical concepts and procedures more speedily and accurately, following extensive practice with different types of problem. There is evidence that at an intermediate level of expertise on some complex tasks, people may get worse before they get better (Lesgold et al.1988). Patel and Ramoni (1997, p93) suggest this may be due to people at an intermediate level having their domain-specific knowledge of concepts and procedures stored in memory in a less efficiently and accurately structured form than experts. So it takes longer to recognize and access relevant prior knowledge.

How could these students increase their level of expertise further? Movement forward from intermediate level can be achieved by extended relevant motivated practice (with feedback) on all aspects of building models for a variety of problem types (Ericsson et al., 1993). Such practice needs to be aimed towards developing recognition of underlying problem categories and formation of sufficiently and relevantly detailed problem representations that mediate between the abstract model and the real world problem context. Students need also to have extended practice to improve speed and accuracy in accessing and deploying appropriate mathematical procedures for particular categories of model and in relating these, where appropriate, to relevant features of the real world problem context.

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LEVELS OF MODELLING COMPETENCE

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Abstract: *Since the publishing of the TIMSS and PISA-results, a more competence-oriented approach of education at school is in the focus of attention of the current discussion and research about didactic. The discussion about mathematical competence places special emphasis on the aspect to apply mathematics to solve different problems of daily life. In this paper the concept of a competence-oriented approach of modelling will be examined and furthermore, a level model of modelling competence will be introduced. The characteristic abilities associated with each level are listed and some insightful examples are provided. The level model will be put in the framework of the concept of mathematical literacy and it will be briefly compared with other models of modelling competence.*

Keywords: modelling, competence, abilities, level model.

COMPETENCE-ORIENTED APPROACH

This paper refers to a competence following the definition of Weinert (2001) in which it is described as the sum of available or learnable abilities and skills as well as the willingness of a student to solve upcoming problems and to act responsible and critical concerning the solution.

If we look up the domain of mathematical competence, a precise definition of the term mathematical competence is provided by Niss (2003). Niss describes mathematical competence as the ability of individuals to use mathematical concepts in a variety of situations, including those that lie within and outside of the normal realm of mathematics, where mathematics can or could play a meaningful role (to understand, to decide, and to reason).

In order to identify and examine this type of competence, Niss distinguishes between eight characteristic mathematical competencies. These characterized competencies, however, are closely related and in some cases overlapping. The presented classification scheme uses the notion of overlapping "competency clusters" to describe the cognitive activities involved.

Competence in the building of models is derived from a wide range of human abilities. These abilities, however, are primarily the same as those deemed essential for the concept of mathematical competence. Furthermore, modelling competence also requires an overlapping set of abilities, those that specifically relate to the act of modelling.

If you look at the teaching and learning of modelling there are at least two possible approaches. One approach aims at describing necessary abilities, skills and attitudes of students, we can call this approach *component descriptions*. The examination of differently shaded competencies is based on so-called *level descriptions*. Klieme et al. (2003, p. 61) call these two descriptions “Komponentenmodelle” and “Stufenmodelle”. This paper follows these distinctions between a list of abilities, skills and attitudes (components) and the examination of different levels of these abilities, skills and attitudes considering these two perspectives as complementary possibilities to describe modelling competencies.

MODELLING COMPETENCE

In the following we will look on modelling competence. Following a definition of the term modelling competencies by Maaß (2004), this paper includes in the term modelling competence those abilities, skills, attitudes and the willingness of students that are important for the modelling process.

Modelling competence includes the following: to structure, to mathematize, to interpret and to solve problems and it includes as well the ability to work with mathematical models: to validate the model, to analyze it critically and to assess the model and its results, to communicate the model and to observe and to control self-adjustingly the modelling process (Blum et al., 2002).

THEORETICAL FRAMEWORK

In the following the theoretical framework of the paper will be shortly introduced. Based on the considerations of a component oriented descriptions on mathematical literacy and modelling competence the authors adopted the competence levels of mathematical literacy to build up a level oriented description of modelling competence.

The construct competence cannot be observed directly. One can only observe students’ behaviour and actions as they solve problems, for example. Competence is understood here in the sense of a variable, from which different values can be reached by observing the behaviour of students. In a pilot study (Henning and Keune, 2004; Henning et al., 2004; Keune et al., 2004) students’ behaviour was observed as they worked on modelling problems with the goal of reaching conclusions concerning the level of modelling competence. The authors include their observations from modelling examples in different levels of school education to obtain a theoretical construction of a level model of modelling competence.

In a second phase the authors set up empirical research to achieve deeper insight of the relations between the proposed levels of modelling competencies and the abilities, skills and attitudes of students.

In the following a level model of the modelling competence will be introduced.

LEVELS OF MODELLING COMPETENCE

The development of the modelling competence is characterized in three levels. The three modelling competence levels are:

Level 1: Recognize and understand modelling

Level 2: Independent modelling

Level 3: Meta-reflection on modelling

This competence level model focuses mainly on cognitive modelling abilities and bases on theoretical considerations and empirical studies (Henning and Keune, 2004; Keune et al., 2004).

The construct competence cannot be observed directly. One can only observe students' behaviour and actions as they work on modelling tasks. Competence is understood here in the sense of a variable, from which different values can be reached by observing the behaviour of the students.

The theoretical assumption here was that methods would at the first level be recognized and understood so that students would be able to independently solve problems at the second level. Furthermore the authors make the assumptions that meta-reflection on modelling would at the very least require both familiarity with modelling and personal experience.

In the following the characteristic abilities that are related to the levels will be introduced.

CHARACTERISTIC ABILITIES

Level 1 – *Recognize and understand modelling* – is characterized by the ability:

-to recognize and

-to describe the modelling process,

-to characterize, to distinguish and to localize phases of the modelling process.

Level 2 – *Independent modelling* – is characterized by the ability:

-to analyze and to structure problems and to abstract quantities,

-to adopt different perspectives,

-to set up mathematical models,

-to work on models,

-to interpret results and statements of models,

-to validate models and the whole process.

Pupils who have reached this second level are able to solve a problem independently. Whenever the context or scope of the problem changes, then pupils must be able to adapt their model or to develop new solution procedures in order to accommodate the new set of circumstances that they are facing. A modest degree of improvement occurs within this level when pupils merely apply various approaches to solve the problem. Whenever the context or scope of the problem changes, then pupils must be able to adapt their model in order to accommodate the new set of circumstances that they are facing (Ikeda and Stephens, 2001).

Level 3 – *Meta-reflection on modelling* – is characterized by the ability:

-to critically analyze modelling,

-to characterize the criteria of model evaluation,

-to reflect on the cause of modelling,

-to reflect on the application of mathematics.

At this third level of competence, the overall concept of modelling is well understood. Furthermore, the ability to critically judge and to recognize significant relationships has been developed. Consideration concerning the part played by models within various scientific areas of endeavour as well as their utilization in science in general is present.

At this level, it is not absolutely necessary to have previously solved problems by means of modelling techniques. This implies that finished models are examined and the inference that was drawn from them evaluated (Jablonka, 1996), while at the same time criteria for model evaluation is scrutinized (Henning and Keune, 2002).

MATHEMATICAL LITERACY AND MODELLING COMPETENCE

The concept of classification levels of modelling competence was developed in order to provide insight into the following important areas:

-Portrayal of the range of requisite human abilities involved

-Coordination of lesson plans and the selection of suitable instructional materials

-Implementation of a criteria based grading scheme for pupils

-Formulation of learning goals (i.e., acquisition of a mathematical competence, improving modelling competence)

The level of modelling competence pupils/ students achieved could be considered as one dimension of at least three dimensions in which a modelling activity takes place.

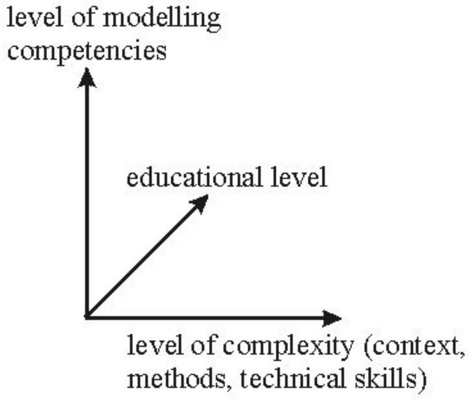


Figure 1: Different views on development of expertise in modelling

The concept of mathematical literacy connects the development of mathematical terms with the treatment of realistic tasks. This connection can be considered as analyzing, assimilating, interpreting and validating a problem, to be brief – modelling. The OECD/PISA (OECD, 1999, p. 41) gives a precise definition of the term mathematical literacy. “Mathematical literacy is an individual’s capacity to identify and understand the role the mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual’s current and future life as a constructive, concerned and reflective citizen.”

The competencies which form the base for the process of such tasks have already been examined. In the works of Haines et al. (2001) component-oriented approaches are applied. Haines et al. distinguish between modelling competences and skills based on the phases of the modelling process.

Based on the works of Niss (1999, 2003), Blomhøj and Jensen (2004) characterize modelling competencies within three dimensions. According to that, the competence acquired by students concerning “technical level”, “radius of action” or “degree of coverage” can vary.

The presented level oriented description of modelling competence can be considered as another perspective on modelling competencies. The level model can be used as a descriptive, normative and meta-cognitive aid when assessing student performance, planning lessons and selecting teaching contents.

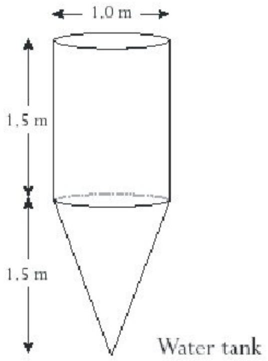
EXAMPLES

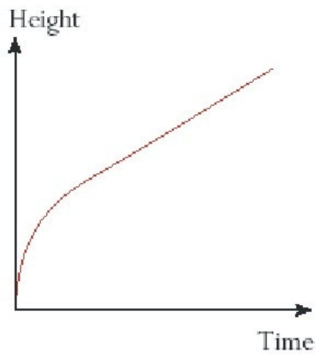
In the following three examples for assessing the level of modelling competence are given. The examples are based on PISA study examples (OECD, 2003) and have been reformulated.

The "Water Tank" Problem:

Consider an actual water tank. At the beginning the tank is empty. Now it is being filled with water at the rate of one liter per second.

What you see here is the result of a model building process used by pupils. These pupils have made certain assumptions about the tank and then sketched an appropriate graph.





a) Describe how these pupils proceeded with the modelling process.

b) What assumptions did they make?

c) What kind of model did they used?

d) Are there any assumptions that were not used in this graph?

e) What could be the next step after sketching the graph?

While solving the tasks the pupils who are confronted with the problem have to demonstrate their ability to recognize that the water tank as depicted is a compound object, recognize that material thickness does not play a role in the solution of the problem, recognize that a qualitative graphical model is used, recognize that the quantitative data given is not used in the model. These are abilities situated in level one.

The second example aims to assess abilities from the second level.

SCHOOL PARTY

It has been announced that a famous band is going to play in the gym at a school party in our school. Almost all the students from your school and many students from neighbouring schools would like to come to the concert. From the organizers of the party you receive the task of calculating the maximum possible number of spectators for the gym.

a) Plan how you will proceed with solving the problem and write out the steps needed for the solution.

b) Complete the task which the organizers gave you. If any details are missing, figure them out by estimating.

The organizers would like you to show your work to the heads of the school in a short presentation.

c) Make up a sheet of key points which you would like to tell the heads of the school.

In this modelling task the pupils have to demonstrate their abilities to solve a problem by using modelling techniques.

The third example is based on the PISA example "Rising Crimes" and has been reformulated to assess the abilities in level three.

The "Rising Crimes" Problem:

Presented below is a table that shows the number of reported crimes per 100,000 inhabitants over a 24 year period.

year	1960	1965	1970	1975	1980	1984
number of crimes	110	200	330	480	590	550

A certain manufacturer of security alarm systems has used this data to create an advertising slogan: "Every 10 years the number of crimes doubles or triples. Buy your alarm system now!"

a) Is the first sentence of the advertising slogan correct? Support your answer.

b) Why did this manufacturer use this mathematical statement?

c) Is it possible to misuse mathematics?

In this problem pupils are asked to demonstrate their ability to reflect critically on the modelling process and its use in a real world application. Furthermore, they have to develop the ability to evaluate the use of models in general.

When considering models and the modelling process, one must be incessantly aware of the possible misuse of mathematics, as well as the social relevance of models, their interpretations, and the predictions that they can make.

CONCLUSION

In the paper a level model of modelling competence has been presented and it has been compared with other descriptions of modelling competence. Important issues for further research are the examination of the level model in different levels of the educational system and the role of the context of the modelling tasks.

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APPLIED OR PURE MATHEMATICS[1]

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Abstract: *In a basic algebra course for prospective elementary and secondary teachers in mathematics, the students were asked if they wanted applied or pure problems for a two week assignment. Most of the 31 students were in favor of applied problems, implying that an applied situation would help them to find greater pleasure in the problem solving process. A wider exchange between formal mathematics and intuitive strategies were observed among the group who selected applied problems. This group also investigated more general solutions for their problems, indicating that applied problems with a natural context, inspires conjecturing and exploring.*

Keywords: Applied or pure problems, formal or intuitive strategies, problem solving, teacher education.

INTRODUCTION

The primary consideration any researcher should do before she or he conducts a study is of course to think about: what am I expecting to find out of this study? I start by declaring that this is not a deep, theoretical research study. It is a small study of how students in a teacher program responded to pure or applied problems, and to different problems in these categories. I will also discuss the possibilities of rich applied problems when teaching mathematics to prospective elementary or secondary teachers of mathematics.

PURE OR APPLIED MATHEMATICS

Mathematics can be categorized in many different ways, for many different purposes. In elementary and secondary school, we usually divide mathematics into branches like arithmetic, algebra, geometry, statistics, and so forth. Another way to divide mathematics is to look at mathematics as an intellectual game and a problem solving activity and to divide or classify the problems into the categories pure or applied mathematics.

In the beginning of the 20th century, the very foundations of mathematics were under intense discussions. In parallel, a split between "pure" and "applied" mathematics developed, probably for the first time. Traditionally, mathematicians were generalists combining theoretical mathematical work with applications of mathematics and often with work in mechanics, physics, and other disciplines. Leibniz, Lagrange, Gauss, Poincare, and von Neumann all worked with concrete problems from mechanics,

[1] Thanks are due to Gabriele Kaiser and Morten Blomhøj, who discussed a preliminary outline of the paper and made several helpful comments and suggestions.

physics, and a variety of applications, as well as with theoretical mathematical questions.

The split was highlighted by the book *A mathematical apology* by Godfrey Harold Hardy, first published 1940. In this book, Hardy expressed emotionally arguments for why one should prefer applied or pure mathematics.

> Pure mathematics is on the whole distinctly more useful than applied. A pure mathematician seems to have the advantage on the practical as well on the aesthetic side. For what is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics. (Hardy, 1992, p. 134)

Also John von Neumann argued why to prefer one side of mathematics in favor of the other:

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second or third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration.

John von Neumann (Simmons, 1991, p. iii)

Evidently these opinions are somewhat opposite. Where Hardy sees the beauty of pure mathematics, von Neumann sees a danger in the development of pure theory sprung from and gradually separated from its empirical sources. One could claim that when we are going from concrete examples to abstractions we are doing this over and over again. And sometimes, like in non-Euclidian geometry or abstract algebra, we surely develop *l'art pour l'art*. Nevertheless, this also works the other way around. Even the purest mathematical ideas finally end up in some application. I we look at Hardy who so strongly advocated for staying pure, and prayed that his research results would stay out of practical use, his results are useful today for example in blood group analysis. The theorem is called the Hardy-Weinberg law and throughout information about details and practice can easily be found via the Internet.

PROBLEM SOLVING AND TEACHER EDUCATION

Most, if not all, of the courses in mathematics that I teach have a problem solving approach. It does not matter if I teach a course in algebra, calculus, discrete mathematics or geometry. Since I mainly teach prospective teachers (pre-service or in-service) I believe it is easier to get my students interested in the subject if I start with a problem from a well known context. The starting point for a problem solving situation should, if possible, be concrete while the ultimate objective should be abstraction or generalization. When deciding on a problem solving perspective, I have chosen the right side described in Figure 1 below. Consequently, many of the problems end up in a mathematical model of a real situation.

PROBLEM SOLVING

Pure mathematics Applied mathematics

To investigate **To construct**

Models

Figure 1: One way to divide mathematical problems into categories

In an algebra course for elementary and secondary teachers, given at Jonkoping University, the students were asked whether they wanted to have an applied or a pure mathematical problem for a two week assignment. The 31 students in the class were encouraged to solve their problem in pairs (obviously one group had to consist of three students). Predictably the majority of the students asked for applied problems, probably based on a "fear" for pure problems and an "attraction" to applied situations. The problems within these categories were selected randomly and the students were asked to carefully describe and explain their problem solving strategy. They were also asked to think about the possibility to generalize the problem.

APPLIED AND PURE PROBLEMS

A "pure" problem could be the following (problem 1):

> There are many special relations between numbers. One such relation is that we sometimes can "turn around" the numbers in two-digit multiplication, and get the same product. One example is:
>
> $$39 \cdot 62 = 93 \cdot 26$$
>
> - Are there more such two-digit combinations?
> - Exactly how many such two-digit combinations are there?

An "applied" problem could be the following (problem 2):

> Sven owns a small boarding house in the mountains. Right now there are just three guests, and the new cook who hasn't met the guests yet, wants to know how old they are. Sven knows that problem solving is the chef's passion, and formulates his answer in the following manner. "The product

of their ages equals 2450 while the sum of their ages is equal to twice your age."

The chef sits down with paper and pencil and starts to solve the problem. After a while, she comes back to Sven and says: "With the information I got from you, I'm not able to calculate the ages of the guest."

"Well" Sven says,"I can add that I'm the oldest person in this boarding house. Can you solve it now?"

The chef says yes. Explain why she could solve the problem after the last piece of information and reveal the involved mathematics hidden in the problem. How old was Sven?

Obviously both problems explore elementary number theory, but the students who got the applied problem seemed more positive about their problem. Without being explicitly encouraged to say so, the students clearly argued that the style in solving an applied problem, a word problem, drives one into a more conceptualized manner of doing mathematics. When translating a situation into the language and syntax of mathematics, one could argue that we always need to develop a mathematical model as a result of the translation. Verschaffel, Greer and De Corte (2000) suggest the use of word problems to engage students in mathematical modeling. One could add, that any applied problem engage students more in making sense of the semantics of the problem and less in doing tedious computations.

DIFFERENT KINDS OF APPLIED PROBLEMS

In the same way as pure problems may differ from applied problems in terms of style and language, applied problems may differ widely in context, and that, in turn, may lead to quite different problems solving approaches. The following two problems share partially the same objective concerning the procedure of constructing and solving a quadratic equation, but in a different ways.

Problem 3: The following problem is quoted from the old Babylonian kingdom: An area A is equal to 1000 area units and consist of two squares. The side in one of the squares is 2/3 of the side of the others square, subtracted by 10. What length are the sides in the two squares?

Problem 4: Angela has 2 one liter bottles. Bottle A contains 1 liter pure orange juice, bottle B is empty. She pours a part of the juice from bottle A into the empty bottle B. Thereafter she fills up bottle B with water, so that bottle B is full and shakes it so the liquid is well mixed. Finally she fills up bottle A with the mixture from bottle B, until bottle A is full again. Calculate the amount of orange juice which at least is in bottle A.

STUDENTS' PROBLEM SOLVING TECHNIQUES

A lot of research is published addressing the fact that productive reasoning in problem solving cannot exclusively be based on formalistic reasoning. A student with access to only formal structures will not easily develop the creativity that is necessary in problem solving.

> The main idea is that the same type of mental attitudes and endeavors which characterize an empirical attempt at solution intervene also at the formal level. /…/ Therefore, even when dealing with axiomatical structures, the mathematical activity resorts to the intuitive forms of acceptance and extrapolation which may assure its required behavioral firmness, its productivity, its dynamic, flexible consistency! ((Fischbein, 1987, p. 23)

The students in my algebra class did not show any such behavior to begin with. About 2/3 of the students decided on applied problems, with a larger percentage of prospective secondary teachers among the "pure" problems group.

Nevertheless, when the assignments were turned in, I could notice that the group with applied problems seemed to have worked much harder, written longer and more developed essays, and used much more intuitive and creative reasoning. They also showed more developed ways to generalize their solutions into other situations.

Viewing a problem solver, we can conceive her or him as someone forced to search through a number of possible solutions witch are not immediately available to the problem solver, but which needs to be produced. The problem solvers task is to generate the possibilities in some reasonable order, testing each and every one, as she or he goes along, until the problem solver finds a solution that satisfies the problem solver or until she or he gives up.

If a procedure exists for calculating a solution in a finite number of steps, or for ordering the search so that the solution is guaranteed after a finite number of trials, then such a procedure is called an algorithm or a formula. It seems as if the students who chose "pure problems" were much more focused upon the search for a formula. Something in the presentation of the problem itself thrive the problem solver to search for a formula.

Student group 1 (prospective secondary teachers):

Here is our explanation about how we arrived to the solution. We used the formula:

$23 \cdot 64 = 32 \cdot 46$ which we translate to $ab \cdot cd = ba \cdot dc$

Algebra: $(10a + b)\,(10c + d) = (10b + a)\,(10d + c)$

$$100ac + 10ad + 10bc + bd = 100bd + 10bc + 10ad + ac$$

$$99ac = 99bd$$

$$ac = bd$$

There is one restriction in the equation, namely that $ac = bd$. All figures must be larger than zero and smaller than ten: $0 < a, b, c, d < 10$.

All figures between 1 - 99 that can be produced by 2 different singular figures, for example $2 \cdot 6 = 12$ or $4 \cdot 3 = 12$ can be placed in the equation $ab \cdot cd = ba \cdot dc$.

Example: $21 \cdot 36 = 12 \cdot 63$ or $48 \cdot 63 = 84 \cdot 36$.

Answer: There are 12 possible combinations.

Summary: As a matter of fact the solution is wrong, there are 14 possible combinations, and the solution is also thin and unconnected. An illustration with help of the multiplication table would probably have helped this group a lot. When suggested to write down and look at the multiplication table in order to find all solutions and finish the assignment, the students expressed an attitude of being at a higher level of expertise and not in need of the multiplication table.

This group finally managed to find a way to complete their solution of the problem to the end. Nevertheless, the students did not explore the problem in full; neither did they try to find generalization possibilities. After arriving to the formula, implicitly expressed in the problem statement, they were quite satisfied to generate solutions to the formula.

Student group 2 (prospective elementary teachers):

Here is our essay about our problem solving journey. We started with playing with the number 2450 and finding the proper divisors and found that $2450 = 1{\cdot}2{\cdot}5{\cdot}5{\cdot}7{\cdot}7$. Three guest means that it is not the smallest factors we need, and since three numbers can add up to twice the chefs age (called CA) in a number of ways, we decided that we must have a table describing all possible outcomes. We also have an intuitive feeling that there is a "trick" or some hidden difficulty in the problem, since the chef needs to get more mysterious information all the time. After some time of brainstorming, we decided to label the guests A, B, and C, and the chefs age CA, and structure our ideas into a table. The table will bring order into our chaos and illustrates the 12 possible outcomes.

Table 1: Possible outcomes of guest ages

	A	B	C	Sum	CA
(1)	1	2×5×5	7×7	100	50
(2)	1	2×7×7	5×5	124	62
(3)	1	2×5×7	5×7	106	53
(4)	2	5×5	7×7	76	38
(5)	2	5×7	5×7	72	35
(6)	5	2×5	7×7	64	32
(7)	5	2×7	5×7	54	27
(8)	5	2×5×7	7	82	41
(9)	5	2×7×7	5	108	54
(10)	7	2×5	5×7	52	26
(11)	7	2×7	5×5	46	23
(12)	7	2×5×5	7	64	32

At first we talked about possible and expected ages for the chef and among the guest. She is probably not 23, and the guests are probably not 1, 98 and 10. We decided to have a "common sense" and intuitive perspective when we analyzed the figures, but without excluding any combinations.

Then we discussed like this. The chef must know her own age. If the chef for instance is 38 years old, she would immediately see that alternative (4) is correct. She could then determine the guest ages as 2, 25, and 49. So why couldn't the chef solve the problem at this stage of the process?

It must mean that the information about her age does not give an unambiguous choice among the 12 possible alternatives. We only have alternative (6) and (12) generating the same age for the chef, namely 32 (which we consider a reasonable age). Consequently the chef is 32 and the guests are either 5, 10, and 49 years old or 7, 7, and 50 years old.

The chef then goes back to Sven and learns that Sven is the oldest person in the house. Compared to the guest's ages, we see that Sven must be at least 50 since he otherwise wouldn't be oldest. If Sven on the other hand would be older than 50, the chef wouldn't be able to solve the problem after given this final information. So Sven must be exactly 50 years old, and the guest's ages are 5, 10, and 49 years. The answer to the question is that Sven is 50 years old.

We believe that this type of problem easily could be constructed for different ages in compulsory school. By constructing numbers that can be divided into proper divisors and at the same time sum up to unambiguous results, it is possible to teach students at different age's basic number theory and problem solving at the same time.

We illustrate this by adding a similar, but little less complicated, problem:

Adam has just moved into a new house in Gothenburg together with his family. Caroline comes by and asks him:

Caroline: How old are your sisters?

Adam: The product of their ages is 36 and the sum is equal to your house number.

After some thinking, Caroline gives up. Adam then tells her that his oldest sister has red hair, and after a while Caroline says that she now knows the ages of his sisters.

How old are Adams sisters?

Summary: This group certainly explored their problem deeply and with great enthusiasm. They even invented more problems along the same way; they wrote about teaching strategies and discussed different ways to assess open problems like this one. They went back and forth with intuitive ideas and formal reasoning.

Student group 3 (prospective elementary teachers):

> Students: We actually guessed the squares first and then found the equation. It was not that hard to see that $30^2 + 10^2 = 1000$ and that 2/3 of 30

minus 10 is 10. It only took us some 10 minutes more to conclude that there are no other solutions among the natural numbers, since $33^2 = 1089$ and $27^2 = 729$.

Algebraic solution: We finally got the equation: $\left(\frac{2}{3}x-10\right)^2+x^2=1000 \Leftrightarrow 13x^2-120x+900=9000$

with a positive root $x = 30$. From the problem we found that the other square must be 100. So the sides in the squares are 10 and 30.

Summary: Besides geometrical drawings illustrating the squares, this group did not go beyond the problem itself and did not try to generalize and invent anything similar. When they found the algebraic solution, they were happy to confirm their assumptions from the beginning and stopped there. They acted as true formula seekers.

Student group 4 (prospective secondary teachers):

We had no idea about how to directly translate the situation into mathematics and to construct a quadratic equation, so we started with bottles in our kitchen and by pouring from A to B and back to A again; we made up the following table:

Table 2: Pouring orange juice back and forth

From A to B	Becomes in B	Back to A	Finally in A
1/5	1/25	1/25	1 – 1/5 + 1/25
1/4	1/16	1/16	1 – 1/4 + 1/16
1/3	1/9	1/9	1 – 1/3 + 1/9
1/2	1/4	1/4	1 – 1/2 + 1/4
X	x^2	x^2	$1 - x + x^2$

So by induction we finally (after quite some struggling) invented the mathematical model $y = 1 - x + x^2$ which we made a graph of on a graphic calculator in order to understand its behavior and to find eventual extreme points. Then we started to discuss the size of the bottles. Did the bottle volume matter? We did not think so, at least not as long as the two bottle volumes were the same.

But if A and B were 2 liter bottles, we would get the formula $y = 2 - x + x^2$ which has an extreme point at $x = ½$ (the same as first formula) but at which $y = 1,75$! Error! We quickly graphed a numbers of graphs for $y = V - x + x^2$, where $V = \{1, 2, 3, \ldots, 10\}$ and arrived to terrifying contradiction! Our reasoning led to the result that the larger bottles, the less impact. At the same time, we knew that the situation should be the same, picturing Angela pouring back and forth in bottles in any size. So where was the error?

At this point, we decided to concentrate on a purely analytical approach. Presume that Angela pours over x liter orange juice to B from A. Then bottle B contains x liter orange juice and bottle A contains $1 - x$ liter orange juice. After mixing the orange with water in bottle B, every part of the mixture in bottle B will contain ax liter of the orange juice from A. Angela pours back as much fluid as she took (the amount x liter mixture) which means that $a = x$ and that bottle A therefore contains an amount of orange juice equal to $y = 1 - x + x^2$ which can be written $y = (x - ½)^2 + ¾$ yielding that y is at least ¾ or 75 %.

When getting this far, we understood that if the bottles are larger (or smaller) than 1 liter, then x and x^2 will be part of that volume and not absolute quantities related to 1 liter. For instance, if the bottles are 2 liter each, we get that 1/5 of 2 liters is 2/5 of one liter and the model will become $y = 2(1 - x + x^2)$ with a minimum at $x = ½$ where $y = 1.5$ or 75 % of 2. So our inductive reasoning from the beginning holds even when the bottles are larger. We illustrate this by a graph for five different bottle sizes.

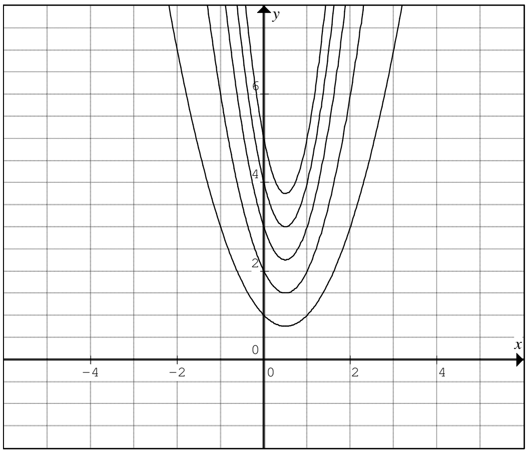

Figure 2: Graphing $y = V(1 - x + x^2)$ for $V = \{1, 2, 3, 4, 5\}$

If the bottles are of different sizes one has to look at every situation differently. If for instance we imagine the situation where A's volume is 1 liter and B's volume is 2 liters, then we will get the following table:

Table 3: Different size on A's and B's volumes

From A to B	Becomes in B	Back to A	Finally in A
1/5	1/10	1/50	1 – 1/5 + 1/100
1/4	1/8	1/32	1 – 1/4 + 1/64
1/3	1/6	1/18	1 – 1/3 + 1/36
1/2	1/4	1/8	1 – 1/2 + 1/16
X	$x/2$	$x^2/2$	$1 - x + x^2/2$

This model has a minimum value at $x = 1$ where $y = 1/2$. It means that if we transfer all the orange juice from bottle A to bottle B, we will get half the amount of orange juice back again, which is the lowest possible concentration of orange in this situation.

Summary: This group obviously found this problem highly challenging and motivating and if there had not been a timeline of two weeks, they probably would have taken their investigation even further. They presented me with a 10-pages long solution/discussion and I'm honestly sorry that I cannot share it with the readers of this paper. At least I hope that I have been able to give you a flavor of what this group managed to do with their applied problem. This group also clearly visualized the strategy of moving back and forth between intuitive and formal knowledge, which identifies a good problem solver in mathematics.

CONCLUSIONS

Even though these excerpts are not extreme in any sense among all the essays I got, they clearly illustrate the different approaches I found when comparing essays from students who worked with different applied or pure problems. It simply seems as if students who worked with applied problems became much more involved and engaged in the problem solving process. The context itself seems to be important, especially when the problem offers possibilities to explore at different directions. Many researchers emphasize the importance the context and the use of language in the discourse have for the understanding word problems. It also seems as if when you can engage the intuitive or "common sense" part of your mind, then you are ready to go deeper into the problem solving process. When analyzing the work of the students, I have found that most students express intuitive ideas about the concepts involved in the problems they are dealing with. These intuitive ideas often play a crucial role in the creative part of their problem solving process, and are obviously needed for the students to be creative and find different ways to a solution.

The richer the problem is, the further the students will be able to go when exploring and generalizing. Evidently, we can discuss the favor of differentiation between contexts, as in formal or more natural connections for a problem. This is illustrated in problem 1 and 2. We can also discuss differentiation within a context, as within a mathematical concept. To be able to switch between an intuitive and formal approach as the students with problem 4 did, forms a good example of how students structures their mathematical knowledge.

Fischbein has acknowledged the vast importance of students learning to master the interplay between formal mathematics and intuitive ideas.

> One of the fundamental tasks of mathematical education – as has been frequently emphasized in the present work – is to develop in students the capacity to distinguish between intuitive feelings, intuitive beliefs and formally supported convictions. In mathematics, the formal proof is decisive and one always has to resort to it because intuitions may be misleading. This is an idea which the student has to accept theoretically but that he has also to learn to practice consistently in his mathematical reasoning." (Fischbein, 1987, s. 209)

This implies huge benefits in using as many applied problems as possible and maybe always try to find applied problems, even when some of them are of the artificial constructed kind I have explored in this paper. It simply seems to be much more interesting to explore even constructed situations. When students are given problems to solve in which more than one route to the solution is possible, and especially when students can at least discuss the problem in pairs, then it seems as if the students benefit much more compared to when solving pure problems.

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CONCEPTUALIZING THE MODEL-ELICITING PERSPECTIVE OF MATHEMATICAL PROBLEM SOLVING

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Abstract: *The notion of modeling occurs abundantly in mathematics education literature, primarily in the context of studies documenting modeling behaviors of students and teachers. However there is a lack of studies related to epistemological issues arising in the teaching and learning of modeling constructs, especially vital for the training of future researchers in the field. This paper explores the complexities, preferences and the variety of meanings that post-graduate students' attach to the notion of model eliciting. Students' conceptualization about model eliciting was influenced by classroom discourse, reflections on their cognitive mechanism whilst engaged in a problem as well as the features of a given problem.*

Key words: Conceptualization; Epistemology; Modeling; Model-eliciting; Problem solving; University researcher education.

1. INTRODUCTION

The importance of modeling has been stressed by numerous curricular documents (NCMST, 2000; NCTM, 2000, NRC, 1998) as well as by mathematics education researchers (Gravemeijer and Doorman, 1999; Lesh and Doerr, 2003; Lesh and Lehrer, 2003). Modeling activities provide students with a glimpse of how mathematical knowledge relates to and is applicable to the real world. Examples of the presence of modeling activities permeate day-to-day human activity, the arts and the sciences. These activities can vary from the simplistic sketch of a novice carpenter to the use of probability-distribution tables in the financial and economic sectors. For instance Poisson distributions are often used in the insurance industry to model the number of claims received by an insurance company. Differential calculus is a classical example of a useful modeling tool in engineering, financial sciences and physics. Regression techniques are routinely applied in the physical and social sciences for the purposes of using the gathered data for predictive purposes.

The literature shows numerous papers that report on studies conducted at the K-12 level on the modeling behavior of students and teachers. However there is a dearth of studies at the tertiary level related to the epistemology of modeling, particularly on the development of understanding of theoretical constructs that arise in the literature. This is an important but under-investigated area of mathematics education research with tremendous implications for the teaching and learning of modeling at the post-graduate level. This is especially crucial since post-graduate students are on the crux of doing research in the field, and are future mathematics educators. Schoenfeld (2000) recommended that in the preparation of researchers, “one must guard against the dangers of being superficial…high quality research comes when one has a deep and focussed understanding of the area being examined” (p.476).

In any discussion of epistemology, the underlying philosophical question is to examine the nature of a given construct, specifically its meaning. The words “model” and “modeling” lend themselves to a variety of everyday interpretations. The meanings can range anywhere from solving word problems, conducting mathematical simulations, to creating representations such as scaled drawings that serve as archetypes for buildings and other physical objects or a hypothetical abstract representation of a situation for descriptive or analytical purposes. The mathematics education literature also uses the generic term “model” to denote hypothesized problem solving models as well as schemes that describe mental processes such as abstraction and generalization.

The difference between the terms modeling and model is analogous to the difference between process and product. Modeling is used to refer to the processes employed *to model* a problematic situation. On the other hand the word model refers to the end product, the end result of the modeling process, typically a physical, symbolic or abstract representation. Recently Lesh and Doerr (2003) introduced the term “*model eliciting*” which encapsulates the terms model and modeling. The danger of introducing a term that encapsulates both process and product is that it converts a dynamic process into a static object (Gascón, 2003). The pedagogical goal of model eliciting activities is to help students create mathematical models when confronted with a problematic situation, which typically involves some data (Lesh and Lamon, 1992). In essence, the term *model eliciting* is used to refer to the Gestalt of process and product in a problem-solving context. Given the wide variety of potential meanings attached to these words, it becomes important for the university educator to discuss the meaning of terms used the in modeling literature as well as to arrive at an objective, agreed upon usage of these terms. The primary questions explored in this paper are:

(1) What meanings do post-graduate students attach to the notion of modeling (particularly model eliciting?

(2) What criteria do post-graduate students use to decide whether a mathematical problem is a model-eliciting situation?

2. A BRIEF REVIEW OF MODELING CONSTRUCTS

For the sake of brevity, only the main constructs that were part of the assigned readings in the study and which recently appeared in the literature will be reviewed. The word modeling has been defined in mathematics education essentially as a framework via which a simple or complex real world situations or systems can be mathematically reconstructed, described, and used for predictive purposes (Lesh and Harel, 2003). Numerous examples were provided in the introductory section of the paper. Model eliciting is defined as a problem solving activity constructed using six specific principles of instructional design in which students "make sense of meaningful situations, and…invent, extend, and refine their own mathematical constructs" (Carlsen, Larsen and Lesh, 2003, p.465). In other words, while the traditional problem-solving goal is to process information with a given procedure, model eliciting is the process itself. The purpose of the process is for students to take their model elicited through solving the original problem and apply it to a new problem. Some examples will help better illustrate the notion of a model-eliciting activity. Suppose students are asked to "rate" the quality of all the potential players on a soccer team and then select the team based on a consensus on the ratings. This task requires students to gather/procure "objective" data related to players speed, endurance, past performance, special abilities and reach "subjective" consensus on the criteria most valued for the selection of the team. This is a model eliciting activity because it invokes the six instructional principles of Lesh et al (2003) namely, (1) the Reality Principle (i.e., Does the situation warrant sense making and extension of prior knowledge/experiences?), (2) the Model Construction Principle (i.e., Does the situation create the need to develop (or refine, modify, or extend) a mathematically significant construct?, (3) the Self-Evaluation Principle, (Does the situation require self-assessment?), (4) the Construct Documentation Principle (i.e., Does the situation require students to reveal their thinking about the situation? (5) the Construct Generalization Principle, (i.e., Is the elicited model generalizable to other similar situations?) and finally (6) the Simplicity Principle (Is the problem solving situation simple?). As alluded to earlier, the construct of "model-eliciting" circumscribes a problem solving situation, its mathematical structure, the mathematical models generated as well as the problem solving processes that are invoked by the given situation. The epistemological question here is given the exposure to the meanings of modeling and model eliciting through readings and discussion, what meanings do post-graduate students attach to these constructs? Are the meanings derived by students identical or are they influenced by other sources besides the explicit definition spelled out in the text? If students have been exposed previously to mathematical modeling in various higher-level mathematics courses, what is their understanding of the construct of model eliciting? Is it different from the construct spelled out in Lesh and Doerr (2003)?

3. METHODOLOGY AND DATA ANALYSIS

The study was conducted with five post-graduate students enrolled in a graduate level mathematics education course on cognition. The post-graduate students had a fairly strong undergraduate background in mathematics and were in different stages of completing the M.S and Ph.D degrees in the mathematical sciences. The author was the instructor of this course. The data was gathered through classroom discourse, written assignments and two interviews (approximately in the middle and the end of the semester). Classroom discourse on modeling was based on a selection of readings from Lesh and Doerr (2003). These discussions were led by one of the students (in rotation) and the instructor. The interviews involved looking at problems from pure and applied mathematics. One purpose of these interviews were to create a problem solving experience based on which students could classify the given problems as model-eliciting problems or not. Another purpose was to create a situation whereby students would reflexively apply the definition of model eliciting to their thought processes while solving the problem. Both the classroom discourse and interviews were audio taped and transcribed for the purpose of data analysis. A variety of complementary data sources were chosen so as to ensure both triangulation of data and that an accurate picture of student understanding could be constructed. Since the author was an integral part of classroom discourse and the interviews, the ethnomethodological approach (Holstein and Gubrium, 1994) was most appropriate to interpret events in the classroom and the interviews. The data from the discourse and interview transcripts was analyzed in iterative cycles for emergent themes. The emergent themes were compared with student writings on the written assignments over the course of the semester to check for consistencies as well as deviations in student understanding of modeling constructs. The student interpretations and understanding of modeling constructs (specifically that of model eliciting) is now presented in the form of a time series of episodes over the course of the semester, followed with commentaries that discuss the episodes and findings.

4. COMMENTARIES, FINDINGS AND DISCUSSION

The following classroom episode took place about half way into the 15-week semester. The discussion was centered on the constructs of modeling and model eliciting, based on the readings in Lesh and Doerr (2003). The edited classroom excerpts reveal the various interpretations made by the students.

4.1 Episode 1

S1: So students make mathematical descriptions of meaningful situations…And is not done in teacher guided way like traditional problems that we are used to. (i.e.,) By saying, okay, I will ask you a leading question and try to get you to the next spot. It is done in more of the attitude of what we think of the constructivist (notion) …let them construct their own knowledge and their own models… (In this example) they were producing a product and their clients were the coaches. So they were thinking about this in a real life situation. So they were the consultants. So then they had to say how did this strategy meet the needs of your client? So they went through the warm-up activity, they started into the model eliciting activity. They are coming up with these strategies…they're analyzing, presenting, formulating ideas.

S2: They find some way to understand what to ask for and what the problem is. Like if you ask them to develop a model of…a right triangle. Most of them have heard it before and they can find things on their campus and such. And they find out that the hypotenuse always has to be longer than the legs and they can actually do some measurements. It becomes their problem. It is not a static thing that they are seeing in the book (or)…You can start with the school and say if you didn't want to go down any hall two times, what kind of room would you set up. You could have a different kind of room and find out that the parity of the vertices matters when you are solving an Euler circuit. You would never have to call it an Euler circuit. And just have them in a concrete situation or build like a mouse maze so the mouse would never run over the same part twice.

S3: So what is the difference between taking a mathematical idea and formula and relating that to reality versus taking reality and translating that into mathematics? Well the one is far more complicated. Taking reality and translating into mathematics is far more complicated rather than taking something in mathematics and corresponding that and coming up with some real life example.

4.1.1 Commentary 1

The classroom excerpts presented in episode 1 are "prototypical" understandings of the students 1, 2, 3 about the notions of modeling and model eliciting. Based on the repeated consistencies in the patterns of understanding of these three students seen in the discourse and interview transcripts and writings, the emergent themes under which their understanding/interpretations were placed were "*Constructing Own Knowledge/Mental Model*"(S1), "*Real life Connections with Problem Ownership*" (S2), and "*Ambivalence between Knowledge Construction and Real life Connections*" (S3). These themes are further developed and analyzed later in this paper. Now consider the following interview episode. These interview vignettes are based on student attempts on the following problem: What is the last digit (i.e., the digit in the units place) when you expand 7^{365}? This interview took place 3-4 weeks after the aforementioned classroom episode.

This problem was purposely chosen because it **was not situated in any context**. While the researcher does not think that this is a model-eliciting situation, arguably, one can make an extremely "subjective' case that the six design principles of Lesh et al., are satisfied. Given that the post graduate students had a strong background in mathematics, the reality principle is satisfied in the sense that the problem is a simple extension of prior mathematical techniques the students may have been exposed to. One could argue that the five other design principles also fit, given the sophisticated mathematical background of these students. The interesting question for the author was how would these students interpret the six principles given this blatantly posed, non-real world situated problem? Would they construct a context/meaning under which this problem could be classified into a modeling situation or simply discard it based on a literal interpretation of the Reality principle. The three emergent themes stated earlier consistently appear in this vignette.

4.2 Episode 2

Student1 [Constructing Own Knowledge/Mental model]

A: So…would you consider this a model-eliciting problem?

S1: I think so.

A: Ok, why?

S1: This one I really came up with a couple of different models that I was using. This one is sort of similar to something that I have done before. I didn't necessarily develop this myself. But because I did a problem that was sort of like this in a math competition. Where you dealt with some super huge numbers with some super huge powers and you had to talk about whether they were divisible by a number or had a last digit or something. So I just started breaking it down into simpler terms. Things that I could use and I could have done by hand if I wanted to. To break it down until I was convinced that I got 7…with logical arguments along the way. So I think I had a pretty concrete model. If I had a different number to different power, I could repeat this process in a similar fashion. And this one the same thing, I looked for the pattern. Once I had the pattern, I just had to figure out what power I needed to get there and then I found the end result. So they are two very different models but both equally valid, I think.

Student 2 [Real life Connections with Problem Ownership]

S2: Personally I don't think this is a model eliciting.

A: Why?

S2: Why? No it is not a bad problem. I don't think only model eliciting problems are good problems. When I see it I think clearly it is a number theory problem and I don't think of number theory problems as model eliciting.

A: So what are the features that a problem should have in order for it to be a model eliciting problem?

S2: I guess I am thinking of it as a word problem that has…I guess because I am thinking of it as a specific real-life problem that someone who didn't know mathematical theory could sit down and solve just using common sense. That is what I am thinking of as model eliciting. I think that a person would be able to sit down and think this through and work on it. A kid could get it. I don't think only model eliciting problems are solvable either. So, the fact that this has a solution and can be solved in different ways makes it an interesting problem. It is a problem that I like. But it's not, in my mind, model eliciting.

Student 3 [Ambivalence between Knowledge Construction and Real life Connections]

S3: Well yes and no. I would not classify this as a true model-eliciting problem. I mean it required me to think and to go beyond just the given information and to think about powers of seven and what the last digits would be. Why is it not really a model-eliciting problem? Because it didn't seem to have much more to it. Once I figured out there is a pattern there was nothing more to it than noticing the pattern and figuring out how the answer I wanted figured into that. Which I think is kind of model-eliciting to some extent. It is not a traditional problem you can't just plug it into your calculator. I mean you are not just given something. So it is not a traditional problem. Model eliciting? I really don't think it is model eliciting.

A: OK

S3: I guess I did come up with a model though.

A: So you think that in order for it to be a model-eliciting problem that there needs to be something more? Because you said something to the effect of once you are done with it you are done.

S3: Yeah, I know the answer now. So given any seven to any power I can figure out what its last digit is going to be. Well I guess it does. I am trying to go back and think of all the definitions that we have had for these model eliciting problems. And it seems to fit most of them. It is not a real world problem. I mean I can't think of any situation where this would be useful to real life. It doesn't have any real world flavor to it. But it did make me come up with some type of model for determining a solution. And I did in fact come up with a generalized solution applicable to similar situations for it so in that respect it is model eliciting.

4.2.1 Commentary 2

As suggested in commentary 1 about the emergent proto-typicality of student understandings of model eliciting, the interview vignettes are another example that illustrates consistencies. For student 1, this problem was an example of a model-eliciting problem because it invoked an *a priori* mental model/process used to solve a similar problem in the past. This process was applicable to other similar problems. Knowledge construction did take place although there was no real world context per se. The context was created by a subjective interpretation of the mental processes elicited to solve the given problem. For student 2, this was clearly not model eliciting because of the lack of real world context as well as the analysis that a solver could not experience any sense of ownership with the problem or be motivated to solve such a problem. For student 2, model-eliciting problems needed to be situated in the real world, and did not require sophisticated mathematical machinery from the solver. Finally in the interview vignette of Student 3, one sees the skeptical and ambivalent view of model eliciting. First student 3 stated that there was no real world context to the problem which went against the reality principle, but on the other hand (like S1) the student valued the mental process invoked to solve the problem, and viewed this process and the generalizability of this process as model eliciting.

Students were given the same problem on a classroom assessment two weeks after the interview, to see if their understanding (and criteria for classification) of model eliciting problems had changed.

4.3. Episode 3

S1: This is a traditional problem. Not only is this not real life in any way, I don't think students would see a need for it. Meaning students would have a hard time relating it to real life.

S2: This is a traditional number theory problem.

S3: Traditional problem Not sure? It only required basic knowledge of modular arithmetic, however if it was being used in a slightly different context or ways than I am used to seeing it, then it is model eliciting.

4.3.1 Commentary 3

Once again the responses of students 2 and 3 was consistent within their proto-typical understanding. However student 1 now imposed the reality criteria (meaning relating to a real-world situation) and classified this problem as not model eliciting.

5. ANALYSIS, DISCUSSION AND IMPLICATIONS

The three episodes illustrate the complexities of teaching modeling constructs in the classroom as well as the nuances in student understanding. In this paper the author has deliberately focused on a single construct, namely model eliciting. Although there were six specific principles outlined as the criteria for a model-eliciting problem, student interpretations of these criteria were not literal in any sense. For student 1 the word model eliciting emphasized the mental processes/models invoked while solving the problem, i.e., the knowledge construction that was taking place. The understanding was focused on the Model Construction and Construct Generalization principles. This general view was modified at the end to accommodate the reality criteria. In the case of student 2, the emphasis was on the Reality principle. For this student, model eliciting only occurred when the problem was situated in a real world context. The mental processes invoked while solving this particular problem were deemed as number theoretic procedures. Finally in the case of student 3 there was a back and forth ambivalence between the Reality, Model construction and Model generalization principles. For this student, the mental processes/model invoked created the reality context. Clearly the meanings and interpretations of the students are subjective and dependent on the given problem. While this is reasonable within a post-modern perspective of letting meaning be subjective, a positivist would raise several objections with such a pedagogical approach. From a positivist perspective, a uniform "objective" explanation is the goal, since students were given a definite meaning of the term model eliciting in the readings. Since this uniformity did not occur, the interesting question confronting the author is to hypothesize the reasons why this occurred? One plausible explanation is the distortion that occurs when a (given) static "objective" term is reflexively applied to ones mental processes in a problematic situation. The dynamic nature of the processes invoked while solving a problem result in an emphasis or preference for one or more of the six principles over the others as seen in the episodes constructed in this paper. In this sense, students actually lived through the definition of model eliciting. Creating a pedagogical situation where students were required to reflexively apply a static definition of "model-eliciting" to the dynamic nature of thinking processes resulted in students' emphasizing one or more aspects of the dynamics. Such an outcome is didactically desirable, in spite of objections that may be raised by positivists because it allows the student to experience the dynamic nature of the definition as opposed to simply viewing it as a static object. However the danger in such a pedagogical approach to the teaching of modeling constructs is in not following up the students "lived" experience with an objective discussion of the construct as a class and a direct theoretical application of the six principles to the given problem and its solution. In doing so, the educator creates a perceptual shift of the definition of the construct for the students, from that of subjectively applying it to ones thought processes to that of objectively applying it to the product, namely its solution. In doing so, one ultimately hopes that the Gestalt of the term "model-eliciting" is fully conveyed to the students learning this construct.

Given this brief exposition and analysis of post-graduate students understanding of one modeling construct namely model eliciting, the question still confronting the author is the vast potential subjective meanings given to and associated with numerous other modeling constructs presently in use in the literature. It is a time consuming effort for university educators to create pedagogical scenarios whereby students experience the meaning of the construct definitions. Although this is a worthwhile endeavor, it is unfeasible to expect complete uniformity in how modeling definitions are used and applied. Is it time for the community to explicitly classify modeling terminology (in a manner akin to terminologies and meaning found in a dictionary of philosophy) or do we continue to adopt the post-modern perspective and let conflicting meanings co-exist? The time is ripe to tackle this issue!

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ASSESSMENT OF MATHEMATICS IN A LABORATORY-LIKE ENVIRONMENT: THE IMPORTANCE OF REPLICATIONS

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Abstract:*Since 1993 in the Netherlands, the mathematics core curriculum for junior secondary schools states that students should develop skills for using and applying mathematics in practical situations. For monitoring purposes, a trend study was carried out using mathematical hands-on tasks in a laboratory-like environment. The study was carried out in 1995 and replicated in 2000. It revealed that this kind of alternative, practical assessment can have a satisfying curricular validity, higher than written tests based on the same curriculum. However, comparability of test results (between students, schools, etc.) depends on the uniformity of test circumstances.*

Keywords: junior secondary, modelling, assessment, validity, reliability, manipulatives.

INTRODUCTION

Three decades ago Hans Freudenthal and his colleagues started to transform the mathematics curriculum in the Netherlands with a treatise, known as Realistic Mathematics Education (RME) (Freudenthal, 1975; De Lange, 1983). RME is characterized by the philosophy that mathematics is an integral part of real-life. Thus, mathematics is taught, not for its beauty, but for its applicability. In addition, mathematics is perceived as an activity and not as a set of rules. As such, mathematics is a creative and organizing activity in which unknown regularities, relations and structures are discovered.

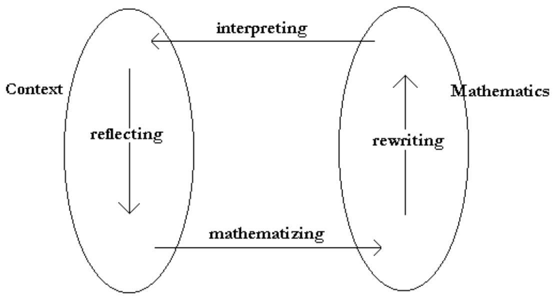

Figure 1: Modelling activities

In RME, contexts and mathematics are perceived as two different worlds, which can be connected through modelling activities (Treffers, 1987). When starting from a context, the student has to strip it from its details and find relations and regularities that result in a mathematical structure, for example a formula, a graph or a table. This activity is called mathematizing (also known as horizontal mathematizing) and the resulting formula, graph or table is called: a mathematical model. The model is part of the mathematical world, and needs rewriting, restructuring and refining to obtain an answer. This activity is called reformulating (also known as: vertical mathematizing). These two activities might need to be repeated a number of times, before a sound mathematical answer is reached. Within the mathematical world it is also possible to further generalize the problem and refine mathematical knowledge (not included in the figure). Returning to the context, the (mathematical) answers need to be interpreted in its context. In this activity, for example, units of measurement (meters or kilograms) need to be attached to the answer, and estimation can be used to verify if the magnitude of the answer is credible in the context. Finally, the answer will only make sense if it is related to the initial question, as there is need to reflect on the process of reaching results.

In 1993, a common core curriculum for Dutch junior secondary schools was legislated. It was largely based on RME, emphasizing data modelling and interpreting, visual geometry, approximation, the use of calculators and computers. Chapters of Dutch mathematics textbooks start with real life situations, in which mathematics is used and applied, instead of concluding with these. National assessment was adjusted to the new content approach. Generally, test items in the RME-based curriculum describe an appealing daily life situation, often with authentic photos to enliven imagination. The test items contain modelling activities, requiring students to mathematize the context (e.g. into a graph), apply mathematical skills to use the model adequately (e.g. derive a solution from the graph), interpret the mathematical answer in its context, and reflect upon the methodology used. Readers who are interested in this approach can examine and analyze test items from PISA, the OECD Programme for International Student Assessment (OECD, 2000), which are in English, and similar to those in the Dutch curriculum.

The new, RME-based curriculum differed considerably from the prior curriculum. A large exercise was undertaken to introduce secondary school mathematics teachers to the new content and its approach. Many workshops on the new curriculum were organized by the curriculum developers, and by teacher training institutes. The national exams were adapted to the new intended curriculum. Nevertheless, the introduction went hand-in-hand with a dilution of the initial ideals. One of the observed weak points in the dissemination of the curriculum was that the assessment practice remained of the written form and did not require students to apply their skills practically, as in small investigation projects. Therefore, a study was designed in which students were tested on their skills to use mathematics in practice, in a laboratory environment. Its objective was to investigate whether, as was the aim of the new curriculum, secondary school students were indeed improving their abilities to apply mathematics in practical situations. The study took grade 8 students as target population, as they had learnt mathematics based on the new curriculum for more than one year. The study also served as an empirical study to investigate what valid and reliable alternative assessment methods can be used to monitor the implementation of a RME-based curriculum.

INNOVATING MATHEMATICS ASSESSMENT

When measuring student achievement in mathematics for a large population, in many cases, paper-and pencil tests have been used. However, these tests have come under debate, as they cannot evaluate all practical competencies from an intended mathematics curriculum. Attempts to alter assessment methods were made, which lead to the definition of criteria for alternative assessment: (a) testing through open questions and for higher order skills, (b) being open to a range of methods or approaches, (c) making students disclose their own understanding, (d) allowing students to undertake practical work, (e) asking for performances and products, (f) being as an activity worthwhile for students' learning, and (g) integrating real-life situations and several subjects (Burton, 1996; Clarke, 1996; Niss, 1993).

In this section, I will concentrate on alternative assessment for applied mathematics and modelling on a nation-wide scale, for example, to monitor curriculum developments. In this area of study, a number of issues have emerged. First, formats such as observation, interviews and portfolio have shown to be labor- and cost-intensive. Second, the interpretation of students' answers can result in unreliable data because of inconsistencies between examiners (Kitchen & Williams, 1993) Especially the coding of borderline answers (which are neither totally correct nor totally incorrect) is conditional to the coders' background (e.g. coding experience, subject matter knowledge, teaching experience, etc) (Zuzovsky, 2000). Despite obvious disadvantages, nation-wide alternative assessments of mathematics have been carried out. For example, countries participating within the Third International Mathematics and Science Study (TIMSS) had to administer a standard written test at grade 8 level (students at the age of approx. 14 year). Additionally, participating countries could opt to administer an alternative assessment, complementing the written test. This TIMSS Performance Assessment consisted of practical investigative tasks in science and mathematics (Harmon et al., 1997). The test was developed from the educational vision that seeks coherence between procedural, declarational and conditional cognition. Students were expected to investigate systematically, being provided with a practical context (manipulatives and instruments). They were tested through open-ended tasks like: designing and executing an experiment, observing and describing observations, looking for regularities, explaining and predicting measurements, etc. The test provided students with a worksheet that guided them through the tasks. Students had to record their answers on the sheet, and hand in products (lumps of plasticine, cut-out models, etc.). The use of manipulatives was considered appropriate as these assist students to better understand the context of the question. Instead of describing real life situations in words, the equipment offered the context directly into students' hands. Especially second language learners and students with lower reading abilities were expected to gain from these circumstances.

In 1995, the test was administered in 21 countries, amongst which the Netherlands. The test raised questions on reliability and international comparability of its results; in the international report a league table of countries was avoided. In the Netherlands, the test was judged as being very valid in light of the new RME-based curriculum, to such an extent that the test was replicated in 2000 (Vos, 2002). Trend results would allow monitoring the implementation of the RME-based curriculum. Moreover, a replication could give experience in analyzing issues on validity, reliability and comparability of alternative assessments.

In 1995, the TIMSS Performance Assessment was administered to a random sample of Dutch grade 8 students (n=437 from 49 secondary schools). In 2000, the test was replicated at a slightly smaller scale because of financial constraints (n=234 from 27 secondary schools). The research questions were 1. *To what extent is there a trend between 1995 and 2000 in the mathematics achievement of Dutch grade 8 students on the TIMSS Performance Assessment?* and 2. *What opportunities and obstacles do exist in large-scale mathematics assessment using hands-on tasks?*

VALIDITY

Validity of a test can be established in various ways. For the TIMSS Performance Assessment in the Netherlands, an expert appraisal was carried out to establish the curricular validity of the test with respect to the Dutch RME-based intended mathematics curriculum for junior secondary schools. Six experts from a variety of mathematics education institutes were invited to assess the test items. Their appraisal showed that eight out of twelve tasks matched well with the intended mathematics curriculum (Vos, 2002). The other four tasks were from biology, physics and chemistry, or a hybrid of disciplines. These tasks were maintained in the test, but were not considered relevant for the measurement of mathematics achievement. Below are the eight tasks, which were considered to match well with the intended RME-based curriculum for grade 8.

The task *Dice* is related to probability: students are given a dice and a transformation rule for each roll of the dice (an even throw: plus 2, an odd throw: minus 1). They are asked to throw 30 times, record their findings, perform the transformation, and explain why one output (the '4') has a higher frequency.

The task *Calculator* is related to number sense: students are given a simple calculator and are asked to discover a pattern in the multiplications of 34x34, 334x334 and 3334x3334. As the calculator holds only eight positions in the display, this is not an obvious task. The second part of the task consists of factorizing 455 into two integers between 10 and 50.

The task *Fold&Cut* is related to symmetry and spatial abilities: using a pair of scissors, students have to cut, with one straight cut only, certain displayed figures. Because only <u>one</u> cut is allowed for each figure, the paper has to be folded (see Figure 2, left). The final item in this task asks students to design a folding plan without actually implementing and testing it.

The task *Around the Bend* is related to scale drawing and finding geometrical rules: students are given a cardboard model of a corridor with a corner (the bend) and have to cut rectangles, modelling furniture to scale. By testing which rectangle fits through

the corridor, they have to find a rule for the critical lengths of the rectangles that fit the bend (see Figure 2, middle).
The task *Packaging* is related to measuring and to the design of nets: students are given four table tennis balls and have to design three different boxes to contain the four balls. One design has to be cut, folded and fixed together with the sides exactly fitting the balls. The other designs only have to be sketched (with clear specifications).
The task *Rubber Band* covers the topics of tables, graphs and extrapolation. In this task a number of washers are attached to a rubber band. Students measure the stretching of the band, which increases with each washer. With only ten washers given, students are asked to predict the length of the rubber band, if twelve washers were attached. This requires students to analyze the decreasing increment of data.
The task *Shadows* is related to geometrical transformations. Students are given a torch, a card and a white screen. They have to project a shadow, twice the width of the object, and find a general rule for the distances between torch, card and screen.
The task *Plasticine* asks for problem solving heuristics. Students are provided with a two-sided (uncalibrated) balance, two weights (20g and 50g) and a lump of plasticine (see Figure 2, right). They are asked to make smart combinations on the two sides of the balance, in order to produce pieces of plasticine of 10g, 15g and 35g. They have to communicate the method used.

Figure 2: Students working on the task *Fold&Cut* (left), *Around the Bend* (middle), and on the task *Plasticine* (right).

In hindsight, we should have consulted a sample of students on the validity of the test. Anecdotal evidence said that students particularily loved the tasks *Around the Bend*, *Fold&Cut, Packaging* and *Plasticine*. In these particular tasks, students had to hand in their answers on a worksheet together with their *products* (cut outs, plasticine lumps). Here, the worksheets had a plastic bag attached with a paperclip (with a label on which the student wrote his/her name), in which the products were returned to the test administrator. One observer noted that students, walking out of the testing session, said students did not feel as having completed a mathematics test; instead, they had fabricated something worthwhile.

Validity of a test can also be checked through an assessment grid, for example the grid designed for assessment of modelling and applying mathematics, as in Kitchen and Williams (1993). The grid contained the following assessment categories: mathematizing, rewriting (generalizing and simplifying), interpreting, and reflecting. All test items were allocated to one of these categories. If appropriate, an item could be fitted into two categories, but then the weight of that item would be spread. Two curriculum experts were asked to categorize the test items independently. Their inter-rater score was 87% and their average results are reported in Table 1. For comparison, a standard written RME-based test for the same level of schooling was analyzed through the same procedure: the *Afsluitingstoets Basisvorming 1999* (Final test for the core curriculum 1999), developed by the *Centraal Instituut Toets-ontwikkeling* (National Institute for Educational Measurement). The TIMSS Performance Assessment showed a better spread over the grid, with a stronger emphasis on mathematizing than the RME-based control test. Often, in written tests, an item already readily states the mathematical formula, which models the context (and thus, these items do not require the construction of a model). Also, the skill to reflect is better covered in the TIMSS Performance Assessment. As a result, the TIMSS Performance Assessment can be considered valid on its spread of required modelling activities.

Table 1: *Percentage of test items in each modelling category, comparison between the TIMSS Performance Assessment and a standard, written RME-based test.*

	Mathe-matize	Rewrite		Interpret	Reflect
		Generalize	Simplify		
TIMSS Performance Assessment	35	20	14	16	15
Standard RME-based test	19	25	13	38	6

RELIABILITY

Reliability of test data depends on a number of issues. First, uniform test conditions must be created. In the TIMSS Performance Assessment, tests administrators traveled from the testing center to the schools with a large box containing all test materials and an abundance of supplies. In this way, the testing was independent of teachers' skills and experience, and independent of schools' equipment. To ensure uniform procedures throughout the measurement, the administrators were trained in how to set up the laboratory environment in an ordinary classroom (in case the school had no laboratory), how to introduce the test to the students, how to communicate with students during the testing session, etc.

Alternative assessments contain open-ended questions, and when testing on a large scale, students' answers need to be interpreted and transformed into a code, which can be entered into a database. The reliability of these data depends largely on the evaluation of students' answers. Students' answers must be interpreted in such a way, that the resulting code is independent of the coder. In the TIMSS Performance Assessment, coders were trained during a one-day workshop on the application of the codes. To verify interpretation differences between coders, two different coders coded a systematic sub-sample of 10% of students' responses independently. In this way, the inter-coder agreement was an indicator of the reliability of coding. This agreement was calculated as the percentage of items on which the two coders agreed with their codes. In the 1995 administration, the agreement on the correctness code ranged between 52% and 100%. The lowest percentage was reached on one item (from the task *Shadows*), where the coders only agreed on 52% of students' answers. In the international protocols, no limits were set for the inter-coder agreement, but in hindsight, 52% should have been considered as too low to yield reliable results.

Another source for unreliable results was the range for the equipment. The TIMSS Performance Assessment Administration Manual (1994) contained extensive inventory lists of the required equipment and materials, including their margins of tolerance. For example, for the rulers it was indicated that these should measure at least 30 cm and could be read to a precision of a millimeter. Another example is the description of the balance, to be used in the task *Plasticine:*

> *"This may be any kind of simple balance, but it should be accurate. It must not have a scale, that is, not calibrated. Balance it without masses (weights) when setting up the station, and make sure that it does not go out of balance with handling. If it is not possible to obtain a balance, one can be constructed from common materials (coat hanger, plastic cups and string)."* (TIMSS Performance Assessment Administration Manual, 1994, p. 30).

All practical guidelines were meticulously followed, both in 1995 and 2000. Within the margins of the guidelines, there were possibilities to make slight adaptations between 1995 and 2000. When organizing equipment for a replication of the test, one cannot always obtain exact copies of the equipment of the prior administration. When preparing for the replication, we did not know whether small equipment differences would have an effect on students' performance. We made two observations that raised our attention, in the equipment of the task *Shadows* and in the task *Plasticine*. In the task *Shadows* a torch is used. The torch used in 1995 gave a slightly vague shadow, while the torch of 2000 gave a sharper edge to the shadow. The latter made student's measurements easier giving them more time for remaining items. Another example of this equipment influence emerged in the task *Plasticine*. In 1995 a delicate metal scale balance was used, while in 2000 a plastic balance was used (see Figure 2, right). Both instruments were allowed within the range of the international guidelines that dictated the test procedures. However, in 2000 the students gained time by the handier equipment, as the new balance reached its balancing point faster. Students would remain with more time for additional items or for reflection on the task.

At the onset of the replication study in the Netherlands in 2000, it was clear that if ever a trend was to be measured, the data needed to be comparable between years. Therefore, it was decided to check with more methods than only an inter-coder agreement. Inter-coder agreement cannot detect effects caused by different laboratory equipment. To detect unreliable and incomparable results, two statistical tests were carried out (Vos, 2002). The results are shown in Table 2. First, for each task, Cronbach's alpha was calculated for 1995 and 2000 separately. Results higher than 0.5 were considered acceptable. On this test, the task *Rubberband* failed. Second, a $\chi 2$-test was carried out to compare the answer patterns of 1995 and 2000. The outcomes, indicated by their significance p($\chi 2$), indicated the probability that answer patterns were comparable. Values lower than 0.05 were considered as indicators of unequal testing circumstances in 1995 and 2000. On this test, the tasks *Rubberband*, *Shadows* and *Plasticine* failed. For the two latter tasks, this incomparability was probably caused by the change in equipment. As a result of the statistical tests, and to avoid distortions of the trend measurement, tasks with questionable data were eliminated: *Rubberband*, *Shadows* and *Plasticine*.

Table 2: *TIMSS Performance Assessment Mathematics tasks, 1995 and 2000 in the Netherlands: reliability, comparability between years, and results.*

Task (number of items)	Reliability Cronbach Alpha		1995-1999 Trend comparability	Students' achievement results	
	1995 (n=437)	2000 (n=234)	p(χ^2)	1995 (n=437)	2000 (n=234)
Dice (6)	0.50	0.64	0.77	77 (3)	74 (4)
Calculator (7)	0.71	0.68	0.99	62 (4)	60 (5)
Fold&Cut (4)	0.83	0.76	0.53	73 (4)	77 (5)
Around the Bend (8)	0.59	0.62	1.00	68 (3)	70 (4)
Packaging (3)	0.61	0.65	0.28	52 (4)	58 (5)
Rubberband (7)	0.58	0.39	0.00	---	---
Shadows (6)	0.64	0.61	0.01	---	---
Plasticine (8)	0.85	0.78	0.00	---	---

Note. --- Dashes indicate omitted results, not satisfying the comparability test.
Standard deviations are shown in parentheses.

RESULTS

Five mathematics tasks (with a total of 28 items) remained suitable for analysis. The achievement of Dutch students on the Performance Assessment in 1995 and its replication in 2000 is given in Table 2, in the right-hand columns. For each task, the average percentage of correct scores on the items is calculated. Compared to 1995, the achievement results did not show significant changes on these tasks. On each mathematics task, the shifts were statistically insignificant. The average percentage correct on all five mathematics tasks in 1995 was 66 (not included in Table 2), which did not differ significantly from the average score correct of 68 in 2000. The results showed that Dutch students in 2000 had not gained practical competencies in mathematics since 1995, despite the increased emphasis on these competencies in the RME-based curriculum. This answers the first research question.

Of course, the null-trend was observed with regret. A curriculum change is a costly affair, and improvements towards its goals (more practical, applicable mathematics skills) are expected. We had to face the fact that educational change does not happen overnight, and it does not happen to its fullest extent. The null-trend could have been caused by the classroom practice, in which students seldom encounter tasks in a laboratory environment. In Dutch classroom practice, hands-on tasks remained scarce and depending on the teachers' initiative. One discoursing policy can be pinpointed: the national assessment practice stuck with a paper-and-pencil format, in which students only read texts about real-life contexts. These tests are easier to organize at a larger scale. As a result, despite curricular intentions, tests offering students tangible real-life contexts (through projects or through manipulatives) are still rare at the lower secondary school level in the Netherlands.

The replication of the test showed, that testing conditions need to be well controlled. If one wants to compare between students, between schools, and also between years, one has to minimize differences in equipment. Small changes in equipment can have a multiplier effect on achievement scores, and thus destroy valuable data. Also, it is important to have different coders, who can code and re-code students' answers at different stages in time. Nevertheless, provided these conditions, alternative mathematics assessment in a laboratory environment is feasible at a large scale. Reliability of data must be scrutinized closely, but the high validity of the test will compensate for this. This answers the second research question.

Anecdotal evidence showed that the TIMSS Performance Assessment was an eye-opener to many mathematics teachers. During the testing sessions, they observed the tasks and how their students coped with these. Some teachers showed their surprise admitting that they had never thought mathematics could be tested in a laboratory environment. They associated manipulatives with 'fun mathematics' as used on the day before holidays. Now, the assessment context created a serious atmosphere. As such, the TIMSS Performance Assessment could be used as exemplary curriculum material, not only in the Netherlands.

In the Netherlands, the replication of the TIMSS Performance Assessment turned out to be a learning experience in many ways. In the first place, only by replicating one can measure a trend and monitor student performance after the curriculum change. From this, we learned that the curriculum change might not (yet) have the in-depth effect that was anticipated. Secondly, the TIMSS Performance Assessment revealed itself as an example for improving teaching and learning. One could imagine that if laboratory-based tests are part of national exams, then teachers who 'teach to the test' might better implement a mathematics curriculum based on modelling and applications. The implementation of nation-wide laboratory-based tests is still far, although the National Institute for Educational Measurement has experimented with it. Finally, replicating the TIMSS Performance Assessment was a methodological learning trajectory. By replicating the test, we discovered methodological weaknesses, which can easily be overcome, if the researcher is attentive and experienced. For example, the test confirmed the potential unreliability of coding open-ended questions. To improve on this point, samples of students' work from prior measurements will need to be safe-guarded for re-coding to enable comparison with the replication study. Also, the narrow margin for laboratory equipment was a surprise. Thus, the replication study turned out to be valuable, because it extended our knowledge on improving reliability of alternative assessment in general. However, as said before, any lower reliability is fully compensated by the high curricular validity.

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