WORKING GROUP 14
Advanced mathematical thinking

CONTENTS

Synopsis of the activities of Working Group 14 ‘Advanced Mathematical Thinking’ 1709
Joanna Mamona-Downs, Maria Meehan, John Monaghan
Calculus and departmental settings 1716
Erhan Bingolbali, John Monaghan
Conceptual change in advanced mathematical thinking 1727
Irene Biza, Alkeos Souyoul, Theodossios Zachariades
Problem solving and web resources at tertiary level 1737
Claire Cazes, Ghislaine Gueudet, Magali Hersant, Fabrice Vandebrouck
The proof language as a regulator of rigor in proof, and its effect on student behaviour 1748
Martin Downs, Joanna Mamona-Downs
Is there a limit in the derivative? – exploring students’ understanding of the limit of the difference quotient 1758
Markus Hähkiöniemi
Characterising mathematical reasoning: studies with the Wason selection task 1768
Matthew Inglis, Adrian Simpson
On some difficulties in vector space theory 1778
Mirko Maracci
An experience of problem solving in mathematical analysis 1788
Giorgio T. Bagni, Marta Menghini
To appear and to be: mathematicians on their students’ attempts at acquiring the ‘genre speech’ of university mathematics 1800
Elena Nardi, Paola Iannone
Relationship between informal and formal reasoning in the subject of derivative 1811
Antti Viholainen
Research and development of university level teaching: the interaction of didactical and mathematical organisations 1821
Carl Winsløw
Mental models of the concept of vector space 1830*a*
Astrid Fischer

SYNOPSIS OF THE ACTIVITIES OF WORKING GROUP 14 ‘ADVANCED MATHEMATICAL THINKING’

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INTRODUCTION

The inauguration of W.G. 14 in CERME-4 marks the first occasion that a working group at a CERME Conference has been devoted to the theme of ‘Advanced Mathematical Thinking’ (AMT). The response was gratifying, in terms both of the number and quality of the papers received, and the commitment shown by the participants at the meetings at the conference. We feel confident that the discussions that took place constituted a firm basis for a secure and fruitful future for this group in ensuing years.

The content discussed was based on the thirteen papers that were submitted and accepted for presentation at the sessions of the group. Twelve of these papers were also deemed suitable for publication in the post-conference proceedings, and appear after this overall report. Every author gave a talk between five and ten minutes, that instigated a discussion about or around the presented paper, lasting typically between 20 to 40 minutes, depending on how the discussion progressed. Naturally, the discussion tended to start closely focussed to the subject of the paper, but evolved into broader topics and issues later. Several times, parallel sessions were organized, allowing some participants to continue their discussions on certain themes at greater length.

The organizing team assigned the papers into certain broad topics so that there would be some consistency in theme for each session. These topics were largely determined by the subject matter of the papers received; however, they also reflect areas that are well represented in the more general education literature on AMT.

Two sessions were devoted to educational frameworks concerning dualities in (student) habits or preferences in mathematical thinking; these reflect perceived differences between the intuitive and the abstract, the procedural and the conceptual, or processes and objects. It was noticeable how many of the contributing authors chose to employ such frameworks. Most of those brought up are well established and widely known, such as Skemp’s instrumental-conceptual model, APOS theory, theory of reification, procept. (However, there were also some less familiar variants that were introduced and explained.) There was interest in how these were applied in particular cases, and in their role more generally. Some concern was expressed about limitations in what such frameworks can provide.

Two other sessions focused on issues more closely tied with specific mathematical content. The mathematical context of the papers received concentrated on calculus / real analysis and vector spaces. (These two areas remain to be the most studied in the literature, despite a broadening of interest over recent years.) Certain images related to the derivative, in particular, formed the mathematical background of several of the papers submitted. There was a tendency for authors to relate their results in terms of general frameworks (as referred to in the previous paragraph), rather than directly connecting students' behavior in terms of the mathematical context.

Also we received several papers that concern the effects on students caused by the settings in which the teaching of tertiary level mathematics takes place, and then the relationship between lecturers and mathematics educators. These concerns are critical for the educators to place their work in a balanced perspective within a college/ university environment, so it was pleasing that the presentations of these papers prompted lively discussion.

Finally, two sessions covered the themes of proof and problem solving. For mathematics education, these two themes are usually taken as distinct agendas, though they are obviously intimately related. We took these two themes together because several papers refer to a particular 'language' that professional mathematicians tend to adopt when communicating. This language reflects both the form of presentation expected for writing down proofs, and as a refined channel for strategy making. Other papers considered more specialized themes, as the role of modeling in problem solving, the extant in which problem-solving processes can be accommodated in web resources, and behavior differences for a logic problem.

The remainder of this report will give a more detailed precis of each session.

THE SESSIONS

Sessions 1 and 2

In Session 1 of Working Group 14, two papers were presented and discussed. Each author gave a 10-minute presentation of his work and results, which was followed by questions from the group.

Matthew Inglis presented his, and Adrian Simpson's, work on *Characterising Mathematical Reasoning: Studies with the Wason Selection Task.* In this paper the authors consider the effect that studying mathematics has on a person's reasoning ability. They look at this through the lens of *Dual Process Theory.* There was much discussion about the *System 1* and *System 2* dispositions of thinking put forward by this theoretical construct. In addition there were several questions and comments on the two hypotheses of Inglis and Simpson on how the System 1 and System 2 of mathematicians is affected by the study of their discipline. Some of this discussion centered on why mathematicians are inclined to make the same (but not the "standard mistake") in the Wason Selection Task.

Markus Hakkioniemi presented his paper on *Is There a Limit in the Derivative? – Exploring a Student's Understanding of the Limit of the Difference Quotient.* The author uses *APOS Theory* in this paper to attempt to analyze students' procedural understanding of the limit of the difference quotient. It was this theory that sparked much discussion among the group. Due to time constraints and the level of interest in APOS Theory, it was decided by the group leaders to start discussions in Session 2 on this topic. It was also decided to revisit Dual Process Theory in Session 2. Finally the group leaders proposed to have small-group discussions in the next session to facilitate more effective and efficient discussion.

Session 3

In Session 3 of Working Group 14, two papers were presented and discussed. Initially it seemed that the theme of the session could loosely be described as *Vector Spaces* but the authors felt that, in discussions, more emphasis should be placed on how students deal with abstraction and the notions of encapsulation, reification, procept and process-object duality. Each author gave a 10-minute presentation on his or her paper, after which the working group split into two subgroups – each one to discuss a particular paper and related themes.

Astrid Fischer presented her paper on *Mental Models of the Concept of Vector Space.* In it she describes a pilot-study carried out to examine students' images of the concept of vector space. In the (sub)-group discussion the following were some of the main topics:

- Predicative and functional thinking were described in more detail;
- It was questioned whether these modes of thinking could be compared with Skemp's instrumental-conceptual model of thinking;
- Encapsulation and reification were further discussed;
- Some methodological issues were raised:
  - What inferences can be made from data on only 3 students?
  - Can a student's thinking on an abstract concept such as that of vector space, be modeled by such a concrete task (that of the Color Machine)?
- It was strongly agreed that the color machine idea was a very nice example and could prove a useful didactic tool.

Mirko Maracci presented his paper *On Some Difficulties in Vector Space Theory.* The following summarize the (sub)-group discussion:

- There was much discussion on various aspects of the paper and several questions put to the author;
- Process-Object Duality of Sfard was discussed, as was Dubinsky's APOS Theory.
- It was questioned as to how useful a theoretical framework can be to explain students' difficulties in abstract mathematics.

Session 4

This session dealt with the followng papers:

Erhan Bingolbali & John Monaghan: Calculus and departmental settings

Elena Nardi & Paola Iannone: Acquiring the 'genre speech' of university mathematics

Carl Winslow: Research and development of university level teaching: The interaction of didactical and mathematical organizations

Although these papers are quite different in several ways each one deals with aspects of the institution that the mathematics is taught in (universities in each case here). Erhan & John looked at the influence of departments on their students' understanding of derivatives. Elena & Paola looked at the mathematical speech and the writing of novice mathematicians. Carl looked at the interaction between research and teaching activities in university mathematics and the didactics of mathematics.

Each gave a 10 minutes summary of their papers. A full and lively discussion followed. Points raised included:

For the paper of Erhan & John: is the theme 'trivial'? - the majority thought 'no'; we'd like to know more about students who do not share their department's perspective.

For the paper of Elena & Paola: Is the question (see the paper) 'trivial'? - 'no', it is at the heart of analysis and took hundreds of years to be resolved. A discussion on symbolism followed which cannot be summarized in a few lines.

For the paper of Carl: How does Chevellard' s praxeological analysis relates to other theories of cultural practice? Can everything be subsumed under this theory?

Session 5

It is noticeable that out of the thirteen papers presented at our group sessions, as many as seven had a theme in Calculus or Real Analysis for their mathematical background. Because of this, we felt that it was appropriate to devote a session on a topic in this field, and we selected two papers for discussion both dealing with images of the notion of tangent/derivative.

The paper by Biza, Souyoul and Zachariathes employs the notion of 'synthetic model' that represents a mixture of existing beliefs and the 'scientific theory'. Beliefs are often based on a few known paradigms rather than definitions. The question is whether students would try to (miss)-apply what they know about the paradigms to a new case, or would they use the new case as a way to assimilate a more general concept. This question was asked in the context of students that only had previous experience of tangents for the circle and conic sections. The students were asked to consider tangents for other curves (some of those were not differentiable). Some students answered simply on the grounds what they understood from their previous experience, others were able to adapt their thinking but was still related to the paradigms known, whereas the remaining students were able to free themselves from the paradigms and articulate more allied to the formal definition.

The paper of Viholainen is concerned with the importance of combining the formal and informal in an effective way, especially through visualization. In this resect, the author reports that students usually do not have difficulties in relating the derivative with the slope of the tangent line, nor in relating a difference quotient with a secant, but they do have a difficulty to obtain a reliable visualization of the limiting process involved. The presentation continued to contrast the different behavior (concerning the understanding of the derivative) shown by two students, one who tended to blend informal and formal reasoning, the other to separate them.

Session 6

An interesting aspect of the intake of papers for the group was that three brought in the idea that mathematicians possess a certain 'language' that their students have to acquire to be successful in advanced mathematics. The important point here is that it is naïve to characterize the mathematician's thinking as being rigorous whilst students prefer informal argumentation. Rather, both depend very much on intuitions and personal interpretations, but the sources are different. For the student, informal reasoning is based on perception and inducing mathematical behavior from a few prototypes, whereas for the mathematician, it is based on evincing meanings from strict definitions and conceiving properties from formally described operations. It is the language that allows the mathematician to access the more refined outlook, so it is important to discuss its exact character and role.

The three presentations take slightly different contexts. The paper by Downs and Mamona-Downs is a theoretical treatise concerning the language that mathematicians use and proposing that this language reflects how they create proofs; many aspects of the creation process is not evident in the presentation. The paper by Nardi and Iannone (also mentioned in session 4) discussed the 'genre' language by asking mathematicians to suggest pedagogical measures that might help their students to attain it. The paper by Tossavainen (presented at the conference but not published here) also describes some traits of the 'language aspect of mathematics'; in particular in how it differs from, or is similar to, a 'real-world' language. This discussion appears within a wider study that employs concept maps to draw up profiles of how novices, advanced students and experts view mathematics and proof.

Session 7

Two papers with a perspective on problem solving at AMT level were presented.

The paper by Menghini and Bagni deals with students' solving behavior when tackling optimization problems concerning 'real-life' situations. The authors identified certain stages for tasks of this kind and used these to analyze the students' attempts. Despite of the standard procedures available from the differential calculus, the modeling of the task environment into a suitable function, and then what to do if the function is algebraically difficult to process, constitute real problem-solving activities. The analysis of the students' protocols revealed that their progress through the more problem-solving aspects of the solution is dominated by raising conjectures and their subsequent evaluation of them. Also it was commented that the existence of a known methodology might restrict the students' flexibility in thinking. This is exemplified by the fact that the students always followed the strategy: model into a function, differentiate and set to zero, whereas in some cases progress could have been made by retaining the geometric setting. Also students, once they modeled their work into formulae, often showed themselves reluctant in referring back to the original environment, and this tendency led to some errors.

In the paper by Cazes, Hersant and Vandebrouk, the authors compare two web resources that offer a stock of mathematical tasks for students to work on. The comparison is made on three lines: the format of the tasks, and how procedural or challenging they are; the range of applications or facets that are represented for any certain mathematical theme; the type of support afforded to the student. One of the most salient differences between the two resources is that for one the students are asked to answer the task and then they can compare the answer with the solution provided, whereas for the other a mark is given for the answer and the student has the option to access a parallel task in order to attempt to get an improved mark. The paper considers the merits and demerits in these two types of support from a problem-solving perspective.

EPILOGUE

The sessions of our group broached themes (applying general educational frameworks; investigating students' conceptual bases to cope with the principles in Linear Algebra and Real Analysis; understanding institutional factors; problem solving and proof) that are both important and need much more research in order to obtain a full comprehension. We are confident that future meetings of the group will build on the good start made on these directions. However, it is opportune to use this epilogue to suggest some further issues that might be useful to discuss in ensuing years.

The divide between the procedural and conceptual is perhaps rather naïve as it is presently represented in the literature. One; the procedural tends to be associated with manipulating sets of operational rules on symbolic systems – this seems hardly justified. Two; AMT is centrally involved in converting systems that are intuitively understood into axiomatic and symbolic systems that provide tools that can extend much beyond what could be achieved intuitively. Three; advanced mathematics often confronts the mathematician with techniques rather that algorithms; techniques often require deep aspects of problem solving to apply them. Perspectives such as these seem to be largely overlooked in educational literature.

The curriculum of a typical mathematics undergraduate program covers a bewildering amount of content. Given the microscopic nature of studying students behaviors in response to particular mathematical situations, it is completely infeasible for AMT researchers to aim for comprehensive coverage for cognitive profiles on all the mathematical material taught. (This may account for a tendency for researchers to keep to general frameworks.) However, it is pertinent to ask how some students, perhaps a minority, do seem to cope with the amount of material. A reason for this may be that they follow basic trains of thinking, based on some fundamental principles, that are repeatedly applied, with slight variations, in many different mathematical contexts. If this is true, then AMT literature perhaps has not gone very far in identifying and examining them.

This being as it may, an even broader issue looms: can we really claim that researchers in AMT have as yet succeeded to present a coherent program clearly demarcating the overall role of the mathematics educator at the tertiary level? A regular series of discussion meetings, such as those provided by CERME conferences, may provide an ideal medium to make this role clearer.

CALCULUS AND DEPARTMENTAL SETTINGS

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Abstract: *The background of this paper is a study of first year undergraduate mechanical engineering and mathematics students' conceptions of the derivative. Test results showed that mechanical engineering students did better on rate of change test items whilst mathematics students did better on tangent-oriented questions. This paper explores the dialectic between departmental setting, lecturers' teaching and student 'positioning'. We report on departmental goals and programmes, lecturers' interpretations of their practices and students' stated preferences for particular conceptions of the derivative. We discuss how these three elements interact and conclude that cognitive functioning is influenced by others, by the setting and by the way individuals position themselves in settings.*

Key words: calculus, lecturers' privileging, departmental settings, engineering and mathematics undergraduates

INTRODUCTION

Most research on students' understanding of calculus and on undergraduates' understanding of mathematics has attended to students' cognitive development, difficulties with and misconceptions of advanced mathematical concepts (e.g., Tall, 1991). Some recent studies have, however, focused on affective dimensions of students' participation in and success in mathematics (e.g., Rodd, 2003). The extant literature of calculus-related studies is largely restricted to these two dimensions. With due regard to the importance of cognitive and affective dimensions, teaching and learning are imbedded in "material environments endowed with cultural meanings; acting and being acted on directly or with the mediation of physical-cultural tools and cultural-material systems of words, signs, and other symbolic values" (Lemke, 1997, p.38). A full analysis of teaching and learning must preserve "as many dimensions of the general phenomenon under consideration as possible, thereby allowing one to move from one dimension to another without losing sight of how they fit together into a more complex whole." (Werstch, 1991, p.121). This study goes a little way towards addressing the 'complex whole' by viewing two groups of undergraduates' understandings of the derivative in the context of their departmental settings, mechanical engineering (ME) and mathematics (M).

Scant attention appears to have been paid to contextual elements and the impact of departmental settings on students' understanding of advanced mathematical concepts. Maull & Berry (2000) is an exception that has commonalities with our study. They examined first and final year mechanical engineering and mathematics undergraduates alongside postgraduate students and professional engineers and concluded that "the mathematical development of engineering students is different from that of mathematics students, particularly in the way in which they give engineering meaning to certain mathematical concepts" (ibid, p.916).

In a similar vein some studies approach teaching and learning in an institutional context ('institution' in a wide sense, e.g. a class, a department or a school/university may act as an institution). Daniels (2001, chapter 5) focuses on institutions as a way simultaneously addressing psychological and sociological issues in education. Some recent French work in the mathematical education of undergraduates is informed by the anthropological approach of Chevellard, which attaches importance to institutional aspects of knowledge acquisition. Praslon (1999), for example, examines a university entrance task on the continuity and differentiability of a function and stresses the important role of institutional values and norms in developing personal relationship with mathematical knowledge, emphasising that individual relationships with particular mathematical objects are shaped by institutional parameters. As Artigue, Assude, Grugeon, & Lenfant (2001) put it:

> *'Mathematical knowledge cannot be considered as something absolute. It strongly depends on the institutions where it has to live, to be learnt, to be taught. Mathematical objects do not exist per se but emerge from practices which are different from one institution to another one'* (ibid., p.2).

Holland, Lachicotte, Skinner, & Cain (1998) also emphasise the importance of the particular social setting, 'figured worlds', and state that in socially and culturally constructed settings "particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (ibid., p.52). These studies suggest institutional values, norms and characteristics cannot be ignored in research on learning; we agree and examine undergraduates' understanding of the derivative with regard to departmental affiliation.

In two recent papers we have focused on aspects of learning and teaching the derivative in mechanical engineering (ME) and mathematics (M) departments. Bingolbali & Monaghan (2004) discuss ME students' tendency towards rate of change aspects of the derivative and M students' tendency towards tangent-oriented aspects. We partially attribute these different tendencies to the calculus practices to which students are exposed in each department. We also argue that these difference tendencies are interlinked with students' 'identity' as mathematicians or engineers. We also note that lecturers 'privilege' different aspects of the derivative when teaching in different departments. This privileging is further explored in Bingolbali (2004) and cultural tools involved in this privileging include examples used in lectures, examination questions and textbooks.

In this paper we explore the ideas in those two papers further. We wish to understand why ME and M students develop different tendencies towards different forms of the derivative concept. In addressing this question we take into account the departmental (institutional) settings in which learning occurs and the teaching. The remainder of the paper presents the context of the research, results (departmental settings, lecturers' practices, students' experiences), a discussion and a conclusion. The discussion section explores the relationship between departmental settings, lecturers' privileging and students' understanding.

THE BACKGROUND AND THE CONTEXT OF THE RESEARCH

The research explored ME and M students' conceptual development of the derivative over the first year of undergraduate studies in a large Turkish university and we do not claim that results from this study generalise beyond the confines of this university. The research employed quantitative (pre-, post- and delayed post- tests), qualitative (student questionnaires and student and lecturer interviews) and ethnographic (observations of semester 1 calculus courses and student 'coffee-house' talk) methods. The overall approach to the research could be described as grounded (Glaser & Strauss, 1967) and naturalistic (Lincoln & Guba, 1985). The pre-, post- and delayed post-tests were applied to 50[1] ME and 32 M first year degree students. The tests were administered at, respectively, the start of the year, the end of semester 1 and the end of semester 2. The tests addressed questions regarding 'rate of change' and 'tangent' aspects of the derivative and were used to gain insights into: how ME and M students' concept images of the derivative developed over the course; how students dealt with rate of change and tangent concepts when questions were presented in graphic, algebraic and application forms. In the pre-test ME and M students performed similarly; there was not a significant difference in their performance. In the post-test and the delayed post-test both groups improved their performance but in different ways: ME students did better than M students on all forms of rate of change-oriented test items whilst M students did better than ME students on all forms of tangent-oriented questions[2].

Although these results show a clear trend they do not reveal why this trend exists. To explore this further we designed two additional items and administered them after the delayed post-test. Due to space limitations we only report on item 2 (see Figure 1) in this paper (see Bingolbali & Monaghan (2004) for further details). The responses suggest that many students 'support' derivative conceptions that are compatible with departmental goals[3].

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[1] 50 ME and 32 M students sat all three tests: pre-, post- and delayed post-test.

[2] These results have not been published yet but are available as an informal paper from the first author. We wish to stress that these results apply to students' correct and incorrect answers, not just their preferences, i.e. institutional setting actually interrelates with cognition.

[3] We use the term 'department' to refer to an academic unit in a faculty and use the term 'departmental goals' to refer to the overarching goals of ME and M study programmes.

Item 2: Two university students from different departments are discussing the meaning of the derivative. They are trying to make sense of the concept in accordance with their departmental studies.
Ali says that "Derivative tells us how quickly and at what rate something is changing since it is related to moving object. For example, it can be drawn on to explain the relationship between the acceleration and velocity of a moving object.
Banu, however, says that "I think the derivative is a mathematical concept and it can be described as the slope of the tangent line of a graph of y against x".
a.) Which one is closer to the way of your own derivative definition? Please explain
b.) If you had to support just one student, which one would you support and why?

Figure 1: An item to explore reasons for rate of change and tangent orientations

RESULTS

The results are presented in three sections. First, the ME and M departments are described by focusing on departmental goals with regard to fostering students, first year courses and the role of lecturers and students. Second, lecturers' reports with regard to their differential privileging of cultural tools in each department are presented. Finally, we report on ME and M students' reasons for preferred forms of the knowledge with specific reference to departmental settings.

Engineering and mathematics departmental settings

Each department has its own goals, practices and features. Of the many ME departmental goals considered as 'the targets of the educational activities carried out in accordance with the mission and vision of department' an important one is:

> The department aims to foster engineering students as those mechanical engineers who have a fundamental knowledge of technological development… who not only analyse but also synthesise, who can have the competence and self-confidence to do research.

There are, of course, many stated goals of ME departments. The important point about the above goal is that it is the overarching goal and is concerned with fostering mechanical engineers. Likewise, the mathematics department also has many stated goals but the overarching goal stated by the department is to foster mathematicians:

> The goal of the department is to foster mathematicians… to provide fundamental know-ledge for those students who want study in mathematics and mathematics-related areas.

There is a dialectic between stated goals and programmes (sets of courses). The courses for both groups of students considered here are presented in Table 1.

First Year Courses			
Mechanical Engineering		Mathematics	
First semester	Second semester	First semester	Second semester
• Calculus I • Physics I • Introduction to ME • Introduction to computer programming • Technical drawing • Ethics of ME (optional)	• Calculus II • Statics • Chemistry • Computer-based technical drawing • Application with computers in ME • Intro to engineering mechanics (optional)	• Calculus I • Linear algebra I • Abstract mathematics I • Introduction to computer	• Calculus II • Linear algebra II • Abstract mathematics II • Mathematical programming

Table 1: ME and M departments' first year courses

The only common course run by both departments are Calculus I & II. These courses, all taught by members of the mathematics department, cover ostensibly similar content but, as Table 2 (compiled from observations of lectures and student course notes) shows, for semester 1 calculus courses, there are telling differences with regard to time spent and examples given over 'rate of change' and 'tangent' approaches.

	Rate of change		Tangent	
	ME	M	ME	M
Duration examples	≈133 minutes (9 examples)	≈11 minutes (no examples)	≈10 minutes (no examples)	≈85 minutes (7 examples)

Table 2: Semester 1 ME and M calculus course content

ME departmental stated goals are also related to applications and practical matters whilst M departmental stated goals are related to abstraction and mathematical thinking, but is this how lecturers and students perceive people and practices in each department? To address this we turn to their views.

Lecturers' interpretations of their practices in relation with departments

How do calculus course lecturers conceive distinctive departmental features and goals and tailor their practices accordingly? Four mathematics and two physics lecturers, who had taught both ME and M students in recent years, were individually interviewed to find out: if they teach engineering and mathematics students in different ways; if they set different types of questions in examinations; and if they used different textbooks for different departments. Lecturers' responses to the interview questions were transcribed, analyzed and translated from Turkish to English. Analysis consisted of repeated rereading of transcripts, noting and categorizing statements which appeared to shed light on lecturers' perceptions of ME and M departments, and students and their mathematical needs. We report the views of five lecturers in this section: L1 taught calculus in the ME department; L2 taught calculus in another engineering department; L3 taught calculus in the M department;

L4 and L5 are physics lecturers who have taught physics in engineering and mathematics departments[4].

All lecturers stated that they made amendments in their instruction and emphasised different aspects of a particular concept whilst teaching in different departments. For instance, when asked if they were influenced by the departments or if they taught in different ways, lecturers stated:

L1 They demand from us some stuff. It is like we use mathematics here and there, we want our students to know this and that so that they can be successful in the coming years' courses.

L2 The starting point and main aim is where maths and engineering students make use of maths. Maths students need to know everything but engineering students only need to know the parts which are useful for them.

L1 stated elsewhere in the interview that he had consultations with the ME department administrators regarding the content of the calculus he was to teach and that ME departments are concerned with the way their students will be using mathematics. L2 differentiates his teaching and makes a distinction between the calculus of engineering and mathematics in terms of 'usefulness for students'. L3 explains the aim of calculus for M students as:

L3 It is how to get students to comprehend theoretical thinking. I mean how to attain a theoretical thought; and to get them to know what proof methods are and how to carry them out. We try to make students comprehend this in the maths department.

Similarly when asked if he includes theorems in M calculus examinations L1 states:

L1 Maths students will be specialists in this area; they need to know this job's reason and logic. That is why you can ask them theorems in their examinations. This is their job.

L3 views the aims of M calculus as introducing students to theoretical thinking and proof methods in mathematics. Likewise, L1 refers to 'reason' and 'logic' issues.

We do not think that these lecturers' perceptions of the aims of M calculus courses can be divorced from the way they view the M department or, indeed, mathematics.

Physics lecturers articulate similar remarks regarding engineering and M departments:

L4 In the mathematics department I tried to give examples concerned with the essence more, while in the engineering department it is more towards to application aspects in the sense that problems can be connected to real life phenomena… I tried to choose some typical questions which are peculiar to this or that particular department.

L5 Topics are presented so that they are useful for the departments' job…, are close to these departments features. And I think this is the right thing. You need to give topics in accordance with each department's feature so that they are useful to students.

It is interesting to note that, as 'outsiders' to both engineering and mathematics departments, physics lecturers view that engineering department's physics should be

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[4] The Mathematics department used to run a physics course in the first year. L4, for instance, taught physics in the Mathematics department for more than a decade. When the study was conducted the department, however, had moved the physics course to the second year's programme.

application oriented whilst mathematics department's physics should be more related to the 'essence' (L4). But why do physics lecturers tend to differentiate the teaching of physics according to the departments and differentiate between engineering and mathematics departments? We return to this in discussion section.

Students' stated preferences

The data obtained through the two additional items shows a clear trend: ME students develop a tendency for rate of change aspects of the derivative whereas M students develop a tendency for tangent-oriented aspects (Bingolbali & Monaghan, 2004). This section attends to the way students forge a relationship between their preferences of forms of the knowledge and the department in which they study. We report on some examples of ME and M students' written responses to item 2 (Figure 1).

Half of the ME students 'supported' Ali (51/49% for item 2 a/b respectively). The other half clearly felt a tension with regard to the 'correctness' of mathematics (Bingolbali & Monaghan (2004) provides more detail on this 'other half'). We, here, will only report on the responses of students who supported Ali. The responses of three such ME students are shown below. They attribute their reasons to: real life; applications; rate of change; engineering.

S1 I am thinking with an engineer mentality. This makes me tend to be close to the practicality and concreteness. I am trying to make what I am thinking and understanding concrete, even when thinking of maths and geometry.What Ali says is closer. Calculating rates of change seems to me more real…since I am going to be an engineer, Ali's idea would be just different because I would be the one who makes mathematics concrete.

S2 Because Ali's interpretation is closer to ME and especially using 'velocity' and 'acceleration' underpins the foundation of my thoughts regarding the derivative.

S3 I guess that Ali would be either an engineer or physicist, and Banu would be a mathematician.

These students' reasoning is compatible with the way their department is commonly perceived. These ME students view their profession as the one which deals with concrete and practical matters, with which they appear to identify themselves.

M students showed a strong preference for Banu's tangent-oriented interpretation (63/78% for item 2 a/b respectively). The responses of three M students supporting Banu are shown below. They attribute their reasoning to: the slope of a tangent; belonging to a mathematics department; interpretation from a mathematician standpoint; the comprehensiveness of the definition.

S4 Banu gives the definition while Ali gives the explanation. I would support Banu because she explains it in a scientific way.

S5 Banu interprets the derivative from a mathematician's perspective, and Ali interprets it from a Physicist's standpoint. At the end of the day, since I am too from the maths department, I find Banu's explanation close to myself…

S6 Banu's one because mathematics is not related to the world we are living in, it is related to the world which we created in our mind.

Support for Banu's interpretation of the derivative includes definition-oriented, scientific and being from mathematics department. S4 and S6 implicitly hint at the exactness of mathematics and S6 even views mathematics as not related to the world.

Although item 2 did not state the department that Ali and Banu belong to, some students tend to 'locate' these imaginary students as engineers, mathematicians or physicists.

DISCUSSION

With regard to departmental/institutional settings and students' development, specifically the emergence of ME students' tendency to rate of change and M students' tendency to tangent aspects of the derivative concept, we consider cultural aspects of both departments and lecturers' and students' responses.

Barab & Duffy (2000) argue that every community has a common cultural and historical heritage which may be manifested through many forms; each community has and develops its own goals, practices, conventions, rituals and histories. This applies to both the ME and M departments, they have their own cultural forms: goals, practices, etc. which have developed over decades, and they continue to evolve.

Both departments have 'stated goals', the 'overarching' goal of each department is to foster future mechanical engineers (mathematicians). Intertwined with these goals both departments have programmes and specific courses. These departments also have some peculiar features which they are often associated with, e.g. engineering is associated with 'practical', 'application', and 'real life' whilst mathematics is associated with 'abstract' and 'theoretical thought'; we shall call these peculiar features 'characteristics'. From an activity theoretic point of view, e.g. Engeström (1987), each department is an activity system, though we do not develop an explicit activity systems account in this paper.

Our data suggests that distinctive departmental characteristics influence (and are influenced by) both lecturers' and students' actions and meaning making. Lecturers perceive the two departments as having distinct goals and amend their instructions accordingly. This does not apply solely to mathematics for engineers or mathematicians. Physics lecturers' views, for instance, differentiate with regard to the physics that engineering and mathematics students should learn, i.e. engineering physics should be application-oriented and mathematics physics should be concerned with the 'essence' suggesting that they have certain perceptions regarding (students in) both departments.

Pre-, post- and delayed post-tests suggest that a great number of students from each department develop cognitive structures, with regard to the derivative, relevant to the perceived mathematical goals of their department. Students' stated reasons for their preferred forms of the derivative also suggest that about half of ME and two thirds of M students develop a personal association (being an engineer or a mathematician) towards particular conceptual forms of the derivative. These ME students associate engineering with 'practicality' and 'concreteness'; they regard Ali as being an engineering student because of Ali's rate of change interpretation of the derivative.

These M students find Banu's tangent interpretation of the derivative more 'formal' and regard Banu as being from a mathematics department. In short, the cognitive structures developed to give meaning to the derivative concept and the stated preferences for given views of the derivative are, for a great number of students, somehow in line with the characteristics of the departments to which they belong.

We have discussed departmental goals, lecturers' privileging and students' tendencies. The important question is how are these interlinked and, in particular, how do individuals (students and lecturers) and the institute interact. We find Holland et al.'s (1998) account suasive. They argue that in socially and culturally constructed settings particular characters and actors are recognised, significance is assigned to certain acts and particular outcomes are valued over others. In ME and M departments, at the observable level, this is manifested in what lecturers report about their instruction and what many students report about their preferred forms of the knowledge. L3, for instance, differentiates between engineering and mathematics students on the basis of their being from different departments; that "maths students need to know everything but engineering students only need to know the parts which are useful for them". In the same vein L2 views that the aim of calculus course for mathematicians is to "make students comprehend theoretical thinking ... get them to know what proof methods are and how to carry out them". Both L2 and L3 statements are broadly supported by observations of lectures and students' lecture notes (see Table2). There is a complex dialectic here between institutions (departments), sets of individuals and values, of which we do not pretend to have but scratched the surface. We present lines of strong influence between these 'players' diagrammatically in Figure 2[5]. One way lines are used to indicate strong influences but further lines of influence could be inserted: all one way lines could be two way, e.g. students do influence lecturers; all three players could/should have loops from themselves to themselves.

$$
\begin{array}{ccc}
 & \text{Department} & \\
\swarrow & & \searrow \\
\text{Student} & \leftarrow & \text{Lecturer}
\end{array}
$$

Figure 2: Strong Influences

We now explore more subtle influences and focus on the student. Lecturers tailor their calculus instruction to the department the student is in. Bingolbali (2004) shows how lecturers' privileging of different aspects of the derivative concept influenced ME and M students' developing conceptions. But students do not perceive their departments only through lecturers. They come to university, to a specific department, with beliefs, values and aspirations. They interpret departmental settings and 'figure out' their positions accordingly. In this connection, for instance, considering student 1's, a ME student, account it can be realised that this student views that certain elements are more valued in their departments, e.g. practicality and concreteness. Conversely, student 4, a M student, values scientific thinking. It is

[5] Our intention here is not to convey the idea that the department is external to students and lecturers; we simply examine these parameters separately for the sake of reporting.

reasonable to assume then that the way students perceive the departments in which they study has an important influence on their developing conceptions.

CONCLUSION

Considerations of departmental settings of ME and M help us to explain why lecturers amend their instructions in different ways and why ME and M students' conceptions develop differently. Each department's characteristics and goals explicitly or implicitly impart particular value judgements with regard to mathematics. Interpretations of these value judgements shape both lecturers and students' perception of, and actions in teaching and learning, mathematics in different departments.

Almost all extant studies of students' understanding of the derivative have focused on cognitive aspects and the individual mind. From this standpoint the results of the pre-, post- and delayed post-tests are startling – *can it be that they actually think in a different way?* We do not dismiss cognitive studies, nor do we ignore the individual, but we feel that they must be seen in *context* – individual cognitive functioning is influenced by others, by the setting and by the way individuals position themselves in settings. From this standpoint differing conceptions of the derivative is not really surprising but is simply an interesting phenomenon to investigate. We are aware that our investigation into this, to date, leaves much unexplained (how student 'positioning' develops as well as accounting for students who do not appropriate departmental stances, e.g., ME students who do not privilege rate of change interpretations of the derivative). We believe, however, that the ideas we have introduced cannot be ignored in future studies of advanced mathematical thinking.

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CONCEPTUAL CHANGE IN ADVANCED MATHEMATICAL THINKING

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Abstract: *In this paper, we argue that the theoretical framework of conceptual change could help us to interpret some of the misconceptions dealing with concepts of advanced mathematical thinking, as the concept of curves′ tangent, which the students have studied in specific cases in the middle high school and they deal with the general case of them in the upper secondary and tertiary level. In this study, we trace the beliefs of the students and the synthetic models, which they create in their effort to assimilate the general concept of curves′ tangent in their existing knowledge of the tangent of circle and conic sections. We make a case that students take for granted properties of circle's tangent in curves, which do not apply in general and they cause the synthetic model that students create to deal with tangent's problems.*

Keywords: Conceptual change, tangent line, concept image, synthetic model, calculus, misconceptions of tangent line.

INTRODUCTION

This study is based on the theory of the *Conceptual Change*. This theory examines the learning process, especially in cases where the new knowledge is incompatible with the prior one (Vosniadou & Brewer, 1992; Vosniadou, 1994). According to this theory, the students very early create initial explanatory frameworks that consist of certain coherent core of *presuppositions*. These *presuppositions* influence *beliefs* that are created through every day and cultural experience. When students face a new knowledge, which is incompatible with the prior one, in their effort to assimilate the new information in their existing cognitive base, they create *synthetic models*, which are a mixture of their existing *beliefs* and the scientific theory.

In this study, we examine the learning development of the notion of curves′ tangent from this theory point of view. We investigate the *beliefs* of the students and the *synthetic models*, which they create in their effort to assimilate the generalized concept of curves′ tangent in their existing knowledge concerning tangent of circle and conic sections.

We make a case that students take for granted properties of circle's tangent in curves, which do not apply in general. These properties are the *generic properties* of the corresponding *concept image* (Tall & Vinner, 1981; Tall, 1986; Vinner, 1991). It seems that students generate a *paradigmatic intuitive model* of circle (Fischbein, 1987). This model remains active after the traditional instruction of the general case of this concept in the upper high school and causes the *synthetic models* that students create to deal with tangent's problems.

THEORETICAL FRAMEWORK

The theory of *conceptual change* examines the process of knowledge acquisition and especially in situations where the prior knowledge is incompatible with the new. According to this theory, the children, in their effort to understand the world around them, create a *framework theory*. This is not a formal theory but something like a *naive* theory, that is an explanatory framework created from first ages and it is consisted of ontological and epistemological *presuppositions* structured in a coherent core. These *presuppositions* are influenced by everyday experience. In most of the cases, students are not aware of the control of the constraints of these *presuppositions* in their interpretation of receiving information and their conceptualization. This *framework theory*, through everyday and cultural experience, causes some *specific theories* (Vosniadou & Brewer, 1992; Vosniadou, 1994). The *beliefs* that constitute a *specific theory*, act as a secondary level of constraints in the process of knowledge acquisition. These *beliefs* and the *presuppositions* that cause them are *intuitive knowledge* with the meaning that Fischbein (1987) gave to the notion.

Many times, these existing *presuppositions* and *beliefs* influence the acquisition of new knowledge and cause cognitive problems. *Conceptual change* theory tries to interpret exactly these problems. In many cases, the new information is incompatible with the existing *presuppositions* and *beliefs* of the student. In these cases, the acquisition of new information needs a radical revision of prior knowledge. In fact, it needs a radical *conceptual change* that is a difficult and time-consuming process of learning. Usually, the students' *beliefs* according to their intuitive nature are too strong and consistent. Consequently, various failures occur in the learning process and some of these create misconceptions that take place in a not arbitrary way. The *synthetic model* is of this kind of misconceptions. The term of model is used as the *mental models*, which is a mental representation generated by a person during his/her cognitive operations when he/she confronts a problematic situation. Especially, the *synthetic model* is a model that reveals students' misconceptions when they try to reconcile new information with their initial *explanatory theory*. These models are a mixture of existing beliefs of individuals and the scientific knowledge concerning the same notion. Actually, the students create *synthetic models* in their effort to assimilate the new information in their existing cognitive base although they are incompatible (Vosniadou & Brewer, 1992; Vosniadou, 1994). Examples of such synthetic models, in the case of science, is the model of the Earth as "a hollow sphere with people living inside it on flat ground" (Vosniadou & Brewer, 1992; Vosniadou, 1994) or, in the case of mathematics, is the model of a fraction as a part of the unit where "the more parts means the less value" (Stafilidou & Vosniadou, 2004).

The theory of *conceptual change* has already applied to a considerable number of cases of science learning. In addition, some recent studies investigate *conceptual change* in the learning process of mathematical concepts. These are referred to the concept of number (Merenluoto & Lehtinen, 2002); to the transition from one set of numbers to a more extensive one (eg. from natural numbers to fractions or rational numbers) (Stafilidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2002, 2004a, 2004b); to proportion (Van Dooren, De Bock, Hessels, Janssens, Verschaffel, 2004) and to infinity (Hannula, Markku, Maijala, Pehkonen, & Soro, 2002; Tirosh & Tsamir, 2004). Many other recherchers have investigated students' previous conceptions concerning mathematical notions and their incompatibility with the corresponding formal knowledge. Fischbein (1987) talked about *intuitions* and their effects in mathematical reasoning, Vergnaud (1988, 1990) mentioned the existence of implicit mathematical concepts and theorems which act as invariants and called them *concepts-in-action* and *theorems-in-action*, Cornu (1991) described *spontaneous conceptions* before formal thinking, Stavy and Tirosh (2000) expounded their theory of *intuitive rules*. Conceptual change approach does not contravene the above theories but it offers a social constructivism perspective and tries to provide, among others, student-centered explanations about knowledge acquisition concerning counter intuitive math concepts and to alert students against the use of additive mechanisms in these cases (Vosniadou, 2004).

In this study, we examine the learning development of the notion of curves´ tangent. The students have studied the concept of the tangent of circle in middle high school. In upper high school, they deal with the tangent of conic sections and later on with the tangent of a curve. The historical analysis of this notion reveals its different aspects as they appeared through the evolution of mathematics science. This historical path could give us a support in our effort to interpret certain answers of students, which could make known their conceptions about tangent line (Artigue, 1990).

The aim of this paper is to interpret the students' misconceptions concerning tangent line from the *conceptual change* point of view. We trace the *beliefs* of the students and the *synthetic models*, which they create in their effort to assimilate the general concept of curves´ tangent in their existing knowledge of the tangent of circle and conic sections.

METHODOLOGY

The participants of this study were 19 first year university students of mathematics, of various levels of performance. They answered a questionnaire printed on paper and afterwards we had an interview with each one of them and discussed his/her answers. All the conversations were recorded during the interviews. Through the previous year, all of these students had a traditional calculus course at high school, which included the concepts of limit, continuity, derivative, tangent line and integral. By the time we interviewed them, they had not been taught these concepts at a university level.

The questionnaire included three parts. The tasks of the first part aimed to investigate student's beliefs about the properties of a tangent. The tasks of the second part aimed to investigate the student's ability to recognize a tangent. The tasks of the third part aimed to test the validity of students' answers in the second part and the persistence of their mistakes.

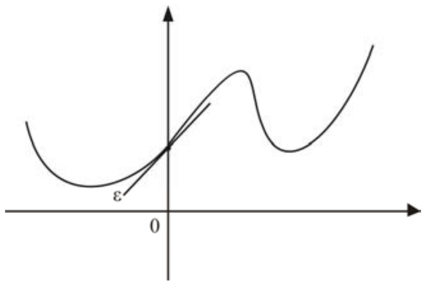
In the first part of the questionnaire, the students were asked to determine whether the following statements are True of False:

A1: *The tangent of a curve at a point $A(x_0,f(x_0))$ is a line having exactly one point common with the curve and it does not split it.*

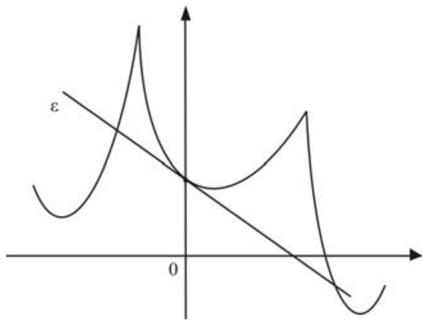
A2: *The tangent of a curve at a point $A(x_0,f(x_0))$, divides the plane into two semi-planes one of which contains the whole curve.*

A3: *The tangent of a curve at a point $A(x_0,f(x_0))$, may have more than one point common with the curve.*

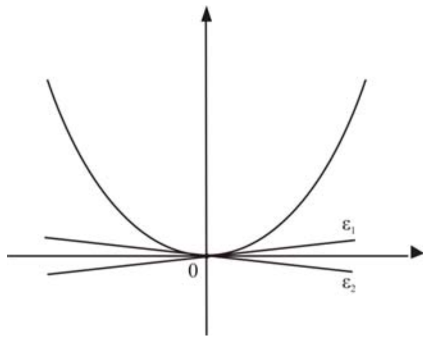
In the second part of the questionnaire, the students were asked to determine which of the drawn lines in the following figure are tangent at point A(0,f(0)).



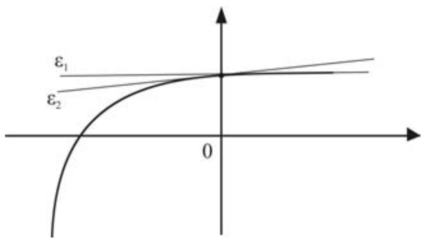
B1



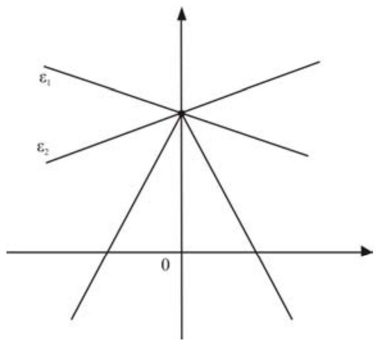
B2



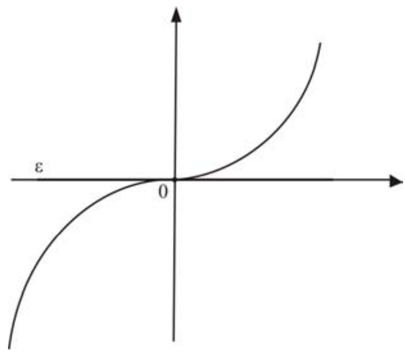
B3



B4



B5



B6

Figure

In the third part of the questionnaire the students were asked to draw the tangent at a point, if there is one, of some curves, similar but not the same as those in the second part. The students answered in this part in a similar way to the second part. They did not draw any unpredictable tangent. So, we do not present here the third part of the questionnaire as it does not provide anything more except of the verification of the results of the second part.

FINDINGS

According to the students' answers to the questionnaire and the interviews, three classes were defined, depending on the extent that the elementary definition of circle's tangent dominates their *concept images* about tangent (table).

First class	Second class	Third class
"circle concept image"	*"circle-like concept image"*	*"curve concept image"*
5	7	7
26%	37%	37%

Table: Findings

The first class comprised five (5/19) students. They had a *"circle concept image"* of tangent. These students, generally, gave wrong answers to the tasks of the first part and they used the properties of circle's tangent to identify the tangent in the tasks of the second part of the questionnaire. Some of them accepted the line ε in B1 graph of the figure as a tangent, but they rejected the line ε in the second graph. In the interviews, they explained their choice by mentioning that in the first graph the other common points are not on view in the figure while in the second they are. The following dialogue indicates an explanation that was given in the interviews:

S: In B2 the line crosses the curve and it intersects the curve more than once… but in B1 it does not.

I: Then what will happen if we extend the line in B1. Won't it look like the one in B2?

S: Can we do that? If this is happened it will no longer be a tangent. But this is not the case. No one did extend it.

Many of these students, also, rejected other correct tangents, as x-axis in B6 graph, which splits the curve while they accepted, as tangents, lines that are not, as ε_1 or ε_2 in B3 graph. For some of them, B5 graph makes an exception to this. The point A is a "corner point" and two of these students remembered that "if the common point is a corner one there is no tangent", so they rejected it without being able to say why (this was the beginning of a fruitful conflict).

The second class comprised seven (7/19) students. They had created a more sophisticated *concept image* of tangent. We will call this *"circle-like concept image"*. They checked the validity of circle's properties locally. For them: "a curve has a tangent at a point, if there is a neighbourhood around this point, where the curve seems like a circle". Most of them gave correct answers in the first part of questionnaire but they could not recognize as a tangent a line that splits the curve, as x-axis in B6, or coincides with a part of it, as the ε_1 in B4. For them, ε was the tangent of the curve in B2, because, as they said in the interviews, the curve looks like a circle, locally. A student was asked during the interview about the tangent of a straight line. Although she knew that the formal definition implies that the tangent at a point is the same line, she replied:

"No, it cannot happen. The straight line does not have a tangent, because the tangent intersects our curve at any neighbourhood around the point"

The third class comprised seven (7/19) students. These participants didn't have any problems to identify a tangent. These students gave correct answers to almost all tasks as they had created a *"curve concept image"* which did not depend on the circle's properties. In the interviews, we asked them to give a formal definition. Only two of them were able to define the tangent at a point as the line which passes through this point and has slope equal to the derivative in this point. All of the students knew that the definition "comes from the derivative" and for this reason they did not care about the validity of the circle's properties. The only criterion for them was: "the point is not corner point". For example one student said:

"I have been taught at school what a tangent line is. I don't remember the formal definition…but I am sure there is not a problem when the line crosses the curve or when the line intersects more than once… but don't ask me why. I don't remember, but I have an intuition which leads me to all the answers that I gave."

DISCUSSION

In order to explain the above findings we have to describe what the students had learnt through their experiences about tangent line. The notion of the tangent line appears in three stages during a student's schooldays. At first, in Euclidean Geometry, students learn the tangent of the circle as a line that has exactly one point common with the circle. An intuitively obvious property of this line is that it has a common point with the circle and divides the plane in two parts, one of which contains the whole circle. Later, in Analytic Geometry, the students are introduced to the conic sections. In these cases, the tangent's definition is more sophisticated: "the tangent in a point A is the limiting position of the secant AB as B approaches A". The "exactly one common point" property remains true in conics, but it is not enough to define the tangent; there are lines, which have one point common with parabola or hyperbola and they are not tangent lines. On the other hand, the "one common point and residence on one semi-plane" property is valid for all cases except hyperbola, where the tangent separates the two branches of it. Consequently, we can say that the property remains true, even in the case of hyperbola, for each branch separately. Therefore, there is no necessity for students to change their previous intuitive images about the two properties of the circle's tangent: "exactly one common point" and "one common point and residence on one semi-plane". A small adaptation of their beliefs is enough to *assimilate* the new knowledge about conics' tangent in their existing knowledge about circle's tangent. In this case, it just needs an enrichment of prior knowledge concerning tangent line.

Finally, in Calculus courses, students encounter the concept of tangent at a point on a curve. At this level, a curve's tangent is defined through the concept of derivative. In fact, this definition is the same as in the case of conic sections. The difference is that none of the above properties remains valid, in general. There are functions that have a tangent that has more than one intersection points with the curve or/and splits the curve into two or more pieces (graph B6).

Analyzing the students' answers, we can say that some of them (first class) use the above properties as the only criterion to identify if a line is tangent. The students of the second class have created a *synthetic model* in their attempts to deal with the tasks of the questionnaire. They know that the general definition of the tangent line does not imply the circle's properties. They also know that the tangent line and its existence, depends on what is happening locally in the curve. Although the two circle's properties are not valid generally, they remain active in their new *concept image* of the tangent line. These students have in mind how a tangent line should "look like". They focus on an area of the curve near the point and test the validity of circle's properties in this part, through their adapted definition. On the other hand, the students of the third group had created an "adequately good" *concept image* of curves' tangent, even though they didn't remember the corresponding *concept definition.* That means that their *concept images* concerning tangent line was not closely dependent on the circle's properties. This is a good basis for their transition to the formal meaning of tangent line especially in cases where the graphical representations become poor in information or they are not trustworthy even in a computational environment like the case of function $f(x) = x^2 \sin\frac{1}{x}$.

Consequently, many of these students have a *concept image* of tangent, involving circle–like pictures. These *concept images* contain a particular representation of tangent that could be called a *generic tangent* (Tall 1986; Vinner 1991). The *generic tangent* acts as a *paradigmatic model* (Fischbein, 1987). It is not an example of the notion of tangent in general but it is a particular exemplar and it is accepted as a representative for the whole class of tangent lines.

In terms of the theory of *conceptual change*, we argue that the ideas related to the notion of the circle's tangent are *beliefs*, which act as a barrier to the process of mastering the notion of curve's tangent. Students usually generate *synthetic models* in their attempts to relate the information they receive about tangent with their knowledge on the circle's properties. This *synthetic model* is a "secondary intuition without formal perfections" that is based on the *paradigmatic model* of circle.

While the number of participants in this study is limited, we believe that it could offer some evidence to support our assumptions. This study suggests that the acquisition of knowledge of tangent line requires a *conceptual change*, which is a complex and discontinuous process. We tend to believe that the main beliefs related to the circle model are "exactly one common point" and "one common point and residence on one semi-plane". These properties are inherited from the circle and they are generic. The circle in this case is prototype and forms their *paradigmatic model* concerning tangent line. These are *secondary intuitions* of students (Fischbein, 1987) but they are not typically correct. They are influenced by their school experience related to circle's properties and they are obstacles to the process of transition to a generalized notion of tangent.

Furthermore, the historical trace of tangent line could give us an interesting point of view of students' difficulties concerning this notion. As Artigue (1991) described, although the first definitions of tangent of circle, firstly, and conics, later on, came too early in the history of mathematics, it was only at sixteenth century when a more general definition of tangent line appeared.

It looks that this transition from tangent of conics to tangent of curve needed a revised thinking about this concept. This innovation in mathematics was not just an addition of new ideas to the previous ones. Actually, it needed the introduction of infinity processes and this was a *revolution in mathematics,* in the sense that Dauben (1984/1992) gave to this notion.

Therefore, a teaching proposal of tangent line could use a revised representation. This approach would take into account the above results concerning *conceptual change* in the case of tangent line and could prepare students from the first stages of the study of this notion for its general features. This could be the local straightness, which is the *cognitive root* for the notion of derivative (Tall, 1989, 2003). The property of *local straightness* refers to the fact that, if we focus close enough to a certain point of a curve, this curve looks like a straight line. Actually, this "straight line" is the tangent line at this point. This property satisfies all cases of tangent lines and it could be facilitated, wherever it is possible, by the use of new technology with appropriately designed software (Tall, 1989, 2003; Giraldo, Calvalho & Tall, 2003).

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PROBLEM SOLVING AND WEB RESOURCES AT TERTIARY LEVEL

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Abstract: *We organised two experimental teaching designs involving web resources in two different French universities. In this paper, we describe these experiments and analyse the students' behaviours. Our aim is to observe whether the use of specific online resources favours the development of problem-solving activities.*

Keywords: Online resources, problem solving, undergraduate mathematics, students activity.

I. INTRODUCTION

The use of computer resources in the teaching of mathematics at the university in France has been institutionally promoted for several years. It led to the production of several softwares and associated teaching designs. We do not intend to make a general study of the use of computers in the undergraduate mathematics curriculum. We are only interested in a special kind of internet resources, belonging to the category of online courses. More precisely, the softwares we study have the following characteristics:

- an important part of them is dedicated to mathematics exercises;
- there is at least one classification of the exercises available (according to topic, level of difficulty, key-words…);
- they propose, for a significant number of exercises, an associated environment (it can comprise hints, correction, explanation, tools for the resolution of the exercise, score, but also corresponding courses…).

We have chosen to examine such products, that we will call "exercises' directory" because of the importance of the problem-solving activity for the learning of mathematics (Schoenfeld 1985, Castela 2000). Previous studies indicate that the work with computer leads the students to a better involvement, increases their motivation, and allows them to work at their own pace (Ruthven and Henessy 2002). But there is for the moment no evidence that the computers can help the students to develop a real problem solving activity, far from a simple drill and practice. As Crowe and Zand (2000) write: «what is undoubtedly lacking is proper evaluation of use, for there is often a serious mismatch between what the teacher intends, and what the student actually does.» (p.146). We intend here to study precisely students' behaviours, in order to answer to the following question: "Can an exercises' directory, with an appropriate associated setting, lead the students to one real problem-solving activity?". We call a "real problem-solving activity" the search by the student of a personal solution. In that process, the student governs him/herself the mobilisation of the necessary knowledge, he/she makes attempts on his/her own, even if a complete solution is not found alone. Only particular exercises can lead to such an activity. There must be no indication of method within the text, and not too many intermediate questions that split the task into elementary steps. But proposing such a text is naturally not sufficient to lead the students to a real problem solving activity. For example, if the exercise is too difficult in regard of the student's knowledge, he/she may simply remain stuck. It is then necessary to propose help, but they must be thoroughly controlled to maintain the possibility of personal search.

The results we will state stem from two experiments, conducted in two different French universities. In section II we give a synthetic description of these experiments, with the main features of the resources involved and of the teaching design, but also with two particular exercises that we retained to observe in detail the students' behaviours. In section III we describe and analyse the students' activity, using specific characteristics of it for each case. We will go back to our initial question in the conclusion (section IV). The observations presented in section III lead us to propose a balanced answer to it.

II. THE TWO EXPERIMENTS

In that section we present the two experiments that led us to the results stated in section III. These experiments are named after the softwares involved: Braise[1] and Wims[2]. Both took place in 2003, with first year students following mathematics major. We describe the two softwares we used, the associated setting and the exercises that we will examine in detail. More than a mere description, we want to emphasise here the use of the grids we elaborated. These grids (presented with more details in Cazes, Gueudet, Hersant, Vandebrouck (2004)) constitute a first step into the study of a teaching design involving such a software.

II.1 Characteristics of the web resources

Table 1 stems from the general grid of the characteristics of a courseware. In the building of this grid, we firstly used more general tools for analysis of web resources, like the one elaborated by Hu, Trigano and Crozat (2001). We did not find in the literature any grid taking into account precise didactical aspects. So we progressively added to the general grid stemming from the work quoted above more didactical categories that appeared relevant for several resources. We finally tested the grid be using it to analyse different products concerning different school and university levels. However, it will probably still evolve, at least because the products themselves evolve very quickly. For the sake of brevity, we only retain here the most salient

[1] Rationale basis of mathematics exercises, http:// tdmath.univ-rennes1.fr
[2] Web interactive multipurpose server, http://wims.unice.fr/wims, developped by Xiao Gang.

elements, for our research questions, of the two softwares we used. We do not examine their mathematical content, but only their structure, with a double point of view: the didactic structure and the technical features.

Table 1: Grid of the characteristics of the resources

	Braise	Wims
Didactic choices	Problem solving	Practising several times on similar exercises, proposed with random elements.
Public	Undergraduates	All levels
Environment of an exercise	For all exercises: courses, descriptions of methods, hints, detailed solution. Summaries of the important points	Depending on the exercise: numerical calculator, computer algebra system (CAS), graphing tools…
Organisation of learning	The students work on the exercises. They access the courses only through the exercises.	The students access to worksheets prepared by the teacher. They can also access freely and solve exercises on topics they know.
Classification of the exercises	Key words: level of difficulty, theme, task. Possibility to avoid specific difficulties.	Search by key words of the theme. The direct search can not really be done by the student; it is a tool for the teacher, to elaborate his sheets.
Random elements	No.	Random elements (numerical values, but also questions) which change at each attempt.
Kind of answers awaited	The answer must be written on a paper and then compared with the solution proposed.	Numerical value or brief mathematics expression.
Feed-back	None.	"Right" or "Wrong"
Marking	No mark	Mark from 0 to 10 for each exercise. The students are supposed to make several tries in order to improve their mark.
Record of the students' activity	Log files giving some details upon students' activity.	

These exercises' bases are very different. One crucial point is the kind of answers awaited, and thus the feed back proposed. It corresponds in fact to different didactic choices. Braise proposes mathematical problems. The answer can not be simple and thus could not be interpreted by the computer. So there is no reason to offer the possibility to write it on the computer. Wims is built to encourage the students to practice several times on similar exercises. For that reason, there is a mark intended to motivate the student to make several attempts on similar exercises to improve his/her mark. It is thus necessary that the student provides a simple answer, which can be interpreted by the computer. It does not prevent some of the exercises from being really difficult. Most of them are at least quite uncommon, because of the CAS and graphing tools proposed.

II.2 Settings associated

We present the settings associated with the use of the two softwares, using again tables stemming from a general description grid of the characteristics of a setting. We distinguish two main axes in that grid. The first is the place of the computer sessions within the teaching design; the second is the role of the teacher and the students during the computer sessions.

Table2: Place of the computer sessions within the teaching design

	Braise	Wims
Mathematical content	Sequences	Calculus
Public	First year – mathematics major	
Proportion of computer sessions	2 sessions over 24	1, of 3, sessions per week for 8 of the 12 weeks
Link between computer and traditional sessions	Synthesis session after the computer sessions.	The computer sessions take place after the corresponding course and tutorial.
Assessment	No specific assessment. A traditional paper and pencil exam.	Half of the final mark is provided by the work on the computer.

Table3: Role of the teacher/students during the computer sessions

	Braise	Wims
Written notes awaited	Logbook; a similar exercise must be written for the following session.	Nothing
Number of students on a computer	Work by pairs (17 students in the whole class).	Work by pairs (36 students in the whole class).
Possibility of work online outside the sessions	Yes	
Role of the teacher	Individual help	
Use of the log files by the teacher	None	
Other	Use of the logbooks by the teacher for the synthesis.	Preparation of worksheets before the sessions.

These settings are very different; they are in fact strongly linked with the final assessment associated. With Braise, the students prepare a traditional assessment, pencil and paper way. Thus they must be prepared to write detailed solutions and proofs. Both the software and the setting are adapted for that aim. In the Wims experiment, the work on the computer intervenes in the assessment. Thus it leads to more numerous computer sessions, and it explains the choice of no compulsory written notes during the sessions.

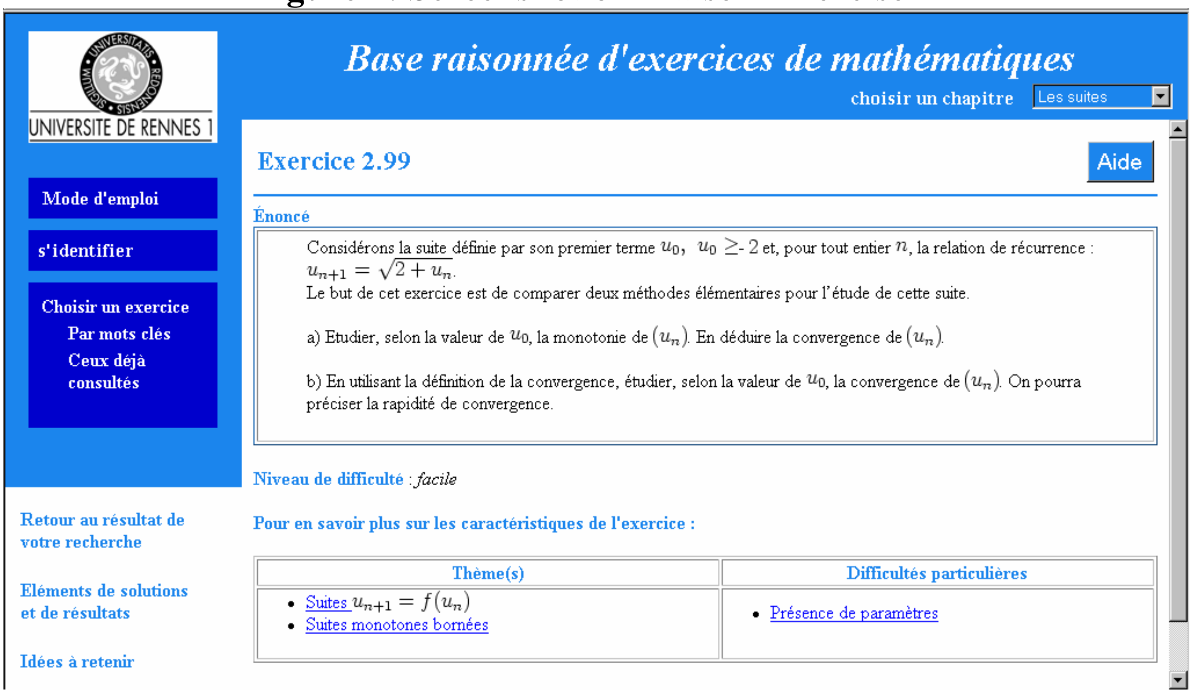
II.3 Specific exercises of each experiment

In each experiment, we will examine in detail the students' activity on a specific exercise. It is indeed necessary to study precisely the students' behaviour to observe if the work on the computer can lead to a real problem solving activity. It depends on the way they use the computer, that we will discuss in the next section; but it also depends on the exercise proposed. Do they only need to apply well-known results, or do they have to produce a personal endeavour, that requires adaptations, mobilisation of similar situations…more specific of a real problem-solving activity (Robert 1998, Robert and Rogalski 2002).

Braise

The exercise we focus on in Braise (exercise B) belongs to the theme "sequences $u_{n+1}=f(u_n)$ ", with the level "easy", and the task "determine the nature of a sequence". Figure 1 displays the corresponding screenshot.

Figure 1: Screenshot of Braise – Exercise B



In Braise, several kinds of helps are available for that exercise[3]:

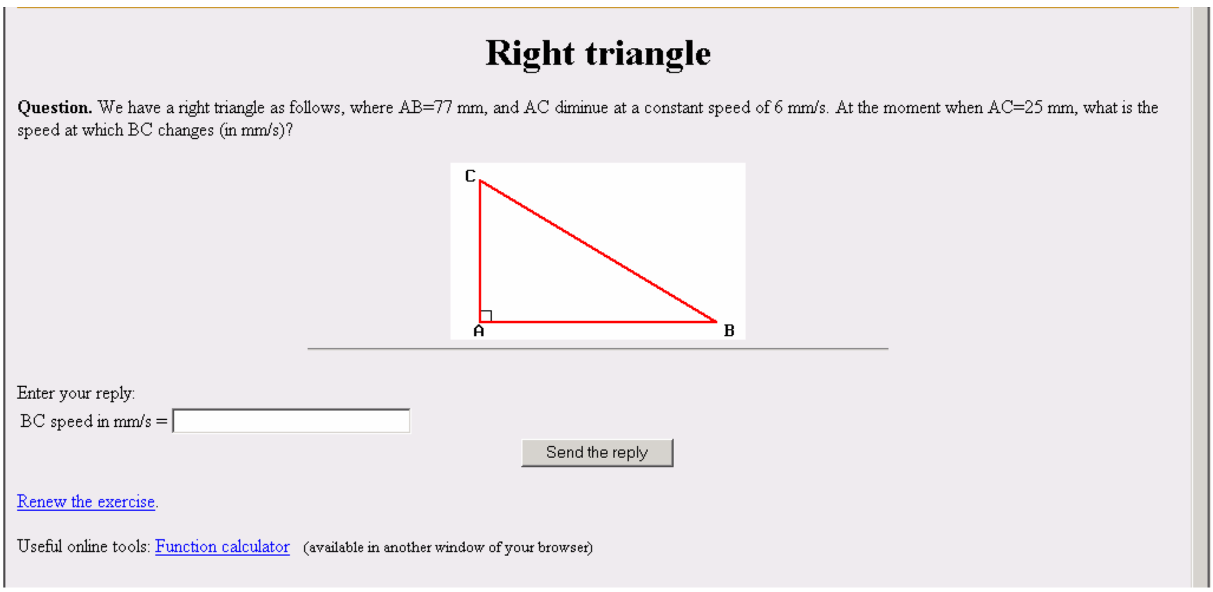
- short courses, recalling general results that can be useful for that exercise;
- description of general methods, that must be transferred by the student to that particular case;
- a graphic help, displaying of the first terms of the sequence;
- hints for each of the two questions.

All these helps are proposed simultaneously; none of them reduces the students' activity to a mere application of properties or routines. However, a detailed solution of the exercise is also available. Some of the students could fake looking for a personal solution, and only try to understand the solution proposed by the computer. The study of the log files will show if it really happens.

Wims

The exercise we focus on (exercise W) requires a mathematical modelling of a geometrical situation. It takes randomly three forms: "right triangle", "circle", or "tower". Figure 2 displays a screenshot of Wims, with the exercise in its "right triangle" form.

Figure 2: Screenshot of Wims- Exercise W



[3] *English version of the text:*

Let (u_n) be the sequence defined as follows: u_0 is given, $u_0 \geq -2$, and for all n: $u_{n+1} = \sqrt{2+un}$

We want here to compare to elementary methods to study that sequence.

a) According to u_0 study if the variation of the sequence. Deduce that (u_n) is convergent.

b) Use the definition of convergence to study, according to u_0, the convergence of (u_n). State the rate of convergence.

There is no help proposed in Wims for that exercise[4]. The students have access to a functional calculator. Even if the answer awaited is numerical, that exercise requires many personal ideas, choices and decisions of the student. He or she has in particular to adapt his or her reasoning to the three possible forms of the text. Thus that exercise is, like exercise B, likely to lead students to a real problem-solving activity.

III. STUDENTS' BEHAVIOUR

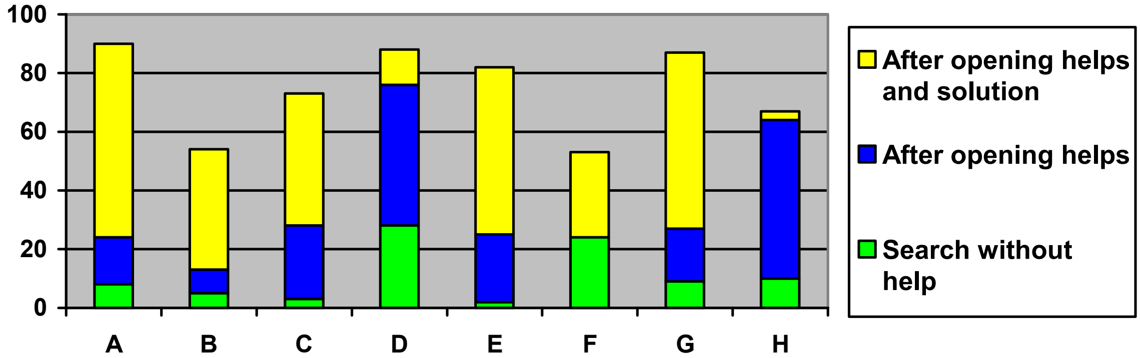
For each experiment, the software provides log files that allow us to follow the students' activity in detail. We demonstrated in the preceding section that the chosen exercises can lead the students to a real problem solving activity. We will now use the log files to examine precisely the students' behaviours, in order to determine if they really develop such an activity, and in particular if they do not use the computer to fake producing a personal solution.

Braise – Exercise B

The log files of the eight pairs of students working on Braise provide the connection time on each possible window (hints, courses, graphic help, solution…). The first thing we observe is a real involvement of the students in the task. The average time spent on the exercise is 1 hour and 13 minutes. In a traditional tutorial session in France, the time spent on such an exercise rarely exceeds half an hour (after that time the teacher usually proposes a solution[5]).

However, the students working on the computer look at the helps, and at the solution, after a quite short time. We present the corresponding numerical values on figure 3. It displays three stages of the students' work: a search without any help; after opening the "helps" window, but not the solution; and after opening the "solution" window. The times are indicated in minutes on the vertical axis.

Figure 3: Synthesis of the students' activity with Braise.



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[4] A similar exercise is proposed with a graphing calculator at the Dutch National examination 2002 (Drijvers 2004)
[5] The observations about traditional tutorials stem from a master's dissertation: Sylvie Le Merdy (2003), "Problem-solving at the university and in preparatory classes".

We observe on that graph a well-known result about the work in mathematics with a computer: the students can work at their own pace. It leads them to very different attitudes. More precisely here, we observe a prevailing attitude shared by the groups A, B, C, E and G. They open the “helps” window during the first nine minutes, and the “solution” window within the 25 following minutes. B, C and E even look at the helps during the first five minutes. It means that they open the corresponding window right after reading the text of the exercise, before any personal attempt to solve it. Some of these students could certainly find at least a partial solution without any hint. For all these students, more than a half of the time is spent after displaying the solution. The more extreme case is B: the students look at the solution after only 13 minutes, just after reading the helps.

However, the time spent after opening the solution’s window is not only dedicated to reading and understanding it. A precise study of the log files shows that all the pairs who opened quickly the solution’s window closed it at one moment. Even if they open it again later on, for these students (A, B, C, E, G), the time spent with the solution’s window open represents almost 60% of the third stage of their work. It indicates that they use the solution as a very detailed hint. According to the direct observation, it seems that they read it quickly and then try to produce a similar reasoning on their own. It is not a problem solving activity in a strict sense, but it is a real personal work of the students.

Besides, all the students wrote questions and remarks for the teacher on their logbook[6]. And they succeeded in solving a similar exercise at home.

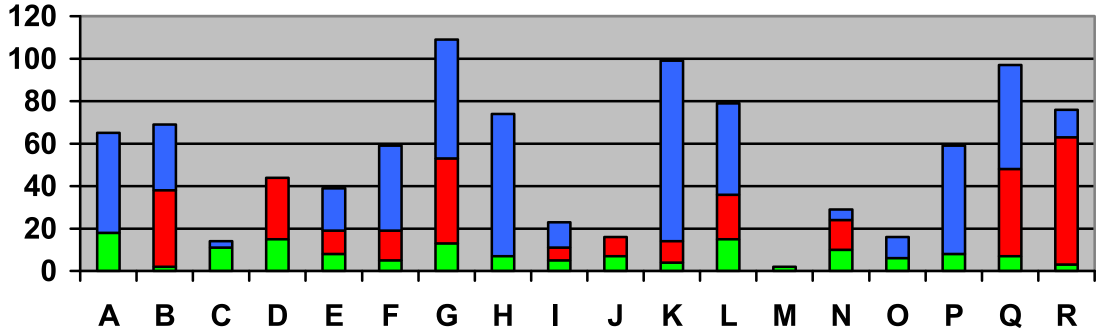
Wims - Exercise W

The figure 4 displays for the 18 pairs:

- The time spent before the first answer (*lowest zone*);
- The additional time needed to reach the maximum mark of 10; it can be zero when the first proposition of the student is correct (*middle zone*);
- The additional time, where the students go on working on the exercise after reaching the maximum mark, which can also be zero when the students do not work after reaching the mark 10 (*upper zone*).

[6] For the sake of brevity, we will not present here examples of these remarks. They will be presented in the complete paper.

Figure 4: Synthesis of the students' activity with Wims.



The numerical values confirm again the real involvement of the students (an average of 54 minutes work on the exercise, including 16 minutes of personal work outside of the organised sessions). They also show a great variety of attitudes between the different pairs of students: the total time spent on the exercise ranges from 2 min (M) to 1h 49 min (G).

The first stage, before making a first attempt of answer, is relatively short: an average of 8 min. All the students reach the maximum mark at one moment, but the time spent on that second stage changes a lot. There is no help from the computer for that exercise, and very often the teacher intervened to provide hints. He sometimes even indicated the solution. That is the reason why everyone reached the mark 10. However, after reaching it most of the pairs went on working on the exercise by themselves during a long time. Some of these were not able to reach the mark 10 again.

Besides, many pairs renew often the exercise in order to obtain their favourite version of the text (triangle, circle or tower). Perhaps some of these guess the right answer for one of the configurations without having the slightest idea of the mathematics results involved.

The final paper and pencil exam comprised the "triangle" and the "circle" version of that exercise, and a new variation of it, with a sphere. The results obtained by the students prove that most of them really understood the exercise: over 36 students, 24 succeed with the triangle and the circle, and 20 of these also succeed with the sphere.

IV. CONCLUSION

Let us go back to our initial question: do these softwares allow the students to develop a real problem solving activity? For a first kind of students, the ones who search by themselves during a long time before proposing a solution, the answer is clearly positive. But they represent only a minority of the students we observed. For all the others, the answer is not obvious. They undoubtedly develop a real mathematical activity, spending a long time working with the solution (Braise) or working on the exercice after reaching the maximal mark of 10 (Wims). They were able afterwards to solve similar problems, so they clearly learned mathematics. Can we claim that they developed a problem solving activity, even if they worked with the solution's window open, or with a solution provided by the teacher? One can answer positively, because all these students needed at one stage to produce a personal solution, adapted from the model. But that question must be discussed. It indicates the need for further studies, especially in order to produce more precise descriptions of the students' activity on one exercise after looking at the solution or at least at a correct answer.

Anyway, we observed in both experiments that the students adopted very different behaviours. It goes further than the usual observations about the possibility of work at their personal pace with a computer. The observations exposed in part III prove that very different working patterns are developed. Some students spend a long time looking for a solution by themselves (Braise) or before proposing a first answer (Wims). On the opposite, others ask very quickly for the solution (Braise) or renew often the exercise (Wims). Moreover, that flexibility in the activities choice and organisation at the exercises' scale has also been observed at the scale of a sequence. Is that flexibility a reason for the greater involvement of the students? Does it help the learning and in which way? And how can the teacher cope with that relative freedom of the students? These questions are mentioned in several research works about the use of computers. We intended to contribute to their study in further research using in particular the grids presented in part II.

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THE PROOF LANGUAGE AS A REGULATOR OF RIGOR IN PROOF, AND ITS EFFECT ON STUDENT BEHAVIOR

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Abstract: *This paper discusses the character of the language in which formal proof is set, and the difficulties for students to appreciate its exact form, and why it is needed. It shall describe the effect that these difficulties have on student attitude towards proof, and how it influences student behavior whilst generating proofs. This will be placed in a perspective of what extra demands there are in producing proofs beyond those that occur in general problem solving.*

Keywords: Language, Problem solving, Proof, Representation.

1. INTRODUCTION

There exists now a substantial amount of mathematics education literature devoted to the theme of proof. Much of this literature focuses on attitudes because there seems to be a lot of confusion in students' minds as to the identity of proof. This paper aims to delve a little into the causes of this confusion, and then to list some effects of it when students actually attempt to construct a proof.

A proof can either be an argument that shows that a conjecture is true, or an argument that is to be constructed for the demonstration of a given mathematical proposition. The prior case no doubt is more satisfactory to constructivist ideals; if you are (re-) creating your own mathematics independently of an 'authority', one never can assume a result is true before proving it. This paper, though, will only consider proof from the second, more institutionalized standpoint, as this reflects the preponderant image of proof that most students have.

The apparent artificiality in making an a-priori assumption of the truth of a statement would seem easily remedied; simply state the task in a different way, 'Is this proposition true or not'. This option is feasible sometimes, but in more cases not; the proposition often expresses something that the student would not naturally think of otherwise, in which case it would seem perverse to introduce it were it was false. In practical terms, the form of the proof cannot be avoided.

Not surprisingly, there is some disagreement about what proof is exactly, and the crux of the uncertainty seems largely to hinge on the theme of rigor. Perhaps it is the fact that proof requires formal treatment that most distinguishes proof generation from general problem solving. However, we can also possibly discriminate differences in broad motives between problem solving and doing proof. Problem solving stresses the process of solution over the actual result, and its main topics are heuristic argumentation, metacognition and control. For a proof, though, the proposal to be demonstrated usually has some integral significance in a mathematical theory or structure. Hence a task comprising a proof is regulated by mathematical needs, so that the argumentation involved may be unavoidably messy and opaque; a task set within the problem-solving agenda is usually designed to give experience in a selected technique and can be chosen to have a ‘neat’ solution. (Further, even the statement of the proposal in a proof may be not easily understood, whereas authors in problem solving usually exhort the use of task environments that should be easily comprehended, see e.g., Schoenfeld, 1985). The tools of problem solving will remain relevant to proof, but proof puts extra demands on the students, some of which we will identify in this paper.

A paper of this length on such a broad topic has to be ‘sketchy’. A longer version will be prepared for distribution at the conference. In particular, explicit illustrations will be given for some of the points raised.

2. THE PROOF LANGUAGE

Paralleling the deep philosophical argument between logicism and intuitionism (Kline, 1972, chapter 51), there is on a more mundane level a pedagogical issue on how ‘rigorous’ should presented proofs be, and to what degree do we expect our students to keep to these standards. It has been noted that even papers found in mathematics journals are far from rigorous (Hanna & Janke, 1996; Thurston, 1994). This realization has encouraged mathematics educators to regard proof as some kind of institutionally imposed system of conventions that is expected to be met when presenting a final argument. However, these ‘conventions’ are not as arbitrary as they might seem; they form a basis of a particular language that has its framework in creating a consistent checking system. In this paper, we shall refer to this as the ‘proof language’. This section sets out a rough characterization of this language.

The most obvious reason for insisting on some conventions in rigor in presentation is to make sure that your reasoning is accurate, and that your argument is in a form that it is readily checked for validity by others (Barnard, 1996). One thing to ascertain is that each implication is clearly identified; in a complicated argument couched informally in words, hidden assumptions can occur. A symbol (=>) greatly clarifies the route that the thinking has taken. It is exactly what can be accepted as an implication that can cause a degree of subjectivity. An implication rarely is done at the level of a simple logical operation, but may involve much assimilated argumentation. Such subjectivity might confuse students, and may, for example, encourage them to use the device ‘it is clear that’ too freely.

Another thing to realize is that in a complicated argument involving many different kinds of objects, it becomes very difficult to trace the relative roles and the status of the objects, and this can cause unreliable reasoning. Some constructs have to be introduced to organize the situation. One obvious measure to make is to develop labeling systems, otherwise confusion about exactly what entity you are talking about will soon arise. The notion of set occurs as a natural organizational device expressing 'what belongs to what'. For any interaction between objects there must be underlying correspondences; in the same sort of spirit of organization, sets are identified to describe the extent for which the correspondence acts, and formal relations, with functions as a special case, are defined to provide a unified mechanism to explicitly express, for each element in one set, what are the associated elements in the other set. The notions of set and formal relation are indeed fundamental in the foundations of the proof language; in particular students must obtain a more constructive handling of functions than that usually reflected in teaching. The clarification of an informal argument also has to attend to restrictions and conditions, and hence quantifiers and statements of ranges for application should appear in the presentation. Hence much of the character of the proof language can be explained simply as a 'sieve' to make sure that an argument is well organized. These precautions of course are equally valid in doing any task, but are particularly pertinent in proof because there the stress is on the argumentation rather than achieving an undisclosed result. (However, by using tasks with an undisclosed result may be pedagogically useful in convincing students of the need of the proof language when they review incorrect answers.).

It would be a mistake, though, to consider what we have dubbed 'proof language' simply as a means of organization, clarification and checking. Modern mathematical theories tend to be abstract, and they develop with the same kind of language. These theories incline to axiomatic systems, which allow the consideration of a whole class of mathematical entities that a-priori has no other identity apart from the conditions implicit in the axioms. However: *Axioms result from necessity, not from some arbitrary decree, and this reason is often misunderstood* (Artmann, 1988). Axiomatic systems certainly introduce abstraction, but they do not necessarily annihilate intuition, as many models of abstraction forwarded by mathematics educators indicate. For instance, Piaget (1973) sees intuition as being closely related to the process of formalization because *...intuition is essentially operational and the nature of operational structures is to dissociate 'form' from 'content'...* This suggests that the proof language is consonant to the way that entities are defined in abstract mathematical theories, and it is the natural medium in which to reason for such theories, but at the same time an informal interpretation can be drawn from abstract constructions that is useful for noticing particular structural features that may bear on a task strategy. On top of this, a whole new level of conceptualization takes place, where property-based notions are isolated and named (e.g., abelian, homomorphism, quotient, order, ordering…) that are eventually assimilated comfortably in casual discourse.

3. EXPLANATION AND REPRESENTATIONS

Students quite naturally become confused about the exact character and role of the proof language. In particular, students are exposed to the proof language that they are told is needed to keep a standard in rigor, yet more intuitive thinking might appear more explanatory. We treat this theme in reference to representations.

In this paper, a representation is any system perceived in the mind that imitates an identified aspect of structure evident in the task environment. The task might then be thought through ‘mentally’ via the representation. The use of representations are of critical importance in doing mathematics, as reflected in the interesting issues particularly linked to proof raised by Epp (1994), and Greeno’ s accompanying commentary. However students who have some exposure to the language of ‘strict’ proof can be resistant in using representations, regarding them as not being legitimate mathematical tools. This phenomenon can be particularly strong for the use of diagrams (Eisenberg & Dreyfus, 1991). This confusion may be allayed by some explanation by the teacher to the student. Using representations means a two step process in argumentation (though in cases where the representation is ‘institutionalized’, such as graphs for well-behaved real functions, these steps may be taken simultaneously). After the representation has been used (first step), the second step will take one out of two paths. First, how the representation accommodates the result may be examined structurally such that an argument may be traced analogously and checked in the task environment. In this case, the representation becomes a catalyst; there will be no trace of it in the presentation. Second, there is something in the structure specific to the representation that allowed the argumentation in the representation to work. In this case, explicit formal relations have to be constructed to integrate the representation to the task environment. Whatever path, undertaking it might be difficult for students to effect. The relative value put on obtaining a crucial idea informally through a representation against presenting it in the proof language seems to change with personal opinion, as a wealth of statements extant from mathematicians indicate. Mathematics education researchers often take a positive view towards the use of representations as these are regarded to ‘explain’ the proof better. This view seems to have deflected attention from the problems that may occur in converting mental frameworks into more formal systems.

The examples of proofs considered in educational papers are often given in terms of representations that make the proposition seem obvious in that framework. Despite the claims of some authors, though, representations are not fully explanatory. One point is that students can be concerned by how to realize the representation in the first place. Also, the ‘obviousness’ of the proposition due to extra structural factors in the representation does not give explanation in the system in which the proposition was first couched. Another point is that the extra structure brought in by a representation often needs messy conversion in the ‘proof language’ if rigor is demanded; on the other hand, Harel & Sowder (1998) point out the difficulty in arguing about what seems obvious. Although representation is an essential tool in mathematics thinking, these factors have to be remembered. We put forward that an explanation for a proof is better described in terms of a global, structurally based image of the system determined by the task environment, in which every step of argument taken is referred to this image. Sometimes it is not feasible to achieve this.

4. PROBLEMS THAT STUDENTS HAVE IN 'DOING' PROOF

Research suggests that many students are poorly equipped in producing proofs. In this section we list some factors that may contribute to this phenomenon. Some are well documented in the literature, some are not so well represented. Right through the list the influence of the proof language will be evident, but some factors are very close to issues that arise generally for problem solving. The list is broken down into three classes for organizational reasons. The list, also, is far from being complete.

4.1 The effect of affect, and types of thinking

Students' perceptions of proof naturally would affect their approach in producing proofs. Also, an individual's natural ways of thinking may be more suited to some aspects of proof than others.

a) Students may feel a lack of motivation in making proof; the 'exciting' thing about mathematics is getting results, whereas the proof format denies this. A new aesthetic towards mathematics has to be formed, i.e., an appreciation must be fostered to the character of the argument used. This, indeed, may be one of the main trends that marks Advanced Mathematical Thinking.

b) It has been noted that students can regard proofs as having a procedural character, in the sense that a proof has to follow *a sequence of steps that one performs* (Moore, 1994). This may parallel the student belief that a task has only one solution, as reported often in the problem solving literature. If the student feels that his / her argumentation has to follow a 'predestined' path, then the flexibility to reason will be restricted, in particular for the more constructive aspects of proof. A significant contribution to this belief may be the rigidity that a presented 'deductive' argument might seem to display.

c) As previously discussed, another belief that students often have is that employing informal descriptions and representations should be avoided in generating a proof. This constitutes a huge constraint in thinking. It should be explained to the students that a strategy can be sought in any intuitive way, and then be converted into the proof language. The ease in achieving this depends very much in incorporating encapsulated objects in the glossary of your informal language, in order to keep it consonant to the formal definitions.

d) A belief of students sometimes reported is that proof production depends only on knowledge (e.g., Harel & Sowder, 1998). This belief may be partly explained by proof often being set within a mathematical theory currently studied. There is a momentum set up where previous results will be employed in the proof. Taking advantage of this requires appreciation of the defined objects in the theory, of the significance of previous results and their proofs, and of how this knowledge is inter-connected. It is significant, then, that many undergraduates show poor judgement in selecting what they should remember for further use (Weber, 2002). Further, they have problems in accessing the knowledge that they do retain. Finally, the expectation that 'importing' knowledge, even taken in a broad sense, will determine the proof in itself ignores the unique structure that every proof possesses.

e) There are many models characterizing general types of style of thinking in doing mathematics. Perhaps the most elaborate directed specifically to proof is by Harel & Sowder (1998). Other papers have classified types of students by their dominant thinking behaviors (e.g., Duffin & Simpson, 2002). The demands in coping with proof, or with particular forms of proof, will vary with the type of student. Important in characterizing types of thinking is the inclusion of a category where argumentation is guided by a structural sense; this is expressed as 'transformational reasoning' by Simon (1996). It is contrasted with deductive and inductive reasoning. The identity of these types of reason, though, would seem somewhat blurred. For instance, as said before, there is a subjectivity in making implications in the proof language, and so in deductive reasoning. Sometimes researchers give an impression that deductive reasoning is just to do with logic; this disregards the fact that, for any stage in a proof, there are many choices, all that are logically valid. Strategy is still required.

4.2 Students' problems in conceiving formal mathematics

The proof language is closely allied to how abstract mathematics theories are expressed, so producing proof requires good understanding of formal mathematics.

a) Many students have difficulty in understanding and handling fundamental concepts such as set and function, and have serious confusions in the basic roles that different objects and statements take; for example, it is not rare for a student to mistake a definition for a proposal to be proved. The proof mode seems to encourage wildly uncontrolled behavior in making implications (Vinner, 1997). The cause of this seems to be an abandoning of sense making once students are faced with the proof language. This will be particularly true for handling symbolic systems.

b) Generally, students are not well versed in the 'fundamentals'. Especially they do not have a good sense in sets and functions; they cannot manipulate them well mentally, nor can they build up symbolic frameworks in order to examine properties of sets and functions for a more analytic treatment. As sets and functions are the basic constructions supporting the proof language, this language will remain foreign to them: *The language is not alive except to those who use it* (Thurston, 1994). Not only should set and function have more attention pedagogically, but their role in constructive and analytical aspects in making proof should be explicitly illustrated to students.

c) A theoretical setting means that the entities that occur in the task environment will be formally defined. Moore (1994) points out students' problems in coordinating proof with definitions when there are associated concept images. The form of definition may not reflect the way that a student understands the underlying concept, and further the associated concept images would lack the language that the definition offers. This means that if a student persists in thinking in terms of concept images only, the resulting argumentation may be logically inconsistent to what can be deduced from the definitions, may not be easily expressed with an accepted level of clarity, and finally may be relatively limited.

d) Some papers note that students do not know how to begin proofs even in simple cases (Moore, 1994; Weber, 2002). The complexity that a proof may have, then, does not seem to explain all the difficulties experienced by students concerning proof. However, characterizing ‘simple tasks’ should be taken with caution. The situation envisioned is one based on a single definition where the proof ‘drops out’ in making a few operations much suggested within the immediate structure implied by the definition. The problem largely seems to lie in students’ inability to ‘unwrap’ definitions and statements (Seldens, 1995; Weber, 2002). Students have difficulty in evoking at will the underlying structure that would expose the handles for first engaging the proof. However, perhaps one should not make out that the operations taken are automatic or procedural, as might be suggested by the phrase ‘unwrap definitions and push symbols’, sometimes used by mathematicians. Actions guided by a sense of structure are ‘natural’, but this is not the same as being procedural.

4.3 General tools and sources for generating proofs

As students gain experience of proof, one would hope that their skills in producing further proofs will be enhanced. To which extent should one expect students to pick up these skills on their own, or do they need pedagogical assistance?

a) Building up a structural understanding of a system is quite often essential in observing a certain feature that is significant in forming a strategy for a proof. The process of identifying such a feature is not easily characterized, but it requires reflection and sometimes becomes evident only after some tentative exploration of the structure and how it impinges on the desired result. The demands on students in achieving this can be high, yet it is disappointing that even in cases where the feature would seem to be obviously apparent the linkage is not made. This theme is relevant for all problem solving and is well captured by Mason’s work on ‘shifts of attention’ and change of focus, (e.g., Mason, 1989).

b) Techniques can accrue both from mathematical theories and from broad practices that transcend theoretical boundaries. We call the latter ‘proof techniques’. Applications of techniques may be easily ‘missed’ if they come in unfamiliar contexts. (A factor in this is that students often associate universal notions too closely with particular theories, and as a result their access to the underlying ideas are restricted.) An appreciation of an abstractly described situation where a technique is applicable may act as a cue, see Mamona-Downs (2002). An important difficulty with proof techniques is that they are not often taught; the student has to pick them up from previously encountered proofs (Mamona-Downs & Downs, 2004). The few proof techniques that do tend to be taught are those that act as overall logical organizers for an argument, such as induction, proof by contrapositive or by contradiction; all these constructions cause problems to many students.

c) Although proofs can be complicated and involve several major stages, tasks set to students involving proof production tend to be either relatively straightforward, or if not, hints are given in order to break down the proof. It is a common phenomenon for students not to use the hints, and because of this they usually do not progress much. The hints are usually in the proof language so that when students are trying to make some sense out of the task environment, the hints may seem to constitute an extra load in accommodating them in an informal framework. (Hints are not necessarily explicitly stated; the very form of the proposition to be proved might suggest the line of analysis to take. Another point is that proof tends to lay down exactly what you need for the proof, and nothing more; this means that one can attempt to identify roles for each entity involved.)

d) Mathematics educators often advocate that students should be placed in pedagogical situations that reflect as well as possible the professional mathematician's practices. One thing that seems to be overlooked here is that the mathematician spends a lot of his time reading, and this will mostly be material in the proof language. To become expert in producing proof needs extensive practice in reading proofs, and this reading should be far from passive. Not only notice should be put on the result, but also on an overall idea why the proof works, including any special device that may have potential application elsewhere. Though it may sound procedural, a teaching strategy that might be useful is to let students to read presented proofs until they feel that they understand them, and then challenge students to reproduce them with the presentation withdrawn. This exercise will involve far more than memory!

5. FINAL COMMENTS

Perusing the points of difficulty listed above, there would seem to be a complex mix of issues concerning general problem solving and dealing with what we have called the 'proof language'. The two rather separate traditions in mathematics education dubbed as 'problem solving' and 'proof', respectively, perhaps have not served well to treat this mixture. In this regard, we may remark how often we refer also to the word 'structure' in the list. Perhaps the notion of structure, as a coherent set of relationships dominating a system, may be regarded as a bridge between the problem solving and proof perspectives, in how the relationships are thought more as informal correspondences in the prior case, whereas they become formal relations in the latter. Structure is then the key in moving between the proof language and an environment more condusive to mental processing.

In some universities we have seen the introduction of special courses in problem-solving and in proof that tend to be very different in character. Problem-solving courses may take the line of the paradigm set by Schoenfeld (1985), and hence follow well-known principles endorsed by educators. On the other hand, courses in proof tend to stress elementary propositional and predicate logic as a basis to talk about proof techniques (for a typical textbook, see, e.g., Garnier & Taylor, 1996). Proof techniques in these courses are about explaining informally the choices the student has in approaching a proof (by type) in logical terms. Presenting this basic logical structure certainly would seem to constitute an important pedagogical undertaking, so it is surprising that it has not caught much attention from the mathematics education research community as yet.

The last two paragraphs suggest a situation where three broad issues concerning proof (aspects of problem solving, affect due to the proof language, logical proof techniques) are treated by completely distinct channels; this state of affairs cannot be satisfactory, and deserves fuller discussion.

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IS THERE A LIMIT IN THE DERIVATIVE? – EXPLORING STUDENTS' UNDERSTANDING OF THE LIMIT OF THE DIFFERENCE QUOTIENT

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Abstract: *Task-based interviews to five postsecondary students were arranged to investigate the students' understanding of the limit of the difference quotient (LDQ). The students' procedural knowledge was analysed using the APOS theory and conceptual knowledge by examining what kind of representations they had of the limiting process and how these were connected to LDQ. It was found that students had two kinds of connections: they could change from one representation to other or they could explain one representation with other. Among the students, all combinations of good or poor procedural and conceptual knowledge of LDQ were found.*

Keywords: derivative, representation, connections, APOS theory, conceptual knowledge, procedural knowledge.

INTRODUCTION

The limit is a central feature of the derivative. But in carrying out algorithms of calculating the limit of the difference quotient (LDQ) there is not much limiting. If there really is a limit in the derivative for a person, one should have some kind of representation of the limiting process. This is known to be difficult for students. In Orton's (1983, p. 236) study the students scored weakest on items of 'differentiation as a limit' and 'use of δ-symbolism'. In these and other items involving the limit, students made a lot of structural errors (ibid., p. 237-240).

In Zandieh's (2000) framework for understanding the derivative, special attention is given to the limiting process inherent in the derivative. In addition to the formal limiting process, Zandieh (2000) also considers limiting processes of slopes of secants, rate of change and average velocity. In her framework, the limit in these different representation contexts can be used as a process or as a pseudo-object. For example, for a student the process in a graphical context may be secants converting to tangents and the object may be the slope of the tangent line at the point. The object is called a pseudo-object because it does not necessarily include an internal structure of the limiting process for the student. In Zandieh's case studies of nine calculus students, the students could often describe the limit as a pseudo-object but considering the limiting process was more difficult (ibid. p. 122-123).

In this paper I study what kind of representations students have of the limiting process inherent in the derivative and how they are connected to LDQ. Analysis of the other representations is reported in Hähkiöniemi (2004) and in Hähkiöniemi (2005).

USING AND CONNECTING REPRESENTATIONS

Traditionally, conceptual knowledge is conceived as knowledge which is connected to other pieces of knowledge, and the holder of the knowledge also recognizes the connection (Hiebert & Lefevre,1986). Procedural knowledge consists of the formal language of mathematics and of rules, algorithms, and procedures used to solve mathematical tasks (ibid.). Slightly modernized characterization of the two types of knowledge can be found in Haapasalo and Kadijevich (2000). In many cases procedural knowledge deals with using some representation and conceptual knowledge about connections from that representation to other representations. In this study the focus is on the use of LDQ and connections from that to limiting representations.

The APOS theory (Asiala & al., 1997) is used to analyse the procedural knowledge of representations. According to the APOS theory (ibid., p. 400.), an action is a physical or a mental transformation of objects to obtain an other object. The action is a reaction to external stimuli and it is carried out step by step without individual's conscious control of the action. When the individual reflects on the action and gets a conscious control of it, the action is interiorized to a process and the individual can describe the steps in the transformation without necessarily doing them. The process becomes encapsulated as an object when the individual becomes aware of the totality of the process and is able to perform new actions to it. A schema is a collection of processes, objects and other schemas.

Asiala et al. (1997) also describe graphical and analytical learning paths to the derivative. Roughly, the graphical path consists of the action of calculating the slope of secant, interiorizing these actions to a process as the two points on the graph get closer and closer, and producing the slope of the tangent as a resulting object (ibid., p. 426). The analytical path consists of the action of calculating the average rate of change, interiorizing these actions to a process as the interval gets smaller and smaller, and producing the instant rate of change as a resulting object (ibid., p. 426).

Clark et al. (1997) noticed that the APOS theory is insufficient to describe students' understanding of the chain rule, because with different differentiation rules students may be on different levels of APOS. Also Hähkiöniemi (2004) noticed that with different representations of the derivative, students may be on different levels. To overcome this, Clark et al. (1997) developed a three-staged framework for analysing the schema development. At the intra stage, a student focuses on a single item isolated from other items, at the inter stage he or she recognizes relationships between different items, and at the trans stage the coherent structure of relationships is structured (ibid., p. 353-354). For example, at the intra stage a student may have a collection of differentiation rules, at the inter stage he or she recognizes that in some way they are related, and at the trans stage he or she considers those rules as special cases of the chain rule (ibid., p. 354). Also McDonald et al. (2000) found that students were on different levels of APOS with different representations sequence. They also applied framework of Clark et al. (1997) and found that it was hard to students to connect these different representations and develop their schema to the trans stage.

The framework of Clark et al. (1997) focuses on the development of connections between representations and thus it fits for the purpose of analysing conceptual knowledge. Also Goldin and Kaput (1996) have developed a classification of different connections. According to them, two external representations may not be physically linked but they may be linked internally in the mind of a person who produced or read them (ibid., p. 416). The link is weak if the individual is able to predict, identify, or produce the counterpart of the given external representation (ibid., p. 416). The link is strong when given an action to one of the external representations, the individual is able to predict, identify, or produce the result of the corresponding action on its external counterpart (ibid., p. 416). Other comparisons of APOS theory and the framework of Clark et al. (1997) to other theories can be found in Meel (2003).

METHODOLOGY

The interviewed students attended a teaching period carried out by the author in the autumn of 2003 as a part of a Finnish grade 11 course Differential calculus 1. The teaching period consisted of five first lessons on the subject of the derivative. Different representations and open problem solving were emphasized in teaching. To introduce the derivative concept, the students were given the following problem: How to determine the instant rate of change at a certain point? Different solutions were discussed with the class. One estimation was to draw a tangent at the point and calculate the slope of the tangent. Another way was to calculate the average rates of changes over diminishing intervals and to estimate what number they approach. The average rate of change was noticed to be slopes of corresponding secants, and they were also called difference quotients. Finally, the limit was determined algebraically and the derivative of the function *f* at a point *a* was defined as $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. It should be emphasized that limit was discussed in the course without $\varepsilon-\delta$-definition.

Research questions

This study explores students' use of LDQ and other limiting processes when solving problems. To analyze what kind of representations the students used and how the representations constitute procedural and conceptual knowledge, the following research questions were set:

1. How do the students use their representation of LDQ?
2. What kind of representations do the students use of the limiting process?
3. How do the students connect the representation of the limiting process to LDQ?

Data collection

The data were collected by task-based interviews to five students who attended the course. Interviews of the subjects Tommi and Niina were carried out right after the teaching period. Samuel was interviewed one lesson after, Susanna three and Daniel five lessons after the teaching period. During that time the teacher of the course continued with the concept of the derivative function and with differentiation rules. Students' previous success on mathematics could be roughly classified so that Niina's and Susanna's success was weak, Tommi's and Samuel's was average and Daniel's was good. In the about 45-minute interviews the tasks discussed in this paper were:

3. Estimate as accurately as possible the value of the derivative of the function $f(x) = 2^x$ at the point $x = 1$.

4. a) Interpret from the figure (Fig. 1) what the quotient $\frac{f(1+h) - f(1)}{h}$ means.

4. b) Interpret from the figure (Fig. 1) what the limit $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ means.

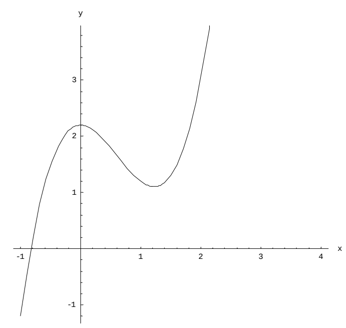
Figure 1: The figure in task 4

Task 3 was chosen because students cannot solve it using LDQ, but at this stage of their learning process, they cannot know this and they probably try to use it. This will reveal aspects of their reasoning while they describe what they would do if they could. This indicates their understanding of the procedure better than if they would only carry out the procedure successfully. For this function, they also need some other method to estimate the value of the derivative. Among these methods they may use those which include limiting processes. Task 4 was designed to explore what kind of limiting processes a student may use and how students connect these with symbolism. The students had not yet faced this form of the difference quotient. Thus they could not only recall what they had seen but they had to reason.

Data analysis

In the students' whole interviews the situations where they used LDQ or any kind of limiting process were located. From these it was analysed how the students used these representations and how they were connected to other representations. This analysis was reflected on the APOS theory so that students' uses of LDQ and limiting representations were classified actions or processes. Based on the analysis of connections, the students were classified to stages of Clark et al. (1997).

In the analysis there are two levels of interpretations: observations of the student doing something and interpretations of these observations. The observations are more reliable than the interpretations. Quotes from interviews are presented, so that reliability may be controlled. The transcripts are translated from Finnish. In the transcripts, "- -" means that the text is snipped and "[]" that one word was not audible.

RESULTS

Niina's understanding

At task 3 Niina first wondered whether the derivative could be estimated by the slope of the tangent but was not sure about it. It seems that she was thinking of LDQ as a method to calculate the slope which she knew to be the derivative:

> I would look at that certain formula. - - That, by which you calculate it. Or those cancelling out. - - I mean that [] difference quotient. - - *(Interviewer gives the formula.)* - - You would substitute the point there.

It seems that Niina's representation of LDQ was a recollection of the formula in the textbook. When the formula was given to her, she could not figure out how to use it, but she knew some actions which should be done, such as cancelling out and substituting the point into the formula. So Niina was at the action level with LDQ.

Niina used representations of the limiting at other tasks when she considered using average velocity to estimate the instant velocity and when she used local straightness to explaining why the derivative is zero. But she did not show any connections from these representations to LDQ. Thus Niina was at the intra stage.

Susanna's understanding

At task 3 Susanna first tried to use differentiation rules and then drew the graph and estimated the derivate as the slope of the tangent. She thought that it was not very accurate. So she was asked if she could figure out a more accurate estimation:

Susanna:	Calculate those limits from both sides of the two. - - I mean one, from both sides of one. For instance, 0.5 and 1.5 *(marks the points at the x-axis)*. And then draw *(marks the points to the graph and adjusts ruler as secant)*. You can't have it very accurately with a drawing like this. - -
Interviewer:	Would you come up with something which would give it more accurately?
Susanna:	Well If you'd take [] the limes, and then x approaches to one. Over there the expression *(writes* $\lim_{x \to 1} f(x) = 2^x$*)*. (Pause.) Then you can substitute it directly over there, substitute the one and then it would be the two.

Obviously, Susanna thought that the 'limes-formula' she used was LDQ but she remembered it incorrectly. Yet she knew that there is an algorithm which can be used to determine the derivative at a point, and later she also indicated that this algorithm gives the exact value of the derivative. So Susanna was at the action level with LDQ.

Susanna also used the representation of secants approaching the tangent so that the common points of the secant and the graph are approaching from both sides to one. This representation did not help Susanna to find a better estimation because the representation is graphic like the tangent. It seems that Susanna was at the action level with the limiting representation. From her image of limiting, Susanna changed to LDQ. It seems that She knew that the limiting represent the same thing as the formula of LDQ. But she did not understand why it is so, which could have enabled her, for example, to notice that there should be a slope in the formula of LDQ. Thus Susanna was at the inter stage.

Tommi's understanding

Tommi seemed to be able to describe the phases of calculating LDQ and to perform it fluently. Thus he had interiorized LDQ to a process. This was demonstrated when he tried to use LDQ at task 3:

> You could calculate it very accurately, I suppose. - - *(Writes* $f'(1)=\dfrac{f(x)-f(1)}{x-1}$*.)* Lets put that *(adds* $\lim\limits_{x\to 1}$ *to the front of the difference quotient).* *x* approaches one. *(Adds* $=\lim\limits_{x\to 1}\dfrac{2^x-2}{x-1}$*.)* How could you cancel out then? - - That's bad. (Pause.) Then you could, of course, start to, like, approach from both sides, like. But I can't remember that, because I have memorized only that formula, trusted that I would manage with it.

In this problem-solving situation Tommi changed from the representation of LDQ to the representation of approaching from both sides. After other estimations, he still wondered how to use LDQ or approaching from both sides. When asked, he explained what he meant by the approaching:

> You could calculate the average rate of change of the function, for example, at points 1.1 and 0.9 and continue to approach 1. Finally, it would become very close to that correct one. – – I don't remember at all how it's calculated.

Tommi mentioned calculating the average rate of change but could not figure out how to use this idea. Thus he was at the action level with the limiting representation. It seems that Tommi associated the symbolic process of LDQ to the action of limiting and that he knew that these representations represent the same thing. Still he did not understand how they are connected and did not use one to explain the other. This was demonstrated also in his solution to task 4b:

> That's almost like the derivative, but there *(points to the denominator)* should probably be minus something, minus one. It could be the derivative of the function at the point one. - - How could you then get the h out of there, the denominator, it goes to zero, that's a bad thing.

It seems that Tommi interpreted the formula as the derivative at the point 1 because it resembles the formula of LDQ ($f'(1)=\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}$). He suggested that there should be $h-1$ in the dominator like in the formula he knows. He also started to wonder how the cancelling out should proceed. So his interpretation was very procedural. When the interviewer guided him to consider the special case where $h = 0.5$ and the changes in *y* and *x*, he was able to interpret the quotient $\dfrac{f(1+0.5)-1.2}{0.5}$ as an average rate of change, and drew the corresponding secant line. Thus Tommi could interpret correctly what the quotient meant in the graph but when the limit was considered, he had problems:

> If it would be, for example, 0.1. Then over here would come [] 1.1 []. It would be little smaller than that *(points to the graph close to the point 1). - - (Uses ruler to find the corresponding point at the y-axis to the point 1.1. Writes* $\dfrac{f(1.1)-1.2}{0.1}=\dfrac{1.09-1.2}{0.1}=-1.1$*.)* It could be the average rate of change from somewhere very close to zero *(points to the graph close to 0)* to there - - to the point one *(points the graph at 1 and sketches a secant through points 0.1 and 1 on the graph).*

Tommi knew that the *h* is approaching zero and he was able to produce a numeric "two-step" ($h = 0.5$, $h = 0.1$) representation of this approaching but he became confused when interpreting what the latter step meant in the graph. This shows that at least in this situation Tommi did not understand why or how the representations of LDQ and the average rate of change were related. Thus Tommi was at the inter stage.

Daniel's understanding

Daniel did not mention the difference quotient at the interview before he was solving task 4a:

> Let the *h* be also one. - - Then here *h* would be the distance *(points to x-axis at [1, 2]).* (Pause.) From that value *(points to the graph at 2)* we subtract that value *(points to the graph at 1),* so it would be this interval, difference of these values *(points to y-axis at [1.2, 3.2]).* So that divided by the lower part *(points to x-axis at [1, 2]).* How does this go? I assumed that this would, of course, be connected to this kind of line *(sketches the corresponding secant in the air).* Oh yeah, is this then? What's the difference quotient? This could be quite close to the difference quotient *(sketches the secant).* This defines also the tangent. I'm not quite sure, don't remember if the formula of the difference quotient was just like this. If it was it, then it would be the slope of that line *(sketches to the secant).* Yes, it comes from here, too. This distance *(points to the y-axis at [1.2, 3.2])* divided by this *(points to x-axis at [1, 2]). - - (Draws the secant.)* So it would be the slope of that line, that's like, how to say it, average derivative at that interval.

It seems that Daniel explained the formula with the division of the *y*- and *x*-intervals. From this he changed to the secant and then to the difference quotient. At this point he mentioned also the tangent, but he probably meant the secant. From the difference quotient he changed to the slope of the secant. Thus Daniel coordinated the slope, the difference quotient and the unknown quotient through the change in *y* divided by the change in *x*. Finally, he gave his own verbalization "average derivative" to the slope of the secant. It is noteworthy that Daniel did not explain the details of the difference quotient or the unknown quotient but only the main idea of $\Delta y/\Delta x$. After that he was able to interpret also what the limit meant at task 4b:

> What does the limit mean? When *h* tends to zero. Obviously the derivative at this point one *(points to the x-axis at 1)* is wanted here. Because the *h* is this distance and if *h* tends to zero, so h would be zero here *(points to the graph at 1).* So it would be to this point, you would get the tangent here *(draws the tangent),* which slope would come out from that formula *(points to the formula at the task).* So you would get the derivative at that point.

It seems that Daniel considered some kind of limiting which results the tangent at the point in question. On the basis of his reasoning at task 4a, this limiting might include secants approaching the tangent. It seems that Daniel understood the limiting as a process because he considered it so fluently. He also used this limiting representation to explain the unknown formula and finally mentioned that the formula would give the slope of the tangent. Even though Daniel did not mention LDQ here, I interpret that he was at the trans stage because he had coordinated the limiting with an unknown form of LDQ.

Despite of his good interpretation, he did not use LDQ to calculate the derivative during the whole interview. After the previous successful interpretation, he was asked if he could calculate the derivative at task 3 based the interpretation. He mentioned the "difference quotient system" but was not able to even begin to use it. Thus Daniel was at the action level with LDQ.

Samuel's understanding

At the interview Samuel's first method to estimate the derivative at task 3 was the following:

> *(Writes* $Df(x) = \frac{f(x) - f(1)}{x-1} = \frac{2^x - 2}{x-1}$*.)* Now you can't substitute one here *(points to x at numerator and denominator)*, because it would be zero here. - - You should find some common factor from there *(points to numerator)*. - - If you could find a common factor from here and the other factor would be *x* minus one, then you could cancel out. - - Then you would substitute one to that what's left.

Samuel knew how to use LDQ, and he could describe the phases of the procedure although his notations were insufficient. Thus he had interiorized LDQ to a process. Although the cancelling out did not succeed, he was able to estimate the derivative. He calculated the difference quotients over the intervals [0.9, 1], [0.99, 1] and [0.999, 1] and estimated that "it approaches to 1.4". After that he used the tangent to explain LDQ and secants approaching the tangent to explain the difference quotient:

Interviewer:	What do these *(difference quotients over intervals)* tell about the function? If this *(LDQ)* is the derivative and these aren't quite the derivative, what do they mean?
Samuel:	*(Draws a graph and a tangent.)* It would really be that. *(Draws three secants approaching the tangent.)* They constantly approach the correct derivative.
Interviewer:	Ok. Ok. Do you have more to say about that?
Samuel:	No, or well, that this is because you can't substitute one here, because it would be zero here, but you can put it however close to mm close to one, but not still one, then there will be no zero and you can calculate this, and that is why it approaches.

Samuel used two representations of limiting in coordination. Obviously, he was at the process level with these limiting representations. He also used the difference quotients over diminishing intervals to explain the formula of LDQ and thus understood how these representations are related to the symbolic representation of LDQ. Thereby, he seemed to be at the trans stage. He used these same representations at task 4a:

> If it was, for instance, 0.2, then this would be 1.2 *(points to 1 + h at the formula)*. - - This is [] or that it is at point one. [] Then if it is 1.2 here. (Pause.) Well yeah. If this was the derivative *(draws a tangent to point 1)*. Then. Mm. If it is 1.2, then it is quite close to 1. - - If we assume that it is 0.2. Then this is 1.2 *(points to 1 + h at the formula)*. Then this *(points h at the denominator)* is 1.2 minus 1 that is 0.2 that is *h*. It is not the derivative, but it is something which. It can be anything, it can be also negative. Well it is something which passes this and is somewhere around this correct derivative *(draws two secants)*. It can [] also lot of if you add a large number to this. - - Something as in the previous task they approach the correct derivative.

Again Samuel used the tangent line to represent the derivative at a point. His key idea to the solution seemed to be understanding that $h = 1 + h - 1$. When he figured this out, he seemed to see the quotient $\frac{f(1+h)-f(1)}{h}$ as $\frac{f(1.2)-f(1)}{1.2-1}$, which was the difference quotient to him. In this key idea he seemed to explain the formula (LDQ) with the numeric representation of the (difference) quotient over a diminishing interval. He also explained that the unknown quotient depending on the value of h means the secant line and that the limit means that they approach the correct derivative, which is the tangent to Samuel. This supports the case that he was at the trans stage.

DISCUSSION

According to the analysis of the student use of LDQ based on the APOS theory, Niina, Susanna and Daniel were at the action level but Tommi and Samuel had interiorized LDQ to a process. All the students also used some kind of limiting representation. Niina used the average velocity and local straightness, Susanna used secants approaching tangents, Tommi used the average rate of change over diminishing interval, Daniel used secants approaching the tangent and Samuel used difference quotients over the diminishing interval with corresponding secants converting to the tangent. In the analysis based on the APOS theory, we found that only Daniel and Samuel seemed to be at the process level with limiting. Other students were at the action level although Niina discussed her limiting representations very shortly. Thus, the students were on different levels with different representations.

However, connections between these representations are the most important thing. It was found that Niina used no connection. Susanna and Tommi changed between their representations but did not understand how they are connected. Daniel and Samuel could explain one representation with another and thus seemed to understand why they are connected. Only Daniel and Samuel seemed to have some underlying structure for limiting. Thus, using the framework developed by Clark et al. (1997), it might be interpreted that Niina was at the intra stage, Susanna and Tommi at the inter stage and Daniel and Samuel at the trans stage of their schema development of LDQ. The main characteristics of inter and trans stages correspond, respectively, to Goldin and Kaput's (1996, p. 416) weak and strong links between two representations because at the trans stage the student should be able to trace changes in one representation to the other. It seems that students' ability to make connections between representations is the main characteristic of their understanding of LDQ. Moreover, the connections which consist of explaining one representation with the other seem to be crucial. For example, a person with fully developed understanding of LDQ should be able to use it to explain secants converting to the tangent, the average rate of change tending to the instant rate of change, local straightness, different forms of LDQ, and other representations of limiting.

These results suggest that regardless of how skilfully a student may be able to use LDQ, it may be that he or she has a weak representation of the limiting inherent in LDQ (cf. Tommi's case). Thus, if we want students to learn also limiting representations, they should be explicitly discussed in teaching, for example, in problems like task 3 where the limiting is needed. It may be also that some students first learn the powerful representations of the limiting related to LDQ and the use of LDQ only afterwards (cf. Daniel's case). In other words, if procedural knowledge is considered as the use of LDQ and conceptual knowledge as connections from LDQ to limiting representations, some students may first learn the former and other students the latter. These two ways correspond to Haapasalo & Kadijevich's (2000, p. 147-154) developmental and educational approaches. In line with this, it was found that Tommi had a lot of procedural and little of conceptual knowledge, but Daniel had it the other way round. And because Niina and Susanna had little of both and Samuel a lot of both kinds of knowledge, it is possible to have every combination of good or poor procedural and conceptual knowledge.

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CHARACTERISING MATHEMATICAL REASONING: STUDIES WITH THE WASON SELECTION TASK

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Abstract: *This paper analyses the nature of mathematical cognition with reference to the recently developed dual process theory account of reasoning. We briefly summarise dual process theory, and then present evidence from a study where mathematics students, mathematics staff and history students were asked to solve the Wason selection task, a standard logic question from the psychology literature. The mathematicians gave a dramatically different range of answers to the non-mathematicians. Using interview data from the same study, we suggest that one of the major differences between mathematical and non-mathematical thought is the ability, or willingness, to use System 2 processes whilst reasoning.*

Key Words: dual process theory, implication, logic, mathematical cognition, reasoning, Wason selection task.

DUAL PROCESS THEORIES OF REASONING

Recently, psychologists have proposed that there are two distinct cognitive units that deal with reasoning. Roughly speaking the first corresponds with intuitive thought, and the second with abstract reasoning. In an attempt to combine many different versions of similar theories (e.g. Evans & Over, 1996; Skemp, 1979), the generic terminology System 1 and System 2, first adopted by Stanovich & West (2000), has become commonplace.

System 1 is characterised by processes that are quick, operate in parallel and are highly context specific. These processes are almost entirely subconscious in nature, only the end product is deposited in the conscious brain. The system is independent of language, is old in evolutionary terms and is also present in animals. System 1 is believed to be a large collection of subsystems that operate autonomously. Some of these subsystems are believed to be innate, whilst others may have been acquired by an experiential learning mechanism (Stanovich, 2004).

System 2, on the other hand, is slow, operates in serial and allows for non-contextualised hypothetical reasoning. It is controllable and conscious, has evolved relatively recently and is unique to humans. It is this part of the brain that allows humans to construct complex abstract simulations that are context independent and depersonalised. Fluency with System 2 is often measured using reasoning tests, and tends to be correlated with measures of general intelligence (although it is perhaps not surprising that one form of reasoning test correlates with another). System 2 is also involved in expressing the output of System 1, and it has the ability to monitor and, possibly, override these intuitive responses, although, as we shall see, this does not always happen.

Although System 1 is innate, it can be developed over time through experience. For example, it has long been recognised that chess grandmasters, as well as having superior analytical skills, have a different way of 'seeing' the chess board to amateur players (e.g. de Groot, 1965). Their experience of chess playing has altered their System 1 heuristics as well as developed their System 2 analytical skills. (See Evans, 2003 for a full review of dual process theories).

EMPIRICAL EVIDENCE FOR THE DUAL PROCESS ACCOUNT

Since the sixties there has been mounting evidence that participants in reasoning experiments do not always respond in a normative manner. Take, for example, the 'Linda' problem (Tversky & Kahneman, 1983). In this task participants were told:

> Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in antinuclear demonstrations.

Participants were then given eight possible descriptions of her present employment and activities, and were asked to rank them in order of probability. Intriguingly, 85% ranked "Linda is a bank clerk and active in the feminist movement" as more probable than "Linda is a bank clerk". Clearly, such a ranking is impossible. Tversky & Kahneman named this the 'conjunction fallacy', and explained it by noting that Linda resembles the prototypical feminist bank clerk more than she resembles the prototypical bank clerk.

Tversky & Kahneman, however, also noted that it would be unreasonable to claim that their (highly educated) participants had conceptions of probability that were largely based on resemblance to prototypical examples. Instead, the dual process account argues that the Linda task's standard response comes from System 1. It is intuitive, automatic and more concerned with social data than formal logic. System 2 cues the opposite response, that which realises that $P(A)$ cannot possibly be less than $P(A \cap B)$. Individuals who respond with the conjunction fallacy, then, fail to successfully use System 2 to monitor and correct their intuitive System 1 output.

There is also neuropsychological evidence that supports the dual process account of reasoning. Goel & Dolan (2003) used fMRI brain scans whilst participants took standard reasoning tasks. They found that responses traditionally associated with System 1 were related to activity in the ventral medial prefrontal cortex, whereas the logically correct System 2 responses originated in the right inferior prefrontal cortex, an entirely different part of the brain. They concluded that System 1 reasoning was influenced by emotional processes.

THE SELECTION TASK

More important evidence that supports the dual process account of reasoning comes from the Wason selection task (Wason, 1968). First published in the sixties, the selection task has become the most investigated task in the whole psychological literature on reasoning.

Participants in the task are shown a selection of cards, each of which has a letter on one side and a number on the other. Four cards are then placed on a table:

The participants are given the following instructions:

> Here is a rule: "every card that has a D on one side has a 3 on the other". Your task is to select all those cards, but only those cards, which you would have to turn over in order to discover whether or not the rule has been violated.

The logically correct answer is to pick the D card and the 7 card, but across a wide range of published literature only around 10% of the general population do. Instead most make the 'standard mistake' of picking the D and 3 cards. Indeed, Wason (1968) suggested that about 65% incorrectly select the 3 card.

There is a vast psychological literature that has attempted to explain why so few participants make the correct selection. Forty years of research has failed to reach a consensus, and the detailed reasons behind Wason's original results remain highly controversial. There are, however, some findings that have been found to be very stable, one of the most robust is the so-called 'matching bias' effect.

Evans & Lynch (1973) varied the structure of the task by rotating the presence of negatives in the rule (for example, they used rules such as "not D $\Rightarrow$ 3" and "D $\Rightarrow$ not 3" as well as the original "D $\Rightarrow$ 3"). They found that participants tended to select the cards that were mentioned in the rule, regardless of the presence of negatives. For the participants, the relevant cases seemed to be those that had the same lexical content as the propositional rule. Evans (2003) argues that this tendency, which has become known as 'matching bias', is a built-in heuristic in System 1. By the dual process explanation, the intuitive response, coming from System 1, is to select the D and 3 cards. It is argued that the standard mistake originates from participants using System 2 to merely rationalise and articulate this selection. As with the Linda problem, it is only if System 2 is actively and effectively monitoring System 1 that the logically correct answer (D and 7) can be produced. System 2 needs to *reason* rather than merely *rationalise* if the logically correct answer is to be found.

To reemphasise, the dual process account suggests that the standard mistake can be explained by a two part process: Firstly, card selections are determined entirely by System 1. Secondly, any System 2 processing that occurs is aimed at rationalising and articulating System 1's output. There is empirical evidence to support this hypothesis. A key prediction of this account is that participants will spend more time inspecting the cards that they select than those that they reject (as System 2 will be rationalising the selections). This was experimentally verified by Evans (1996) using a computer based mouse hovering technique, and by Ball, Lucas, Miles & Gale (2003) who used a sophisticated eyeball tracking system to measure inspection times.

The two routes that lead to the correct answer and to the standard mistake are shown in figure 1. It should be noted that figure 1 is somewhat misleading as, as mentioned above, System 2 is used when rationalising and expressing any output from System 1. However, in figure 1 it is shown as playing no part in route 1 to emphasise that it is not involved in the *reasoning* process.

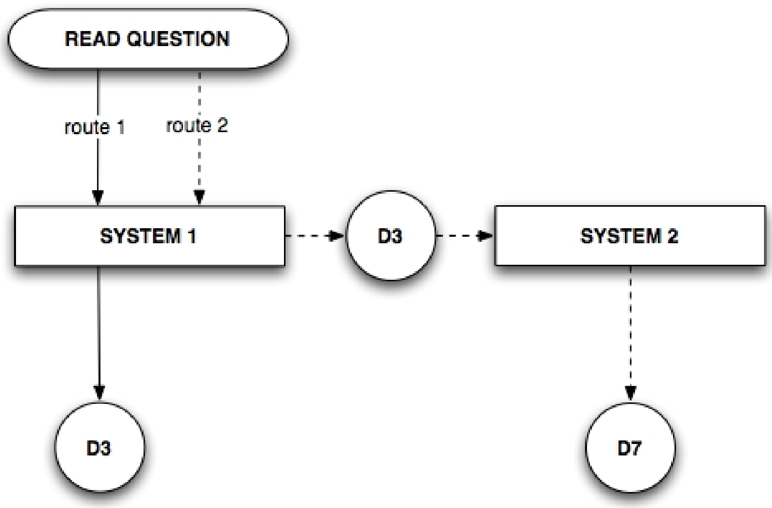

Figure 1: the routes that lead to the standard mistake and to the correct answer.

One of the most famous and striking results in the selection task literature is that performance can be dramatically facilitated by placing the task in a thematic context (e.g. Cosmides, 1989; Wason & Shapiro, 1971). Dual process theory explains this facilitation by suggesting that, in these thematic contexts, both System 1 and System 2 output the same answer. That is to say that only the abstract version of the task requires System 2 reasoning for its solution.

Its important to note that dual process theory is, to a large extent, neutral regarding the many competing theories that have attempted to explain performance on the Wason selection task. For example, mental models theory, mental logic theory, the pragmatic reasoning schemas theory and relevance theory can all be comfortably situated within a dual process framework.[1] Each of these theories can be seen as attempting to explain *why* either System 1 or System 2 produce the output that they do. In this sense, dual process theory is less a theoretical framework, and more a framework for theoretical frameworks.

[1] It is less easy to situate the so-called social contract theory (e.g. Cosmides, 1989; Leron, 2004) within a dual process framework, as it would appear to dramatically underestimate the role of System 2. In any case, this explanation has been heavily criticised in recent years, and there is mounting empirical evidence that appears to contradict it (e.g. Sperber & Girotto, 2002).

As the above discussion illustrates, some of the most common reasoning misconceptions from the psychology literature can be explained by the failure of System 2 to adequately monitor and correct the intuitive output generated by System 1. This paper is concerned with the interplay between the System 1 and System 2 reasoning of successful mathematicians. Has day-to-day exposure of deductive reasoning modified mathematicians' System 1 heuristics in a manner similar to the chess masters, or is System 2 the key to the differences between mathematical and non-mathematical reasoning?

METHODOLOGY

We administered a version of the Wason selection task to three groups: mathematics undergraduates, mathematics academic staff and history undergraduates. The history students were used as a control group, as it was assumed their degree would contain little or no mathematical reasoning. It is worth noting that this is common practice, in many comparable studies the general population is represented by psychology undergraduates. We are not aware of any selection task studies that have used a more representative group.

We adopted an internet methodology. Potential participants were sent emails explaining the experiment and asking them to take part. If they agreed, they clicked through to a website which recorded their answer, whether or not they had seen the task, and their IP address. There are clearly drawbacks to this methodology, however space restraints prevent a full discussion of the issues and solutions adopted here. The reader is referred to Inglis & Simpson (2004) for an in depth discussion of such matters. However, since this publication, our sample has since been expanded to include additional students from two additional high ranking UK universities.

The precise wording we used was identical to Wason's (1969):

> Four cards are placed on a table in front of you. Each card has a letter on one side and a number on the other.
>
> You can see:
>
> **D** **K** **3** **7**
>
> Here is a rule: *"every card that has a D on one side has a 3 on the other."*
>
> Your task is to select all those cards, but only those cards, which you would have to turn over in order to discover whether or not the rule has been violated.

Along with the quantitative based study we conducted a small number of clinical interviews with both mathematicians and historians who were not involved in the quantitative study. A standard 'think aloud' protocol was used, the interviews were audio tape recorded and transcribed for analysis.

RESULTS & DISCUSSION

The results are shown in table 1. Looking at the table reveals that there are significant differences between the mathematics and history students' range of answers (χ^2=100, df=8, p<0.001, ϕ=0.480). The mathematicians have a significantly higher success rate, although given the supposed importance of logic in mathematics, at only 28% for students and 43% for staff, it is perhaps surprisingly low.

	Maths Students		Maths Staff		History Students	
D	108	35%	5	24%	27	22%
DK	1	0%	0	0%	0	0%
D3	19	6%	1	5%	41	33%
D7	88	28%	9	43%	10	8%
DK3	0	0%	1	5%	2	2%
DK7	40	13%	3	14%	1	1%
D37	10	3%	2	10%	8	7%
DK37	23	7%	0	0%	23	19%
non-D	23	7%	0	0%	11	9%
n	312		21		123	

Table 1: The selections made by the different groups.

Critically, for our analysis, very few of the mathematics students or staff made the 'standard mistake' – that of selecting the D and 3 cards. Only 6% and 5% of maths students and staff respectively made this mistake, compared to 33% of history students. Instead, by far the most common mistake made by mathematicians was to select only the D card.

Recalling that dual process theory claims that the 'standard mistake' originates in System 1, this result would appear to have deep implications about the nature of mathematical reasoning. What explains the absence of the 'standard mistake' from the mathematicians' range of answers? What explains the proliferation of the D selection from mathematicians?

There would appear to be two reasonable hypotheses:

Hypothesis 1. Exposure to mathematics on a daily basis modifies System 1 heuristics in a manner similar to the chess grandmasters described earlier.

Hypothesis 2. Mathematicians' System 1 tends to operate in the same way as for the general population. But exposure to mathematics on a daily basis results in an increased tendency to use System 2 for monitoring and possibly modifying the output of System 1.

The two hypotheses are illustrated in figure 2. As before, System 2 is not shown to be involved in hypothesis 1 to emphasise that it is not involved in the *selection* process.

If the first hypothesis were the case, it would appear that matching bias – the heuristic that seems to be responsible for the 'standard mistake' – is a casualty of exposure to mathematics. We are currently working on a study that is designed to test whether this mathematicians' exhibit matching bias in more varied contexts.

If, however, the second hypothesis is the case, and given that a majority of both mathematics students and staff failed to select the correct answer, it would seem that the System 2 processes of many mathematicians are not as efficient at detecting logical mistakes as one might think.

The 'think aloud' protocol data that we collected provides some support for the second of these hypotheses. In the following extract Wolfgang, a postgraduate mathematical economics student, has just been handed a version of the selection task.

Wolfgang	[Reads the task] Hmmm. OK, first I know that I don't have to… that I definitely have to turn over the right one [the D].
Interviewer	The D?
Wolfgang	Yeah, the D, because that's the rule of course […] and the next thing, is the 3 has to be checked for, because when they are… [long pause] no, no, I don't have to, no I don't have to.
Interviewer	You don't have to? Why not?
Wolfgang	Because… because there can be another symbol, and if it does not violate the rule then there is a D and every other thing is not said, so it is not necessary to do it.

Wolfgang's initial reaction is that he needs to pick the D and the 3 cards. If dual process theory is to be believed, this is the result of the presence of a matching bias heuristic within Wolfgang's System 1. However, quickly and unprompted, he uses his System 2 to monitor and check his intuitive System 1 reaction. After several seconds of System 2 reasoning he corrects his answer, and decides that the 3 card is unnecessary, even if his explanation is somewhat incoherent. He eventually went on to find the correct answer.

Wolfgang was typical. Victor, a mathematics undergraduate

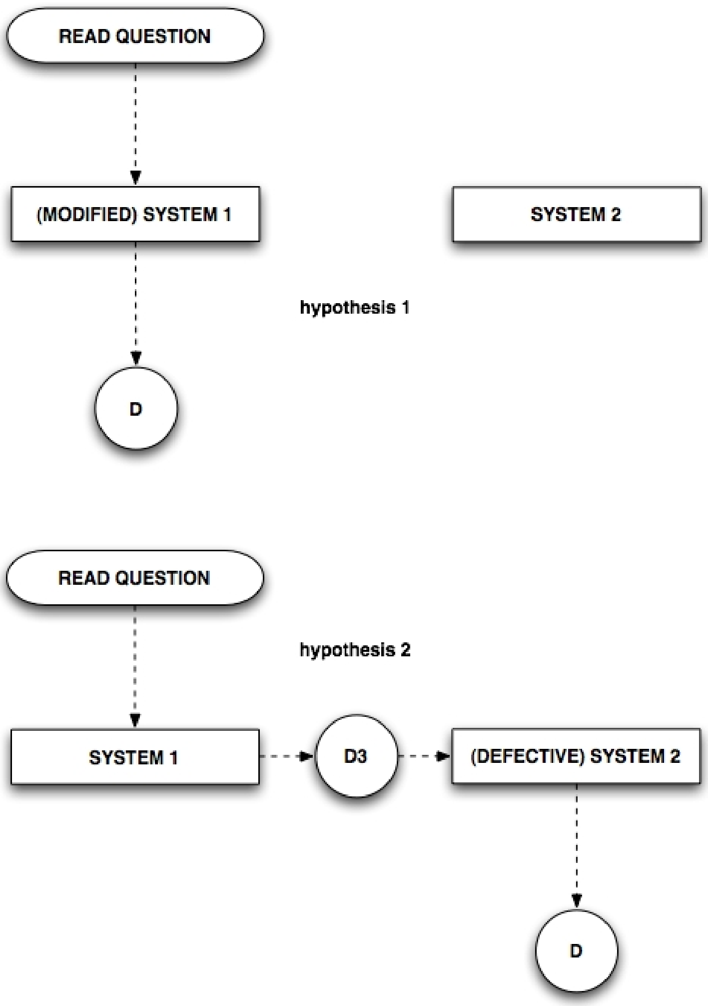

Figure 2: Two hypotheses that explain mathematicians' selection of the D card.

at the end of his first year, responded similarly:

Victor	[Reads the question] Umm… you should turn over the D and the 3… then… they're the only ones.
Int.	Why?
Victor	Because it doesn't say anything about K's or 7's.

Interestingly, Victor even gives a matching bias based explanation for his selections. In fact every mathematician interviewed (n=6) gave D and 3 as their initial answer. However, they were able to use System 2 to modify and correct this mistake. Sometimes this took prompting. For example, here is Yasmin (a first year mathematics undergraduate) responding to the task.

Yasmin	I'd take the D and I'd have, umm, the 3. Because, if that had the D on the other side then the rule would be… [laughs].
Interviewer	What are you thinking?
Yasmin	I've got myself tied up in a knot!
Interviewer	OK, well, not to worry. […] What's your thinking about the 3?
Yasmin	Umm, OK, I would leave the 3, you wouldn't need to turn that over. Because, yeah... but I would need to turn the other two over, I'd have to turn over the K and the 7 to see if there was a D on the other side.

Evidence from the, admittedly small scale, qualitative study would seem to indicate quite clearly that the initial reactions of mathematicians are the same as everyone else. Their System 1 heuristics are as influenced by matching bias as the general population. Given this, the quantitative data garnered from the internet study would suggest that mathematicians tend to be highly effective at using System 2 to monitor and criticise their System 1 reasoning.

However, given that the mathematicians appear to be using System 2, why do a large percentage of them fail to select the correct answer, D and 7? If they are using System 2 to detect and correct System 1's mistaken decision to select the 3 card, why don't they overwhelmingly detect the necessity of the 7?

Despite the proliferation of anecdotal or historical analyses of how mathematicians do mathematics (e.g. Hadamard, 1945; Lakatos, 1976; Tall, 1980), there is very little empirical research in the area. Based on the limited evidence presented here, it would seem that the understanding and use of the modus tollens deduction ($P \Rightarrow Q$, not Q, $\therefore$ not P) is not nearly as important to success in mathematics as one might think. More research is needed in this area.

CONCLUSIONS & SUMMARY

The quantitative data presented in this paper clearly indicates that there is a significant difference between mathematical and non-mathematical cognition. The mathematics students' and staffs' range of responses to the Wason selection task was significantly different to those from history students, who were taken to represent the general population. In particular both mathematics staff and students were significantly more likely to make the correct selection, and significantly less likely to make the standard mistake. Instead, the mathematicians' standard mistake was to select the D card on its own.

We argue that this result has implications about the nature of mathematical cognition. In particular, if a dual process account of reasoning is adopted it would seem that either mathematicians have a modified System 1, or that they are much more likely to use System 2 to modify and correct their System 1 reasoning. Evidence from a small scale qualitative study provides some support for the second of these hypotheses. That is to say that one of the key differences between successful mathematicians and the general population is that the mathematicians appear more able to, or more willing to, activate their System 2 whilst reasoning.

However, the evidence also suggested that, for many mathematicians, their System 2 processes might not be as proficient at detecting some logical mistakes as one might think. Crucially though, since the mathematicians involved in this study were all highly successful, it would seem that this lack of proficiency does not have any serious effect upon their mathematical abilities.

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ON SOME DIFFICULTIES IN VECTOR SPACE THEORY

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Abstract: *This paper reports on the first stage of a research project still in progress aiming at identifying undergraduate and graduate students' difficulties and errors in solving Linear Algebra problems. Some students' difficulties related to very basic notions of Vector Space Theory are shown and analysed in terms of process-object duality.*

Keywords: Vector Space Theory, Process-Object Duality, Advanced Mathematical Thinking.

THE CURRENT STATE OF RESEARCH IN LINEAR ALGEBRA EDUCATION

As many researchers underline (Dorier, 1998; Harel, 1990; Tucker, 1993), the importance of Linear Algebra in many fields of mathematics, science and engineering is widely acknowledged by both mathematicians and scientists, who consider Linear Algebra as an important mathematical prerequisite for undergraduate students in science and technology. Coherently Linear Algebra courses are basic for a wide variety of disciplines at the tertiary level: mathematics, physics, computer science, engineering and so on.

Nevertheless up to the 90's Linear Algebra has not seemingly raised the interest of the community of the researchers in math education. Since then different studies on the subject have been carried out, all of them highlighting a number of difficulties in teaching and learning Linear Algebra at the tertiary level. As a result, Linear Algebra education has made great advances: those studies have contributed to explain why so many students fail in Linear Algebra courses and have led "to some well founded 'recommendations' and 'good advices' concerning the practice of teaching" (Dorier and Sierpinska, 2001; p.270).

In their survey on the state of research in Linear Algebra education at the tertiary level, Dorier and Sierpinska (2001) reviewed and proposed a classification of the most advanced studies in the topic[1]: (i) epistemological and historical analyses (Dorier, 2000; Robert and Robinet, 1996), (ii) analyses of the language(s) of Linear Algebra (Dorier et al., 2000; Hillel, 2000), (iii) analyses of the characteristics of thinking required for the understanding of Linear Algebra (Alvas-Dias and Artigue, 1995; Sierpinska, 2000) and (iv) on the teaching practices and teaching experiments

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[1] The Authors explicitly state that their survey "does not claim the title of an exhaustive review of research on the teaching and learning of linear algebra at the undergraduate level" (Dorier and Sierpinska, 2001, p. 255).

(Harel, 2000; Rogalski 2000;). We can add the following more recent works to the last category: Gueudet, 2004; Trigueros and Oktaç, in press and Uhlig, 2002.

One can notice that Dorier and Sierpinska do not mention any studies focusing on specific concepts of Linear Algebra in their classification. Seemingly researchers share the view that difficulties in Linear Algebra (no matter what concepts are involved) are due to general features of the field or to the axiomatic approach usually used in teaching Linear Algebra. Indeed, as far as we know, the only exception is constituted by Nardi's study on students' concept-images of span and spanning set (Nardi, 1997).

This apparent 'characteristic' of research in Linear Algebra education contrasts, for instance, with research in calculus where many studies are devoted to the cognitive analysis of specific concepts such as those of limit, continuity, function and so on.

Finally let us note that, even if the Authors of the cited survey do not claim to be exhaustive in their overview, one has the impression that few studies on the subject have been carried out in European countries, but in France, in contrast with the claimed importance of Linear Algebra and with the number of reported difficulties. As for Italy, this impression is confirmed by a recent survey on Italian research in math education (Malara et al., 2000), which does not report any Italian study in Linear Algebra at the tertiary level.

In this report we present some indications from a study aiming at investigating undergraduate and graduate students' difficulties in Linear Algebra trying to grasp the specificity of the concepts involved. In particular we focus on the notion of linear combination and show some students' difficulties arisen from our study together with a first analysis in terms of process-object duality.

THE PROCESS-OBJECT DUALITY

As Gray and Tall observe "the notion of actions or processes becoming conceived as mental objects has featured continually in the literature" (Gray and Tall, 1994, p.118). Out of the number of studies which adopt such a perspective we briefly present the framework carries out by Sfard (1991, 1994) on which we ground the analysis developed in our study.

Sfard (1991) claims that abstract notions "can be conceived in two fundamentally different ways: *structurally* - as objects, and *operationally* – as processes" (Sfard, 1991, p.1). According to her the fact that the same representation and the same mathematical concept may be conceived both structurally and operationally apparently pervades the whole mathematics (Sfard, 1994, p.193).

The two main features of Sfard's perspective are that: (i) her distinction refers at once to individual cognitive processes and to the historical formation of mathematical concepts, (ii) "the terms operational and structural refer to inseparable, though dramatically different facets of the same thing" (ibidem, p.9), that is the reason why Sfard speaks of duality rather than dichotomy.

As far as concept formation is concerned, the basic tenet of Sfard's theory is that "the operational (process-oriented) conception emerges first and that the mathematical objects (structural conceptions) develop afterward through reification of the processes" (Sfard, 1994, p.191). A certain mathematical notion should be regarded as fully developed only if it can be conceived both operationally and structurally.

OUR STUDY: AIMS AND METHODOLOGY

Aims

As mentioned above, our study aims at investigating difficulties and errors of undergraduate and graduate students in mathematics when solving Linear Algebra problems. More precisely we focus our attention on basic notions of Vector Space Theory: linear combination, linear dependence/independence, generators and so on.

We devote our attention mainly to the notion of linear combination, which, although central in an axiomatic approach to the Vector Space Theory, is rarely - if ever - object of specific activities in teaching practice and seems never to be explicitly taken into consideration in research.

Methodology

Our study was organized in two different experimental phases:

The first phase was carried out during the academic year 2001/02, with exploratory intent, and involved five first-year students in mathematics (age 19-20). Students have been individually interviewed after the end of their first semester in Linear Algebra, during which they were introduced to the basic notions of Vector Space Theory via an axiomatic approach, and whilst they were attending a second more advanced one (Linear Transformations and Matrices, Triangulation and Diagonalization, Jordan Form, Bilinear and Hermitian Forms, Spectral Theorem). Students were presented with 3 different problems to be solved in individual interviews. A first report on this phase can be found in Maracci, 2003.

The second phase, developed in the academic year 2002/03, involved 12 students in mathematics: 5 first-year students, 4 final-year students and 3 PhD students. Students were presented with 2 problems in Vector Space Theory (one of the problems was proposed in the first phase too) to be solved in individual sessions. As for first-year students, they were interviewed after having attended two semesters in Linear Algebra. Before being interviewed, students were asked to give definitions and examples of some basic notions of Vector Space Theory.

In neither phase time constraints were imposed over the problem solving sessions, which were recorded. As for the second phase the analysis of the collected data is still in progress.

The analysis of the transcripts of the interviews highlights a number of students' difficulties concerning basic notions of Vector Space Theory, we will show some of them in the following sessions.

One of the proposed problems

We present here the only problem used in both phases of our study. The excerpts from students' protocols, which we are going to quote, concern that problem:

PROBLEM. Let V be a R-vector space and let u_1, u_2, u_3, u_4 and u_5 be 5 linearly independent vectors in V. Consider the vector $u=\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$.

Do there exist two 3-dimensional subspaces of V, W_1 and W_2, such that $W_1 \cap W_2 = \text{Span}\{u\}$?

Do there exist two 2-dimensional subspaces of V, U_1 and U_2, which do not contain u and such that u belongs to U_1+U_2 ?

The answer to both the questions is that such subspaces of V exist.

The problem could be approached in at least two different ways:

Approach 1. One might try to describe the conditions which the subspaces must fulfil in terms of their possible generators. For instance one could notice that the subspaces $W_1=\text{Span}\{u, u_1,u_2\}$ and $W_2=\text{Span}\{u,u_3,u_4\}$ verify the conditions posed in the former question are verified, and that the subspaces $U_1=\text{Span}\{u_1,u_2\}$ and $U_2=\text{Span}\{u_3+3u_4,u_5\}$ verify the conditions posed in the latter one; many other pairs of subspaces fulfilling the required conditions can be constructed in analogous ways.

Approach 2. Alternatively one could notice that the definition of u as linear combination of the vectors u_i is not necessary to solve the problem (the only relevant information concerns the dimension of V), which thus could be re-formulated as follows:

Given a real vector space V, whose dimension is greater or equal to 5, and a vector u in V:

Do there exist ...

Such a formulation could lead one to observe that the answer to the second question - which as we are going to show revealed to be hard for the interviewed students - is positive in R^3 and as a consequence in any real vector space with dimension greater or equal to 3.

We presented these two approaches to give the idea of the wide spectrum of possible solutions of the problem. Of course we did not expect that the students involved in our study approached the problem according to the latter way – and indeed none of the interviewed students followed that approach. In fact, as a consequence of the school practice, students construct the idea that any information given in the text of a problem is relevant for solving it and can not be neglected. Moreover, because of the focus of our study, we are particularly interested in solutions consistent with the former approach which obliges students to deal with the notion of linear combination.

Finally, though the proposed problem could be considered unconventional because more information are provided than needed for the solution, let us remark that such information are not really superfluous: approach 1 shows that all the hypotheses can be effectively used to construct a solution of the problem. It is not necessary at all to neglect any of the hypotheses.

PROTOCOLS

In this paragraph we present some excerpts from the interviews of 4 students, who in our opinion shows the same difficulties related to the notion of linear combination.

Since the analysis of the collected data is not complete yet, we can not claim that the reported difficulties are widespread or shared by all the students involved in our study. Nevertheless we think that the presented cases are worth considering.

We structure our presentation dividing the students' excerpts into two groups: the excerpts of the students whose conclusions focus on linear combinations (group A) and those whose conclusions focus on the dimensions of subspaces (group B). As mentioned above, we think that, despite their different focuses, the students' claims reveal the same difficulties.

Group A: *May a linear combination of 5 linearly independent vectors be written as linear combination of 4 linearly independent vectors?*

Protocol A1: Fra. (first-year student, a.y. 2001/2002, medium-high achiever[2])

Fra has just correctly answered the former question of the problem and she is now approaching the latter. Since the very beginning she expresses the feeling that the answer to the second question is negative.

> **74 Fra**: I think that it is not possible because u is linear combination of 5 linearly independent vectors... and if one can write it as... that is it should be an element which can be written as the sum of an element of U_1 and of an element of U_2, and then it should be linear combination of at most 4 linearly independent vectors... now let's see

Then Fra seemingly faces the task of proving her assertion or at least of ascertaining its truth ("let's see", item 74). In fact she does not prove it, on the contrary her assertion is never put under discussion and it is used some minutes later as the conclusive and decisive argument in her solution to the problem.

> **86 Fra**: [...] I find that anyway *u is written as linear combination of 5 linearly independent vectors... then I cannot write u with only 4 linearly independent vectors*

> **92 Fra**: anyway u is written as linear combination of 5 linearly independent vectors, then I cannot write it as belonging to this sum $[U_1+U_2]$

[2]The labels "low", "medium" or "high" achiever are assigned on the basis of the marks got in the Linear Algebra examination.

Protocol A2: Ann (final-year student, a.y. 2002/03, high achiever)

Quite surprisingly – in our opinion -, the same view as Fra is shared by Ann, a final-year student who, after having correctly answered the former question, approaches the latter noticing:

> **83 Ann**: now, I have a lot of doubts about that possibility... because... *because in order to write u I need 5 linearly independent vectors*, to write it as an element of the sum of two 2-dimensional subspaces *I can use at most 4 linearly independent vectors*, because being linearly independents... but I could use other...

> **89 Ann**: because I thought that I write u with 5 vectors and I need all the 5, I cannot do without them because such vectors are linearly independent... on the contrary if U_1 and U_2 have to be 2-dimensional spaces, they can be generated by at most 2 linearly independent vectors... then if u belongs to U_1+U_2 I use at most 4 vectors. The point is that u, I know that u can be written with respect to another basis. [...]Thus on the one hand I am quite sure that it is not possible, but on the other hand I have no idea how I can show it

Ann is quite confident in her claim, her only concern is about the possibility of expressing u with respect to another basis. After stating these assertions Ann spends several minutes to try to prove them without success. It is worthwhile noticing that Ann correctly expresses the relations between u and the possible generators of U_1 and U_2 in algebraic language, but the possible efficiency of this approach is hindered by Ann's confidence in her previous evaluations. Only when she decides to question her assertions rather than to accept them without discussion, the algebraic approach leads her to solve the problem.

Group B: *What dimension should subspaces containing linear combinations of 5 linearly independent vectors have?*

Protocol B1: Pie (first-year student, a.y. 2002/03, medium achiever)

Pie has positively answered the former question even if his justification is quite lacking and inadequate. His approach to the latter question, as well as his argumentation, is perfectly consistent with those of Fra and Ann. Differently from Fra and Ann, Pie's attention is focused on the possible characteristics of the two subspaces rather than the possible characteristics of the vector u, as a consequence he is led to conclude that the dimension of U_1+U_2 should be at least 5 (item 24).

> **23 Pie**: u has to belong to their sum [U_1+U_2]... thus u can be written as the sum of a vector of the former [U_1] and a vector of the latter [U_2]

> **24 Pie**: but u is constituted by 5 coordinates and thus by 5 linearly independent vectors, hence belonging to the sum it should be a linear combination of all these 5, this means that the basis*, I mean the dimension has to be at least 5*; on the contrary the dimension of the sum cannot be

greater then 4 - and I get it only if the intersection of the two subspaces is empty – thus this leads me to conclude that such subspaces do not exist

This argument is conclusive in the opinion of Pie, who confirms his view some minutes later when he reconsiders all the session.

31 Pie: [...] as for the second question, I can say, I think that my proof is rigorous enough and the answer is no, I am satisfied with that.

Protocol B2: Jas (first-year student, a.y. 2002/03, medium-high achiever)

Although the premises are different, the same conclusion is drawn by Jas (item 48) even with greater emphasis.

Jas approaches the problem starting from the last question. One of her first remarks concerns the uniqueness of representation of the vector u.

4 Jas: [...] a vector u is given, such that $u=\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$. It is written in a unique way

8 Jas: then... let's see, I must... I find u in just one way

The uniqueness of representation of u leads Jas to state that u belongs to U_1+U_2 only if vectors u_1, u_2, u_3, u_4 and u_5 belong to U_1+U_2 too.

13 Jas: [...] u has to belong to their sum [U_1+U_2], let's see. I don't think that it is possible, because if I take... let's see [...] in order to get that u belongs to their sum [U_1+U_2] I have to find in this sum at least both u_1 and u_2 and u_3 and u_4 and u_5 [...] but if U_1 has dimension 2 then I get that it does not contain more than 2 linear independent vectors which I can suppose to be u_1 and u_2

Starting from the above argument and from considerations concerning the dimension of U_1 and U_2, Jas supposes $U_1=\mathrm{Span}\{u_1,u_2\}$ and $U_2=\mathrm{Span}\{u_3,u_4\}$ and negatively answers this question. She similarly approaches the former question of the problem and as a consequence she fails to solve it. After that, she reconsiders all the session and states:

48 Jas: [...] at this *point u cannot be... it cannot belong to any proper subspace whose dimension is either 2, or 1, or 3, or 4, it [u] can only belong to a 5-dimensional subspace* [...] or to something still greater.

Surprisingly this last assertion does not cause any cognitive conflict in Jas.

ANALYSIS OF THE PROTOCOLS

As mentioned in the previous section, we think that the 4 reported excerpts reveal the same difficulty of students: the different conclusions which students A (Fra and Ann) and B (Pie and Jas) draw – we mean that (A) the subspaces do not exist because u can not be express as linear combination of 4 linearly independent vectors, and (B) the subspaces do not exist because otherwise their sum should have dimension equal to 5 - are in fact two sides of the same coin. The difficulty that in our opinion emerges from the presented data is that of conceiving linear combinations simultaneously as objects and as processes.

In order to better clarify our hypothesis let us consider what kind of solution students were expected to produce given the approach they adopted. We already observed that the subspaces U_1=Span{u_1,u_2} and U_2=Span{u_3+3u_4,u_5} fulfil the conditions posed in the second question of the proposed problem. The approach adopted by our students, i.e. to try to construct U_1 and U_2 starting from the assumption that u has to belong to U_1+U_2, should have led them to an analogous solution. To produce such a solution requires to conceive u as $\sqrt{2}u_1-\frac{1}{3}u_2+(u_3+3u_4)-\pi u_5$ (rather than $\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$), to construct the vector u_3+3u_4, to consider it and work with it in the same way as u_1, u_2 and u_5, and finally to construct the two subspaces.

The point here is not "simply" to conceive a linear combination as a process (in our case $\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$): in fact the given linear combination evokes a ‘privileged process’, in which the given vectors u_i participate with the same role and status and which is not effective to solve the problem. That ‘privileged process’ might conflicts with the way u should be thought of, that is as the process $\sqrt{2}u_1-\frac{1}{3}u_2+(u_3+3u_4)-\pi u_5$, where the ‘sub-process’ u_3+3u_4 has to be isolated, conceived as an object and then re-thought together with u_1, u_2 and u_5 as inputs of another process.

Moreover students refer to u as ‘a linear combination of 5 linearly independent vectors’ rather than as $\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$. The former signifier (‘a linear combination of 5 linearly independent vectors’) is more informative than the latter (‘$\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$’) - explicitly stating the linear independence of the vectors u_1, u_2, u_3, u_4 and u_5 - , but ‘natural language’ - the system of representation to which that signifier belongs - does not provide the same effective means of treatment and transformation as algebraic/symbolic system. As a consequence, though the expression ‘linear combination’ seemingly may allow to think in terms of processes, the possibility of working with and on the processes themselves results drastically limited. We wish to recall that Ann succeeds in expressing in algebraic terms the relations between u and the eventual generators of U_1 and U_2, but her approach results totally ineffective as long as she continues to refer to u just as ‘a linear combination of 5 linearly independent vectors’.

Alternatively, students’ reported failures might be interpreted as due to their difficulty to associate an object (a vector) to all the possible processes (linear combinations) that produce it. This question raises an interesting point, however as already underlined when presenting the problem, the hypothesis that u is equal to $\sqrt{2}u_1-\frac{1}{3}u_2+u_3+3u_4-\pi u_5$ is probably interpreted by students as a fundamental information which can not be neglected. We hypothesize that, as a consequence, even those students capable to associate an object with the family of the processes producing it, should still face the task of relating those possible processes with the given ‘privileged’ process in order to solve the problem. Thus such students could face the same difficulties as above.

SUMMARY

In this report we presented some difficulties related to the notion of linear combination arisen from a study still in progress. Our study confirms what is widely documented, i.e. that students meet with difficulties even when facing elementary notions of Linear Algebra.

More in details we highlighted some students' failures, which according to our hypothesis – consistent with Sfard's framework of the process/object duality - could be consequence of students' difficulty to perceive linear combinations simultaneously as objects and as processes.

The qualitative nature of our study does not allow us to claim that the reported protocols are representatives of students' difficulties in Vector Space Theory , anyway we consider indicative of the intensity of the problem the fact that difficulties so strictly related to very basic notions of Vector Space Theory emerged from the activity of medium, medium-high and high achievers. In particular we think that the presented excerpts pose the problem of seriously taking into account notions up to now neglected both in teaching and in research.

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AN EXPERIENCE OF PROBLEM SOLVING IN MATHEMATICAL ANALYSIS

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Abstract: *In this paper we discuss some results of experiences concerning activities of problem solving and mathematical modelling in the field of calculus. In particular, while also taking into account the didactical contract, we discuss the importance of the formulation of the problems considered and its influence upon the activity of the students, in particular their ability to formulate conjectures.*

Keywords: Calculus, Modelling, Problem solving, High School.

1. INTRODUCTION AND FRAMEWORK OF THE RESEARCH

Often, in scientific literature, *problem solving* is associated with *mathematical modelling* (Blum & Niss 1991): such an association is plausible in that problem solving activities are often derived from external contexts (the approach called *realistic mathematics education* can be seen in Gravemeijer & Doorman, 1999).

Undoubtedly the pupil's approach to a situation-problem is linked to the creation of a mental model of the situation (Zan, 1991-1992). Many authors affirmed that the fact that a problem reflects a situation which is imaginable in all its details can facilitate the construction of the mental model and hence render resolution simpler (Johnson-Laird, 1983). Although some research works (D'Amore & Martini, 1997) are critical of this assumption.

In this paper some experiences are described in which students in their final year of high school studies (the reference is to the Italian 'Liceo': 18-19 year-olds) try their hands at modelling problems within mathematical analysis. In our aims, linking modelling to problem solving was a way to propose "complete processes" in which pupils were expected to confront themselves with: the formalization of a problem given in a verbal form, the formulation of a conjecture relating to the values taken by a certain variable, the analysis of the involved parameters, the explicitation of a function connecting the variables, the relative study and the interpretation of its variation within the context of the starting problem.

The research was centred on the elements which favour the production of conjectures on the part of the students and an aware verification of those. Our hypothesis is that adhesion to the reality of a problem is not in itself a stimulus for giving a semantic value to the functional aspects of the quantities which come into play: It is therefore necessary to examine some changes in the formulation of the problem itself and to consider their influence in the light of the didactical contract. It should be immediately emphasised that in the first experience the formulation of the problems was "closed" and essentially required the determination of a value; subsequently the formulation of the problems was varied as deemed fit. As pointed out by several researchers in the last decades (Pimm, 1987, Laborde 1991), the formulation of a problem is relevant to the didactical contract (Sarrazy, 1995).

2. SITUATION AND METHODOLOGY

The first phase of the experience was carried out in two final year classes at scientific high schools by two teachers, Gabriella Aprilini (class A) and Massimo Plateroti (class B). The second phase of the experience was conducted in the successive school year by G. Aprilini in the class of a colleague (class C). The syllabus introduced previously, in all the classes, concerned the basic concepts of analysis (functions, limits, derivatives). The students, generally, had acquired sufficient mastery of the techniques and were able to apply them to draw the graph of a function.

Each class was divided into six groups according to ability levels. Each group was assigned a different problem: the "most able" group in each class was assigned the most demanding problem (n.6). The individual groups were asked to resolve the problems, to report their reasoning in as much detail as possible and to record conjectures, doubts, difficulties and strategies adopted while carrying out the task.

None of the classes involved had experience of *problem solving*, meant as the assignment of not easy and non-standard problems, to solve which pupils have time at their disposal, can discuss, try, check ...(see the classical work: Lester, 1983). The students had some experience of writing descriptions of procedures for resolving a problem but the request to comment on the errors made and difficulties encountered was new. This latter request was important as a conduit for information between teacher and student as well as a tool in promoting students' reflection on their thinking (Cooley, 2002). The teachers took on the role of observers, moving between the groups, checking on the directions taken and only intervening through simple pointers where it was considered opportune in order to overcome an impasse.

The students were given two hours to solve the problem. In the first experience (closed problems) there was also a second moment in which each group was assigned the study of the same function resulting from the task they had previously carried out so as to interpret the model in relation to the problem allotted to them. Moreover each group examined the corrected work, and in the course of a two hour lesson, together with the teacher, they commented on and justified the dynamics encountered in solving the problem. The second experience (in which the formulation of the questions was varied) took place over a single two-hour session.

The problems distributed, many of which well-known, were chosen together with the two teachers. The didactic aims of the experience were those which are classically attributed to problem solving and to modelling (we refer once more to: Blum & Niss, 1991), principally: to have the students construct an appropriate model of the situation proposed, thus highlighting their ability to apply the knowledge they had acquired; to frame the problems proposed within a wider cultural horizon; finally, to permit the students to "do mathematics" in an autonomous manner, bringing into play a mastery of mathematical processes. It was especially desired that the model was to be represented by a "not banal" function (Artigue, 1998).

Problem 1. An artisan wants to construct a cylindrical box of fixed volume V, opened from its top. (*Classes A and B*): What radius must the base have and what height must it have in order to minimise the quantity of plate used? (*Class C):* How does the quantity of plate used vary in relation to the dimensions of the box? *(note: in the course of the work the teacher suggested taking into consideration the height and the radius of the base of the box*).

Problem 2. An oil pipeline is to be constructed to transport oil from a platform A in the middle of the sea to a port B on the coast. It is known that the construction costs are equal to N €/m on dry land and M €/m at sea. (*Classes A and B*): Which pipeline route has the lowest cost? (*Class C*): How does the cost vary as the route varies?

Problem 3. 352 square centimetres of a page of a book are occupied by printed text. The upper and lower margins must be 2 cm in width and those at the sides must be 1.5 cm. (*Classes A and B*): What must be the dimensions of the page in order to minimise paper consumption? (*Class C*): How does the paper consumption vary in relation to the dimensions of the page?

Problem 4. A box is to be constructed from a rectangular carton of dimensions a and b by cutting out four small squares at the corners. (*Classes A and B*): Calculate the side x of the small squares in such a way that: (a) the volume of the box is at a maximum and (b) the surface of the box is at a maximum. (*Class C*): How does the volume of the box vary in relation to the side of the small squares? How does the surface vary?

Problem 5. A billiard ball of radius R is placed in a cylinder with radius $r = 10$ cm and water is poured in just sufficiently to cover the billiard ball. (*Classes A and B*): Calculate the radius R of the billiard ball so that the quantity of water to be poured in is at a maximum. (*Class C*): How does the quantity of water to be poured in vary in relation to the radius of the billiard ball?

Problem 6. Two villages are located on the same bank of a river. A farmer has to go from one village to the other filling a pail with water from the river along the way. (*Classes A, B and C*): where is it most convenient to draw the water so as to take the shortest possible route?

While being inapplicable in practice, the texts nevertheless provide a "real" situation which has to be "mathematized". The initial formulation (classes A and B) perhaps emphasized the activity of modelling and optimization more then the one of problem solving. According to our expectations, pupils discussed about the interpretation of the text (although the problems had a close formulation, some data had to be inferred by the students), and about the way to solve it. Contrarily to our expectations, the concrete problem to be modelled did not provide pupils with a general overview of the situation; pupils worked from the beginning "to find a result", without thinking in advance of possible optimal values. The second formulation (class C) was chosen in order to underline the shift towards a wider activity of problem solving. This difference in formulation proved to be effective in encouraging pupils to formulate hypotheses. I.e. the didactical contract was influenced by the formulation. In some sense the "open formulation" entitled pupils to perform "open activities".

Even though our aim was not the full study of the deep connections between problem solving and modelling, we also expected that problem solving in connection with a model would favour a sort of cognitive unity (in the sense of: Garuti, Boero, & Lemut,1998) between the phase of mathematization and the opposite phase of concretization. As we will see, this did not happen, in both experiences.

3. FINDINGS

Here we report an analysis of students' protocols, dividing it up according to some criteria agreed upon with the teachers who oversaw the experience. For reasons of space we will not include images or extracts taken from the protocols of the students.

First Experience (Classes A and B)

Correct analysis and comprehension of the text: Slight uncertainties came out at the beginning, above all with regard to what the problem required. These doubts were overcome thanks to discussion and after careful reading of the text. Here are some examples (the translation is ours):

- those solving P4 in class A observed: "*We met with some initial difficulties in the interpretation of the text. It was not totally clear whether, to construct the box, the small squares taken from the extremities of the carton were also to be used or even whether only these were to be used. We attempted several different proofs in order to compose the box…*"

- at the beginning the group A dealing with P1 was unable to understand what was being asked for in the text and commented: *"It is held that parameter h leads to a saving of plate and therefore the height must be found first of all. On the other hand, the others believe that the problem should be solved by finding both r and h…"*

It was not always the case that the natural language adopted in the formulation of the problem favoured an immediate comprehension of the text, as in fact was observed by the students with regard to P3, class A: "*The greatest difficulties encountered principally regard the final question concerning the paper consumption...*"

Each group carried out an appropriate analysis of the problem and some put forward considerations regarding limit cases. Class B solving P2 stated: "*Given that the costs at sea and on land are different then the limit routes with the longest possible distance on land and the longest possible distance at sea are to be studied. If C coincides with D, that is if the route by land is the cheapest, then the cost will be...*"

The group B working on question n.6 stated*: at the beginning we located the villages at the same distance from the river*, *but we then immediately realised that that was a very particular position...*"

At the end of the first reading of the text only a few students formulated hypotheses regarding the final solution. Albeit contorted, the reasoning of the students working on P2, class A, is interesting: "*If we say DC=x then we know that as x increases the route at sea will be greater. Naturally it should not be assumed that a shorter route is cheaper than a longer route with lower costs. When M<N, i.e. when the sea route is cheaper, the reasoning was put forward that... the route ACB with C coinciding with B is the cheapest"*. As can be seen, the students considered such initial reasoning as hypotheses of work to be verified as they carried out the task and commented: *"the difficulty encountered was to make the calculations add up so as to confirm the logical procedures..."*

Probably not all the groups felt the need to formulate conjectures in that the problem posed rather required the *finding of* something that results from calculations.

Selection and use of variables and parameters. No particular difficulties emerged in formulating the correlation between the data given and in deducing from amongst those which were unknowns and which were to be used as arbitrary parameters.

Group A solving P6 was not put off when faced with a text short of data and chose to use as parameters the distance AB between the villages, the distance of the villages from the river and the distance d between the projections of the points A and B on r.

Again with regard to P2 the students in class B proved to have very clear ideas. They point out that "*assuming DC=x, then we must minimise the total cost Y=... The cost varies depending on: the distance, the values M and N..."* Let us highlight the fact that the students were considering the parameters M and N as variables to be discussed and the distance DB as fixed for construction.

There were also those (P1, class A) who manifested a kind of fear of working with parameters: "*Some perplexity arose regarding the use of the volume as a parameter since it was thought that this would result in simplification. However, on close rereading of the problem it was seen to be necessary for the final result."* The "hope" of obtaining a solution that is independent of the parameter can be perceived.

The students regularly neglect the indication of the domain of the variable. On being asked about this, the students confirmed the initial impression: they were beforehand aware of the necessity of such a step, but in practice they neglected it.

Identification of strategies. The strategies used to arrive at the solution proved to be pertinent, essentially correct, though quite "standard". All the groups worked by adopting algebraic formulations.

The three members of group 6, class B, initially worked independently, each of them seeking a different formulation on their own. After some individual approaches they started to work together and decided on employing an algebraic formulation.

The students dealing with P1 pointed out that "*the only problem encountered was where to put the x in the attempt to reach a faster resolution of the calculations*".

Only in one case we found an attempt to use co-ordinates: the P6 group in Class A tried to solve the problem placing on the Cartesian plane the triangle formed by the two villages and the place were the pail is filled. But soon they abandoned this method using letters to denote the cotes of the triangle and looking for an algebraic formulation of the conditions. Nobody thought at the classical geometrical approach which makes use of reflection.

Those working on P3, class A, once they had identified the procedure, dealt with their task confidently: "*In our opinion the objective of the problem is to identify the function that expresses the area of the page. Once the minimum area of the page has been calculated it will be possible to go back to the dimensions of the page.*"

In the final analysis it can be observed that the students' procedures were very linear and that, when they had worked out what they had to do, they proceeded in a systematic manner without turning back on themselves. Moreover, close attention was paid to the formal aspect and the language.

Mastery of the calculation techniques. The work in groups certainly favoured the accuracy of the calculations and the correct use of the tools of infinitesimal calculus.

The P4 group in class B discussed the parametric inequality without difficulty and identified the solution: "*To obtain the solution we have to study the sign of the first derivative; since the coefficient of x^2 is positive, the solutions of the inequality $12x^2 - 4x(a+b) + ab \geq 0$ are external...*"

Neither was the group dealing with P2 frightened off when faced with an irrational parametric inequality, and after a few steps they were able to reach the solution.

Group P6, class B, found a way to get round the obstacle of the calculations which at first stopped them in finding the zero of the derivative: *"Faced with irresolvable calculations due to the presence of too many parameters, we substituted the parameters with some numbers and we saw that the Δbecame simpler; this device allowed us to understand how to simplify the parametric equation".*

Identification of the correct solution and its evaluation on the basis of the constraints imposed by the problem. While the groups identified the correct solution, they did not always set themselves the task of verifying its adhesion to the data and to the constraints imposed by the problem or of interpreting it in relation to the situation.

This was the case once more with regard to P6: the students found the solutions to the equation which they obtained by putting the first derivative equal to zero and, observing that one of these was negative, they deduced that necessarily the other must be the solution they were looking for. However, once the solution had been found, they were pushed by the teacher to find a different conclusion. After some attempts based on calculations, they finally discovered that the angles α and β included in the outward and return routes to the river were equal!

This last example acts as a cue for reflection: students, even the most capable of them, do not autonomously feel the need to "check the results". Once the solution has been obtained, they seem to forget where it came from and what it represents and what requirement it satisfies. The simplifications resulting from the calculations make them very confident about the accuracy of the result, though reaching the right answer does not always mean that the significance of the problem in all its aspects has been grasped. But the example cited above leads us to understand that the students, appropriately encouraged and guided, are able to go further.

Interpretation of the function as the graphic representation of the problem being studied. Once they had arrived at the function $y = f(x)$ that models the problem, the groups had no problems in representing it on the Cartesian plane, but they were not equally efficient in inferring from the graph informations regarding the phenomenon being studied or in drawing opportune conclusions.

Those solving P4, even in their accuracy and correctness, seemed not to realise that the function made no sense outside the interval $0 < x < 2/b$.

The same contradiction was also fallen into by those dealing with P3, who first rightly observed that: "*From the graph it can be seen that the negative part of the function should not be taken into consideration since the problem requires the dimensions of the page, and hence also its area, and as such this cannot be negative.* Then however they drew the function on all the Cartesian plane and observed that: "*Having positioned x on side AB, it can be seen that by increasing the dimensions of the side the area also increases*. This comment is very vague and expressed in this way is wrong in that it is only valid where the function is increasing.

The P5 group, class B, showed instead to be especially competent in the attention given to the formal language by specifying the detailed steps necessary for the study of the function. In addition, through the following comment they correctly highlighted the part of the graph which made sense in relation to the problem: "*Since the radius is limited to between 0 and 10, the part of the function to consider is that going from 0 to 10. Increasing radius x also increases the difference in the volumes y up to the maximum value. From that value onwards the difference in the volumes begins to fall, in the limit case in which r=0, the difference between the volume of the sphere and the volume of the cylinder is equal to 0"*. Then, in the comment on the graph in connection with the situation represented by the problem, they contradicted what they had just said with some odd observations: "*in making the limit of the function which tends to* $+\infty$ *we are considering the radius of the sphere as if it were infinite and the* *result obtained (-□) allows us to understand that the sphere doesn't allow water to go into the container*".

Therefore, in dealing with the passage from the concrete to the abstract the students did not encounter, or at least so it seems, particular difficulty; on the contrary, they remained perplexed when dealing with the inverse situation. In conclusion, we are able to agree with what has already been observed by A.H. Schoenfeld, who noted that students are competent when it is a question of deducing and when it comes to constructing, but they often compartmentalise their knowledge; so a large part of their knowledge therefore remains unused (Schoenfeld, 1986).

Second experience (class C)

This second experience shared most major findings with the previous one. So we limit our considerations mainly to the

Production of conjectures. This second phase revealed numerous attempts to formulate a priori conjectures.

Those working on P2 were unable to interpret the text and the requirements; after some attempts based on some calculations they observed that "*some data is missing*" and that the "*question is not very clear"*. They even confused the oil pipeline with the route of the tanker in writing*: "We establish that the route of the oil pipeline is without doubt a straight line and that the route in sea is greater than that on land"*.

The students dealing with P3 found the function being studied and, not having a clear idea about how to proceed, arrived at the maxima and the minima only. Feeling the need to be returned to firmer ground, they wrote: "*we determined that it is an optimisation problem and so we decided to study the function to determine the maxima and the minima"*.

Those dealing with P4 put forward a great number of conjectures (not always correct ones). At the beginning they pointed out: "*We thought that as the small squares increased the area of the box diminishes and hence their values are inversely proportional"*. Their drawing shows the correct assumption and shows that they understood that they had to study the function, even though they did not in fact do so and only the calculated the maxima and minima.

Those solving P6 also came up with a good number of conjectures, some fanciful ones like: *"The space covered by the farmer, if we consider that the two villages are both next to the river, and that this latter is straight, does not change depending on at what point of the river the farmer draws the water, at most it would be best to draw the water as near as possible to the destination village"*. The students first set the problem in terms of a straight river, then, realising that the text does not clarify the position of the villages in relation to the river, they examined various hypotheses. At a certain point they began to narrow down the various possibilities, they decided to consider the river as straight and examined several configurations, starting out from the extreme cases. They sought to compare the routes, but since they were unable to reach a conclusion they observed: "*An attempt was made to show that*

AC+BC, where C is a point that varies within the stretch, is least when AB is the largest side of the triangle ABC. In the case in which the villages are equidistant from the river the solution is clearly the two sides of an isosceles triangle.

Students working on P1 discuss giving different values to height and radius of the cylinder. They did not understand that the volume is a constant. When they realized (probably listening to the other groups), that they have to find a minimum, they derived considering the radius as a variable and the height as a constant. They did not understand the result and finished the work. Students working on P5 wrote the formula $\square r^2 h$ for the volume of the cylinder, and $\square r^2(2R)$ for the volume of the water. They gave a lot of values, also, as they wrote, "for the limit cases" and noticed astonished that they have to subtract the volume of the ball from $\square r^2(2R)$. They wrote the correct formula and declared to be finished. The teacher intervened suggesting to look for the maximum volume of the water. The students correctly derived their formula and stopped again, without any comment.

We see that pupils in fact produced hypotheses on the optimal solution. But they did not come back on those hypotheses once they have found the result.

Some teachers' observations of a general character:

-the students in all the classes worked autonomously and faced the problem situation as "protagonists", putting in considerable effort: their enthusiasm over having "done mathematics" in a different manner was visible; instead, a decrease in the level of enthusiasm could be seen, in classes A and B, when they were asked to complete the study of the function and to interpret it in the context of the problem;

-the fact of working in a group made the students more confident with regard to calculations and procedures and allowed the weaker ones to come out of their isolation since it provided them with greater faith in their own abilities;

-the attitudes of the groups when faced with the problem and the challenge that this implicitly involved were of different natures: the "more able" in some way felt the effects of this role allotted to them.

4. FINAL REFLECTIONS

Of course, our experience considered a small number of students, so we will avoid over-interpretations. Even though, as previously noticed, our aim was not the full study of the connections between problem solving and modelling, the activity of problem solving within modelling has undoubtedly had positive didactic effects. Some of these are to be attributed to the problem solving situation, like the effort put into looking for the solution and in writing up the text, the relationship of collaboration in the groups, the recovery of the less able students, and the greater confidence brought about by this exchange, as they themselves declared.

We also consider that it is possible to attribute the relative facility in the use of the parameters to the modelling situation, as for example the fact that pupils were first able to assign arbitrary values to the quantities which came into play.

Yet, in the first experience (classes A and B) there were some findings which make it impossible to judge the contact with concrete problems as being entirely positive: the students worked in a rather "scholastic" manner. They did not, generally, come up with many conjectures: they often easily identified the variable in play and they obtained the value that yielded the maximum or minimum, but they were unable to interpret the function found in the context of the problem posed.

In the second experience a good number of students concentrated on the phase of conjecture and of free exploration of the problem: many hypotheses were put forward even if these were not then verified. On the other hand, once the function had been identified, there was a return to "technicality" and hence a removal from the context.

It is of some purpose to observe that the problems introduced were close (in time) to the treatment of the analysis presented in class: this may in part be the cause of the immediate return on the part of some students to technicality in the second phase of the experience too. In this sense the experience differs from the reinvention approach described by Gravemeijer & Doorman (1999) and can instead be framed within what Blum & Niss (1991) called *applied problem solving*, i.e. a direct application of already developed "standard" mathematical models to real situations.

With regard to the activity of conjecture, we instead maintain that the *didactical contract*, expressed by the different formulation of the problem, played a prominent role. Despite of the unusual kinds of classroom work and of problems, the explicit request of a maximum or minim value reminds students of the classical task of solving exercises in calculus. Even in the second experience the word "maximum" used by the teacher, or simply the establishment of the formula of a function, leads to start calculations and to come back to a well-known way of working. This phase also recalls the use of certain representative registers (Slavit, 1997).

Within the area of analysis a conjecture concerning the optimal values of a variable should help the identification of a function and the interpretation of this in the given context, while proving the conjecture generally consists in a calculation of maxima and minima, which is in effect technical. In any case the students do not seem to be aware of having eventually provided a demonstration of their conjecture, also because they do not see the need for this (cfr. Balacheff, 1991).

In the experiences proposed, the fact of dealing with problems related to concrete contexts did not contribute to giving a semantic value to the study of the function, but the formulation (in the second phase) of the same problems in an "open" manner favoured the activity of conjecture. However, this too proved not to be sufficient to "close the circle" and to return, after the study of the function, to the initial problem, and hence to the validation of the hypotheses initially advanced. It did however permit reaching the optimal solution even without an explicit initial request, and describing on paper the reasoning followed in a direct, concise and linear manner.

Moreover, perhaps it is our mentality which leads us seeing the process concluded only with the contextualised study of the function. In fact, it is not always the case that the procedures based on visual representations, or generally a visual approach, are the simplest in terms of learning (Furinghetti, 1992; Presmeg, 1986); the possibility of guessing the results on the basis of analogy and generalisations that involve visual registers can indeed be a delicate operation (Bagni, 2000) and the distinction between this process and the processes of abstraction should be clear.

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TO APPEAR AND TO BE: MATHEMATICIANS ON THEIR STUDENTS' ATTEMPTS AT ACQUIRING THE 'GENRE SPEECH' OF UNIVERSITY MATHEMATICS

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Abstract: *Mathematics has its own 'genre speech' (Bakhtin, 1986) and success at university level is often linked with its acquisition. We conducted a series of themed Focus Group interviews with mathematicians from six UK universities. Pre-distributed samples of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of bibliography were used to trigger an exploration of pedagogical issues. Here we elaborate participants' reflections on samples of data in which students attempt to express, in what they perceive as formal mathematical writing, a relatively simple, or even obvious, fact. We conclude with a brief reference to the participants' views on how teaching at university level can facilitate students' appreciation and adoption of formal mathematical practices*

Keywords: undergraduate mathematics education, mathematicians, genre speech of university mathematics, collaborative research.

The discussion presented in this paper[1] originates in a study which engaged mathematicians from across the UK as educational co-researchers; in particular, the study engaged university lecturers of mathematics in a series of group interviews, each focusing on a theme regarding the teaching and learning of mathematics at university level that the literature and the authors' previous work acknowledge as seminal. Discussion of the theme in each interview was initiated by a Dataset that consisted of: a short literature review and bibliography; samples of student data (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of the authors' previous studies; and, a short list of issues to consider. Analysis of the interview transcripts largely followed Data Grounded Theory techniques and resulted in thematically arranged sets of Episodes – see elsewhere (e.g. Iannone & Nardi, 2005) for details on the study's rationale and methodology.

The data presented here originate in Episodes from the discussion of the theme *Formal Mathematical Reasoning: Students' Perceptions of Proof and Its Necessity.* In these the discussion revolved around:

[1] This text originates in an early draft of a book that Elena Nardi is currently working on. It is a part of a longer paper that will appear in Portuguese in (Nardi, in press). For an interpretation of the data presented in this paper with a focus on the difficult transition from school to university mathematics see (Iannone, 2004).

- the mathematical question in Table 1, a question given to Year 1 students in the early weeks of their course. ‘Suggested solutions’ are distributed by the lecturer once the students’ work is marked:

Example from Exercise Sheet 1, Week 2, Autumn Semester 2000

Let $x \in R$ have the properties that $x \geq 0$ and $\forall n \in N$, $x < \frac{1}{n}$. What is x?

Suggested solution

Adequate answer for now: $x = 0$. Better answer: since $x \geq 0$, if $x \neq 0$ we must have $x > 0$. But if $x > 0$ then for n large enough (bigger than $\frac{1}{x}$), $\frac{1}{n} < x$, which contradicts the assumption about x. So x must be equal to 0.

Table 1

- Two examples of students’ written responses to the question (Student J and Student W). The responses discussed here were selected from a pool of responses by sixty Year 1 mathematics undergraduates whose writing we had collected as part of a previous study:

2 $x \in \mathbb{R}$, $x \geqslant 0$, $\forall n \in \mathbb{N}$, $x < 1/n$

as $n \to \infty$ $x \to 0$ hence for $\forall n \in \mathbb{N}$ $x = 0$

Student J

And:

~~2) $x \in \mathbb{R}$ $\{x \in \mathbb{R} \mid x \geq 0, \forall n \in$~~

2) $\cancel{x \in} A = \{x \in \mathbb{R} \mid x \geq 0, \forall n \in \mathbb{N}, x < \frac{1}{n}\} = \{0\}$

Student W

- And, finally, a short list of issues for consideration by the participants prior and during the interview (Table 2):

Examples of issues to consider:

This exercise was given to the students in the second week of their first semester at university. The question does not ask explicitly for a proof that $x = 0$; it simply asks *what is x?* Still, can we infer anything regarding the students’ beliefs about proof from their responses? In our sample most students identified zero as the value of x; several students simply offered $x = 0$; others attempted a justification of the claim.

• What can we infer about student W's employment of set-theoretic language above? Why would he choose to write his answer in set-theoretic language?
• What can we infer about student J's employment of convergence symbolism above? Why would she choose to write her answer in such as way?

Table 2

Discussion in the interviews was broadly in three parts: an examination of the question setter's pedagogical intentions; an examination of student responses to the question and identification of problematic aspects of their learning as evident in their writing; and, an exploration of pedagogical practices towards assisting the students with overcoming these difficulties. Here we briefly outline the first part and elaborate the second and the third.

The discussion amongst the participants sets out from the concrete context of a specific mathematical question and examples of student writing. However it soon becomes about an issue that is commonly known to cause difficulty amongst mathematics undergraduates in the beginning of their studies (e.g. Sierpinska, 1994): the need to *appear* (use the norms of formal mathematical notation such as quantifiers, the equals sign etc.) and *be* (use the norms of formal mathematical reasoning, such as provide justification / proof for claiming the truth of a statement) *mathematical* in their writing. This transition in the participants' discourse from the specific to the general was a strong characteristic of the data. As the power of this discourse – rarely have there been studies in which mathematicians' pedagogical discourse is explored in such microscopic ways – seems to lie partly in these transitions, we believe that the methodology of the study succeeded not only as a forum of exploration of mathematicians' pedagogical perspectives but as a basis for collaborative educational research.

Before we launch into a presentation of the participants' views on their students' writing, let us outline a caveat expressed by some of them concerning the mathematical question (Table 1) that initiated this discussion: the question setter's intentions, as outlined to us by the lecturer who taught the course and was also one of the participants (first: an opportunity to develop an intuitive insight into real numbers and the limiting process[2]; and, second: an opportunity to practise Proof by Contradiction) may not exactly have come across with transparency to the students – especially the latter one. The participants suggested that this gap of communication with students is a common source of concern generally and, specifically here, it may

[2] The question-setter elaborated this as follows: setting this question is also a good way for identifying those students who imagine that x might just be an incredibly small positive number, smaller that anything that can be thought of - as literature also suggests (Tall, 1992) a surprising number of them in fact do think that. This, he continued, is the context in which a statement like 'x might be something like one minus point nine recurring' is a fantastic opportunity to open the door to the whole of Analysis. Another motivation for him was to see the students' reaction to something like that, prior to their introduction to the Archimedean Property; to see whether they can convince themselves that they believe the Property without even hearing it stated.

have skewed the students' understanding of what response to this question was expected of them.

Indeed a scrutiny of the student responses revealed that a few students did use an argument by contradiction but we were alerted by the question-setter and other participants, some of whom are also advisers to these students, that that was the case often only after consultation with their adviser; that, for most students, it didn't even occur to them that this was a method of proof; or, that this was a mathematical question at all! But, we wondered, the question doesn't really seem to be asking for a proof, does it? To this the question setter responded that a proof would need the Archimedean Property and that if the grounds are not clear, students may get confused about what is asked of them - an idea we also discussed in (Nardi, 1999). The presence in the 'suggested solution' of at least two ways of providing an answer that is satisfactory is more or less the equivalent of admitting that things are pretty ambiguous as to what is expected, at least at this stage. Overall the question setter's evaluation of the exercise of setting this question was as follows: the question is doing a slightly better job at developing intuition on real numbers, but doesn't necessarily help the understanding of the game of proof very much.

Note to the reader regarding presentational conventions in the text in the following sections: the participants' reflections are presented as 'quotations', indented, in smaller font and in inverted commas. The text within these 'quotations' is a consolidation of verbatim quotations from across the board of our twenty participants. The rest of the text represents our own voices as the researchers and authors of this paper. The references to literature, attached either to 'our' text or the 'quotations' aim at highlighting places where we believe there is resonance between our view - and / or the participants' views - and other relevant works. To suggest that one unified perspective on us, the literature and the participants is possible – or even desirable – would be facile and deprive the conversation our study wishes to contribute to of the richness that often emerges from difference. The aim of the paper is two-fold: to contribute to the substantive conversation regarding students' adoption of formal mathematical expression by bringing their teachers' interpretations of some the difficulties involved in this adoption; and, to represent the complexity and sensitivity of the pedagogical perspectives demonstrated by the participants in the course of these interviews (a research focus currently gaining momentum within undergraduate mathematics education – e.g. (Weber, 2004)) - see concluding paragraphs for a rationale for the second of these aims. We believe that it is crucial that readers approach this work and the work with which we hope to follow it up having this multi-layered set of intentions in mind.

MATHEMATICIANS' REFLECTIONS ON THEIR STUDENTS' WRITING

We now turn to the participants' discussion of the students' responses. The discussion was triggered by an observation in our own analysis of the students' scripts: in resonance with similar work in the area – according to which students do not necessarily recognise the need for proof (Dreyfus, 1999), but they recognise that we want them to give a proof (Almeida, 2000) – this analysis had demonstrated that most students stated that x = 0. Many provided no justification but a significant number attempted justifications such as the ones in the two examples above. We wondered what this fact implies about the students' ideas at this stage of what is an acceptable mathematical argument.

> 'Most of them do not just say x = 0: they try to say this in different ways. This is something more than just stating the answer. Still they do not necessarily know what to do. After all they are used to being asked to show their workings in school, not provide logical explanations. There is a switch from the one to the other that needs to take place at this stage of their studying (Iannone, 2004). Student J's and Student W's responses have something in common: in both cases the students are having a go at giving something that looks like an argument but they are terribly greedy at using the symbols that are provided in the question and are reluctant to provide anything extra. Sometimes they seem to think 'this stuff must be hard so we have to write longer answers' but most of the times they provide short answers (Pugalee, 2004) when their lecturers, mathematicians themselves and markers of the work, want more.'

In spite of a widespread impression that this is a problem pertaining mostly to Year 1 students our participants insisted the problem probably carries on beyond Year 1 and elaborated what also Anna Sierpinska (ibid) recognises as a misunderstanding of the uses of mathematical writing:

> 'It is difficult to convey to students that a string of symbols is not a sentence in English: so, if you throw in a few words in English, why would you expect anyone to make sense of it as if it *is* in English? We communicate mostly in sentences and it is reasonable to expect the students to communicate their mathematical ideas in this way too. However, when confronted with this idea, they often say that this is not what they think is expected of them; that the expectation is for producing strings of symbols, stripped of ordinary words as much as possible; and, that they just drop the words which appear to them to be obvious or superfluous. Overall there seems to be a misunderstanding here about what mathematical writing means.'

And regarding the origins of this misunderstanding:

> '…these may lie in school where the English around the mathematical writing is often seen as irrelevant: what is marked as right or wrong is the calculation part of a student's response and the rest is ignored (Nardi, Iannone & Cooker, 2003). As a result, students become accustomed to dropping the sentence in which their mathematical statement is embedded and end up producing a text that, stripped of rudimentary syntax and punctuation, is in fact unreadable. But in school this same text would probably achieve full marks. And it is very difficult to get them to see the logical framework of a proof, the shape of an argument separately from the little calculations. Especially when the calculations are absolutely trivial and the triviality may trick them into circumventing the logical flow of the argument. Admittedly, professional mathematicians do this sometimes when we present trivial results to students because we ourselves have

compacted the argument into an apparently obvious and concise statement (Burton & Morgan, 2000). But students still have to understand the logic behind it all and therefore have to stick, for a while, with these often longer, unpacked statements.'

The above seem to be the case here with the more or less obvious fact that x must be zero.

'Student W's extraordinary and slightly weird response is a good example. Something is going on in his mind, just before that final equals sign. But he hasn't made the point, he hasn't written it down. Where is the sentence which makes the point underlying the equals sign? He was probably under the influence of some introduction to Set Theory notation in the first weeks of the course and casually drifted into using this notation. It is impressive also because it is syntactically correct but it is also symptomatic of the idea that putting a response somehow in mathematical language makes it more convincing.'

In the rest of the data the case of another student, who presented his point in terms of Mathematical Induction, was an extreme demonstration of the above idea.

'This may give the impression of essentially random mathematics. It is not even certain that by the end of this the student actually believed x is zero. The student is probably taking things from recent lectures and applying them. Or maybe it is a case of previous questions asking students to use Mathematical Induction and they arrive at this one assuming it will be another one of those questions and force their answer to fit the Mathematical Induction mould.'

As for the rationale behind finding a mathematical mould that students think fits the lecturer's expectations of them, the participants summarised it as follows:

'Students do not wish to take the risk of explanation in ordinary English because, the more they write, the likelier it is they will get marked down for the wrong explanation. So, they provide the answer - zero is, after all, the right number here - and are as economical as possible with justifying this answer. They thus minimise the risk of having their logic faulted by the marker. Furthermore they often just provide the answer also dressing it up in, for example, set theoretical language, like Student W's quite impressive and perfectly inscrutable answer! What is being said here is probably that it is obvious that the answer is zero, but the student, aware that saying 'obviously' is not sufficient, pursues a presentation that has credibility.'

When we suggested that another difficulty here, touched upon earlier, may be that it is hard to convince students to unpack the argument behind statements they make, especially ones they perceive as obvious (Housman & Porter, 2003) reactions were as follows:

'In fact the more trivial the mathematics, the tougher it is to convince them to do so. Say you set a triangle, give the two angles and ask for the third. They know that there is a calculation that they have to do. The bright students just write down the answer because they have that calculation straight away in their heads. Persuading them to write down the trivial calculations is tough. This is symptomatic of the same thing going on here.

> The brighter students are unable to write down the convincing part of the answer. They lose track of the fact that their task is to communicate this conviction. And that the argument, of course, in mathematics is at least as important as the number in the end.'

Responses like Student W's almost beg the question what the student actually means here.

> 'The absence of the answer to this question is what is frustrating about the monologue within a piece of homework like this and what brings about a longing for more interaction with the student *[see also concluding section]*. Very similarly to the questions left hanging within Student J's response and her resort to the language of limits. One speculation about the origins of her response is school and some primary contact with the language of convergence which makes her feel comfortable with using it. As with Student W's writing, it is impressive even though it doesn't seem to mean much. The presence of 'hence' and the quantifiers may make one think that rescuing this script is actually possible.'

Returning to the earlier point of the importance of inculcating in the students the need to engage with more verbal mathematical writing (Pugalee, ibid.), some students did attempt that, for example Student LF. This student seems to have thought through the answer that x is zero a bit more than, say, Student J who also seems to think that $1/n$ can get very, very small.

2) $x \in \mathbb{R}$ have properties $x \geq 0$
and $\forall n \in \mathbb{N}$, $x < \frac{1}{n}$ ∴ n can be infinitely small
∴ x smaller than all fractions
∴ $x = 0$

Student LF

Student LF's '*n* can be infinitely small', '…smaller than all fractions …' are quite articulate expressions of her attempt to grasp the idea of why x must be zero. One may also see a vaguely similar attempt in Student L's response below where there is some trace of engaging with an argument by contradiction with the '?' and 'suppose':

2) $x = 0$?
Suppose $x > 0$ $n > \frac{1}{x}$ $\frac{1}{n} < x$.
∴ $x = 0$.

Student L

> 'Indeed in these two responses there is an effort to communicate which Students J and W seem to have forgotten: more so for Student LF who does not seem to mind the risk of putting in a possibly wrong explanation. Of course one should not overlook the weaknesses in her script either: this would be more promising if 'therefore *n* can be infinitely small' bit was not there; if 'because' was there instead of 'therefore'; if *n* was

> x; and, if commas were used to convey the structure of the statements! The overall point is that students need to be able to express what is probably not too problematically (in the case of Student LF at least) in their minds already.
>
> Technically speaking there is little in these responses to suggest any syntactic structure: no commas, no beginning of sentence, no full stops. Is this a transcription, one may wonder, of the sort of ways of writing-what-you-think in some unedited fashion? Which is clearly not in terms of sentences. Where does this way of writing come from? There might as well be here a parallel with doing research mathematics: there we are writing some odd notes, grappling with a problem, and we are not terribly fussed whether someone will ever see it; it is not the polished version (Burton & Morgan, ibid.). Then we experience something similar to what some of these students experience: having to express this thought in a presentable manner; only that, for them, this is grappling with the unknown. It may also be the case that students forget that their script is ever going to be read and marked by another human being. So they resume their efforts by merely presenting the bits and pieces floating around in their minds without fully complying with the obligation for presentability.'

There seems to be a longing in these words for a mathematical writing consisting of fully fledged, clear sentences - the type of writing that Social Sciences or Humanities students are more encouraged and expected to develop perhaps?

> 'It is the teacher's obligation to convey to students as clearly as possible the benefits of communicating their mathematical thoughts in this way. And this persuading can be a long and painstaking process which also needs to be systematically carried through.'

So, confronted with the difficulty of this task, what can one do?

MATHEMATICIANS' REFLECTIONS ON STUDENT-SUPPORTIVE PEDAGOGICAL PRACTICES

Faced with the inscrutability of the writing in the above examples, a longing emerged from the participants' discussions for more interaction with the student, for a dialogue in which when the student claims that 'x is obviously zero', the teacher is there to ask why, to encourage them to convince themselves and others (Mason, 2002) that they really believe this. The overarching proposition made by the participants was that for fostering in students the significance of mathematical literacy through increasing interaction with them. Pedagogical practices that can facilitate the students in doing so included (we elaborate some of these practices elsewhere, e.g. in Iannone & Nardi, 2005):

- the "editorial" work a teacher does on a student script

> 'It is important to put a positive spin on what one reads, see through what the student does, sort out the things which are correct and smooth out the less useful pieces of reasoning. Scripts like the ones above are alarming in their lack of coherence but, in responding to them, one should look for that little nugget of understanding and then work backwards from that: for example, insert the key linking words that are missing

and suggest that in following submissions of homework the students should consider the use of 'if', 'then', 'it follows that' and so on.'

- exposing the students to the weaknesses in their prose, for example, in a deliberately less pressurised-for-marks environment of a group tutorial

'Exposing the students to the weaknesses in their prose – for example by asking them to read aloud what they have written in a group tutorial – can also be very effective as the students immediately start to question their writing. A less pressurised context for doing that is working with students on exercises which are not going to be assessed, to build up their confidence and get to know how to approach what will be assessed work. This does involve of course some extra preparation as well as going around the class and giving students verbal encouragement and indications on what they should be writing down in that type of exercise. Being over their shoulder while they write is a very intimate experience, for both student and teacher, and it allows immediate access and response to how they write: it gives them that little bit of contact at the right time - not too early and not too late.'

- negotiating the polishing of students' prose, in public, for example on the board, until it reaches a commonly acceptable level

'… *[in case the students appear not very comfortable with this otherwise potentially effective strategy]* they can get flustered a bit to start with but they soon get used to it and begin to appreciate the opportunity. But here is a cautionary note on the matter: there is always the danger that, in talking to them, basically the teacher tells them what to write down. So they may respond to a request for verbalising their thinking in their writing by inserting 'more words in between' because they are conditioned to respond to a teacher's request, but the significance of this request may escape them. And, given their school background[3], they probably feel that they should always be calculating and that it is the calculations that count. Gentle pushing in a public arena (the blackboard in a tutorial for example) until they write down a coherent argument would be one way of fostering in them the significance of mathematical literacy (Bullock, 1994).'

- emulating the significance of transparent, elaborate and coherent mathematical writing in the teacher's own prose

'Our own prose needs to reflect the expectations from the students' prose: omitting parts of the writing because it is naff, or too trivial (for a more expert reader) in the suggested solutions distributed to the students is a bit lazy and short-sighted in terms of effecting change on the students' approach to mathematical writing. In fact, we often ask students to do (and write) something trivial in order to stress that we ask them to look at something they have always been doing – in their graphic calculators, on MAPLE etc. – from a philosophically different perspective; to engage with something that looks deceptively familiar but to do so in an almost totally unfamiliar way (Sierpinska, ibid.).' *[In the question in Table 1, for example, the students are asked to engage with*

[3] See http://www.nc.uk.net/index.html for information on UK's National Curriculum. Also: Nardi, Iannone and Cooker (ibid.) for more views from UK mathematicians on this.

something they are only now beginning to experience: the need to justify even the most obvious statement; the need to get used to the idea that just finding the answer is not enough anymore]

Some participants offered the last point particularly with a caveat:

> 'The importance of distributing impeccable suggested solutions to students should not be overstated though: a text, and, in particular, a teacher's text, is a distillation of mathematical thinking that, by the time it reaches the student, is almost dead; it has become part of script. The main strategy for altering students' perceptions of what is acceptable mathematical writing is painstaking interaction where they are offered the opportunity to see that mathematics is about arguments; and to see these argument coming together.'

IN THE WAY OF CONCLUSION

The above presentation relies heavily on the 'quotations' (see Note earlier in the paper) from the participants. This is not merely a stylistic choice: it serves as a reminder of the overall intention of the study to contribute to the highly needed rapprochement of the worlds of mathematics and research in mathematics education. These two worlds – whose members are both intrigued by and having a commitment to the improvement of mathematical learning – need to meet, confer and generate negotiated, mutually acceptable perspectives more often. Through a demonstration of the rich pedagogical perspectives that are evident in these 'quotations' this heavy- on-data presentation is intended also as a response to stereotypical views (Holton, 2001) that see practitioners as non-reflective actors who rush through content-coverage in ways often insensitive to their students' needs and have no pedagogical ambition other than that related to success in examinations and audits (analogously these stereotypes also see researchers as irrelevant theorisers with a suspiciously loose commitment to the cause of mathematics and incapable of 'connecting' with practitioners). In the realm of these stereotypes mathematicians and researchers in mathematics education, oblivious to each other's needs, skills but also idiosyncrasies of their respective epistemological worlds, have no choice other than to remain indifferent, and even hostile, to each other (Sierpinska & Kilpatrick, 1998). Through the presentation we have attempted in this paper, the effect is intended to be a hopefully not too unrealistic proposition: that of partnership (Jaworski, 2003).

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RELATIONSHIPS BETWEEN INFORMAL AND FORMAL REASONING IN THE SUBJECT OF DERIVATIVE

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Abstract: *Formal mathematical knowledge is often interpreted in an informal way. A common way to do this is visualization. This research consists of task-based interviews of two university students majoring in mathematics. The goal was to study students' understanding and utilizing of connections between the formal and informal sides of mathematics in the case of the derivative. The results reveal that the students understood the informal interpretations of the derivative and the difference quotient well but had difficulties in understanding the visual meaning of the limiting process of the difference quotient. One of the students tended to use informal reasoning a lot and also used it simultaneously with the formal reasoning. The other student used formal reasoning more frequently but did it often separately from the informal reasoning.*

INFORMAL AND FORMAL SIDES OF MATHEMATICS

According to Fischbein (1994, 231) mathematics should be considered from two points of view: on one hand mathematics is formal, deductive rigorous body of knowledge and on the other hand mathematics is an activity of a human. Theoretical mathematical knowledge can nowadays be considered as a *formal axiomatic system* with no dependency on empirical world. The basic tools of this system are concluding rules of mathematical logic, undefined elementary concepts, axioms, definitions of other concepts and theorems with their proofs. However, despite of the formal nature of the ultimate mathematical knowledge, the process of working with mathematics is often very informal. When deciding what is the ultimate truth the formalism is in the conclusive situation, but in inventing new ideas and in applying mathematics one needs reasoning based on intuition, associations, mental images and illustrations. This is the informal side of mathematics.

In mathematics - especially in advanced mathematics - it is possible to argue things both by informal and by formal methods. The formal methods usually mean direct communication with the formal axiomatic system of mathematics. So, in the formal arguing it is possible to use only axioms, exact definitions, earlier proven theorems, undefined elementary concepts and fixed mathematical logic. A formal proof done by this kind of method can be considered to be an absolute justification and mathematical knowledge obtained this way can be considered as an objective truth, *but only relative to the formal axiomatic system* (Goldin 2003, 187). So, the truth value of a mathematical proposition is independent from individual preferences or interpretations, social conventions, negotiations or subjective conceptions (ibid. 187), but formal arguments have the conclusive role in deciding what finally is true and what is false.

Multifaceted *internal representations* of concepts are important in mathematical thinking, because they help to make sense of abstract mathematics. According to Goldin (1998) internal representations can be verbal/syntactic, imagistic, formal notational, strategic/heuristic or affective mental assosiations of concepts. A pure formal thinking is based only on formal notational representations, but in informal mathematical thinking any kind of internal representations can be utilized. However, informal thinking and informal arguments lack the level of rigor.

Informal mathematical thinking requires that one gives informal meanings to formal definitions of concepts and formal mathematical claims. These meanings can be called *informal interpretations.* Usually informal interpretations help to make concrete the meanings of the formal definitions and the formal claims, but they -like informal arguments- at least to some extent lack the level of rigor and so they are not included in the formal axiomatic system of mathematics. Informal interpretations have very significant role in informal thinking and in informal arguing: informal arguments are very often based on informal interpretations.

Visual thinking is probably the most usual type of informal thinking in mathematics. Visualization can be considered as a method of seeing the unseen (Arcavi 2003). In problem solving visualization supports and illustrates formal results, resolves possible conflicts between formal solutions and intuitions and helps in re-engaging and recovering conceptual underpinnings which may be easily bypassed by formal solutions (ibid. 223-224). Visualization -as informal thinking in general- has a very significant role in inventing ideas in problem solving situations. Also mathematicians draw diagrams and use them in a systematic way when attempting to solve problems unknown to them (Stylianou 2002, 315). The study of Stylianou reveals that in the problem solving processes of mathematicians steps of visualization and analysis take turns very systematically and are strongly connected to each other (ibid.). But many students cannot utilize their visual representations to advance in their problem solving (ibid. 315).

Raman (2002 & 2003) has separated *private and public arguments* in mathematical thinking. Raman defines private argument to be an argument engendering understanding and public argument to be an argument being sufficiently rigorous for a particular mathematical community (Raman 2003, 320). This is quite similar classification to informal and formal: informal justifications are often built by private arguments, whereas in formal arguing usually public arguments have to be used. According to Raman, private arguments are usually based on heuristic ideas and public arguments are, in fact, procedural manipulations from assumptions to the claim. She defines a *key idea* to be the factor connecting private and public arguments: a key idea is a heuristic idea which one can map to a formal proof (Raman 2003, 323). According to Raman s research (2002), in proving processes mathematicians can find the key idea, but students usually consider private and public arguments separately and so they do not have a key idea.

GOALS AND METHODOLOGY

The goal of this study is to investigate the relationships between informal and formal mathematical thinking. The data of the study consists of videotaped task-based interviews of two university students majoring in mathematics. The tasks used in the interviews concerned the derivative and differentiability of real-valued functions of one variable.

The research questions of the study were:

1. How well do the students understand the connections between the formal definition of derivative and some informal interpretations of it?
2. How do the students use informal and formal reasoning when they solve mathematical problems?

In question 1, the goal was to investigate how the students interpret informally the concept of derivative and how well they understand the reasons why these interpretations are justified. So this included also the question how do they interpret informally the concept of difference quotient and the limiting process in the definition of derivative. In question 2, the goal was to investigate in what kind of situations in problem solving processes the students use formal reasoning and in what kind of situations they use informal reasoning, what kind of difficulties students have in the formal and in the informal reasoning and how do the formal and the informal arguments influence each other.

The students were interviewed one at a time and they were asked to think aloud during the interviews. Blank papers and a pen were provided. The goal of videotaping was to investigate what, when and how do they write or draw on paper and so the videocamera was focused on the paper. The definitions of derivative, differentiability and continuity were given to the students, because this was hoped to help students to concentrate on things essential to the research questions. The definition of continuity was given in the form based on limits, but so-called epsilon-delta -definition was not given.

The interviews were based on four tasks, which are presented in the next chapter. After Task 1, there were short discussions about the relationships between continuity and differentiability and before Task 4 there were also short discussions about visual meaning of eveness and oddness of a function. In the interviews the interviewer did not comment or revice work of the interviewee, but he could ask some additional questions or ask the interviewee explaining more detail.

Both interviewees were majoring in mathematics. In the following chapters the names Anna and Ben refer to them. Anna has studied 120 and Ben 99 credits (ECTS) of mathematics and both have also succeeded quite well in their studies.

TASKS AND DESCRIPTIONS OF PROBLEM SOLVING PROCESSES

In this chapter and in chapter Results some excerpts about the students speech are presented. These have been translated from Finnish to English, but we have tried to keep the contents unchanged.

Task 1: *What does the limiting process in the definition of derivative mean visually?*

Anna: Anna explained the same thing first without a picture and then with a picture. In both explanations she began by fixing h in the definition of derivative and then interpreting difference quotient as the slope of the secant line. Then she explained that when h becomes smaller the secant turns so that it becomes the tangent.

Ben: Ben began by drawing a picture. He drew a graph of a function and two points on it (not the secant line). He pointed first to the difference between the values of the function, then to the difference between the values of the independent variable and explained that by using these differencies the slope of a line could be formed. (He did not mention the word secant.) Then he simply explained that when h in the definition of derivative approaches zero, the line becomes tangent.

Relationships between continuity and differentiability

The interviewer asked if all continuous functions are differentiable or if all differentiable functions are continuous. Both students answered without hesitating that differentiability demands continuity but not vice versa. Then they were asked to give an example of a continuous but not differentiable function. Both students gave the function $f(x) = |x|$. Anna s argument as to why this function is not differentiable was that its graph has a corner. Ben s argument was that the left-hand and right-hand limits are not equal at zero. Ben did not specify if he spoke about limits of the difference quotient or -for example- limits of the derivative.

Task 2: *Why the function in Figure 1 (function whose graph has a jump) cannot be differentiable?*

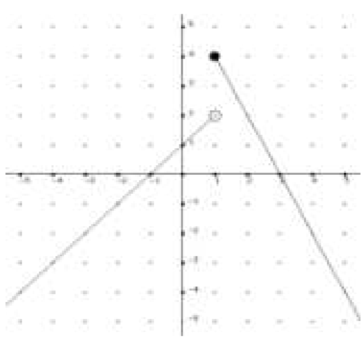

Figure 1: The graph used in Task 2.

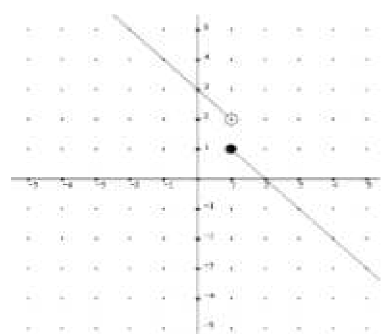

Figure 2: The graph of the kind the interviewer drew to Ben in Task 2

In this task it was not allowed to use the relation between continuity and differentiability.

Anna: The first argument given by Anna was that it is not possible to draw a unique tangent line at the point where the graph has a jump. She did not reason further. Anna s second argument was that at the point of discontinuity the left-hand and right-hand limits of the difference quotient are not equal and so the limit does not exist at this point. It is not clear, how she concluded the left-hand and right-hand limits of the difference quotient to be unequal.

The interviewer asked Anna to explain at which phase the attempt to determine a limit of the difference quotient at the point where the function has a jump would fail. This was a difficult question for Anna. After a long thinking she said that there was no problem in determining the right-hand limit, but she was not able to say what the left-hand limit is. However, she thought it to be unequal to the right-hand limit.

Ben: Ben spoke again only about limit . He explained that, because the one-hand limit is negative and the other-hand limit positive at the point where the graph has a jump, the limit does not exist and so the function is not differentiable. This refers to Ben confusing the limit of the difference quotient for the limit of the derivative. Then the interviewer asked Ben to explain how to determine the left-hand limit at this point. Ben explained that on the left-side of the point limit is everywhere constant. He spoke about negative h and how it approaches to zero, but he also spoke about slope which remains a constant.

”That h approaches, it does not mean anything, the limit is constant everywhere. If we think by this previous method, so that we move the comparision point. In this case we take the left side and so h is negative and then if we move it towards zero until it doesn't reach zero the slope will remain constant, because it's a straight line.”

Then the interviewer drew a graph of a function having equal left-hand and right-hand limits of derivative at the point of discontinuity (see Figure 2) and asked Ben to explain why this function is not differentiable. This question seemed to be difficult for Ben. He took the definition of derivative and concluded on the basis of the graph that the numerator in the expression of difference quotient approaches some non-zero value when h approaches zero from left and zero when h approaches zero from right. This embarrassed Ben. His argument for the question was further that the left-hand and right-hand limits were not equal, but he was not able to say anything more.

Task 3: *Let f: R -> R.*

$$f(x) = \begin{cases} x^4 \cos(1/x^3), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is the function f differentiable?

Anna: Anna realized that outside zero the function is differentiable because x^4 and cosine are differentiable. She tried to calculate the value of the derivative in zero, but she did not find any suitable method for doing this. Then she decided to explore continuity of the function at zero. She did this because she knew that if the function is discontinuous then it cannot be differentiable.

Anna tried to find the limit of the function at zero. After a short thinking she realized that when x approaches zero x^4 approaches zero, but $\cos(1/x^3)$ oscillates between -1 and 1 and so it has not limit. It was very difficult for Anna to conclude how the product of these functions behaves near zero. Finally Anna said that due to the oscillation of the cosine the product function has no limit at zero and so it is not continuous.

Ben: Ben began by drawing the graphs of x^4 and $\cos(1/x^3)$. Then he concluded that near zero x^4 flattens the product function. On the basis of this Ben concluded that the product function is differentiable at zero. He also said that it is differentiable outside zero. When the interviewer asked Ben to argue the differentiability at zero more detail, he realized that by his argument only the continuity was reasoned.

Ben tried to find a visual argument for the differentiability at zero, but he did not manage to find it. Then he began to calculate the value of derivate at zero by using the definition of derivative. He managed without difficulties to get the expression of difference quotient at zero to the form $h^3\cos(1/h^3)$. As above he concluded that near zero it is flattened to zero by h^3.

Task 4: *Even function:Graph of the function is symmetric with respect to the y-axis.*
Odd function: Graph of the function is symmetric with respect to the origin.

Let G: R -> R be a differentiable even function and such that g'(1)=2. What can you say on the basis of this about the value of g'(-1)?

Anna: Anna was little uncertain about the meaning of symmetry with respect to origin, but when the interviewer explained its visual meaning to her, she said that she understood it.

First Anna drew an example of a graph of a differentiable even function. She drew a parabola. Then she drew a tangent line at the point x=1 and another tangent line at the point x=-1. She concluded the general result that derivatives at equal distance but on opposite sides of the y-axis have to be opposite numbers. She made this conclusion only on basis of the picture and her first argument was that it seems to be so . Then Anna analyzed the visual meaning of eveness and became convinced that her conclusion was correct. However, she was not able to explain carefully her arguments.

Anna understood that the conclusion had to be argued also by using the formal definition of derivative. However, she did not see how the proof should go. Anna wrote -without difficulties- the formal definition of eveness of a function (f(-x)=f(x)). Anna tried to prove the result in the general form (not only at the points 1 and -1). She wrote the expression of the limit of difference quotient first at the point x_0 and then at the point $-x_0$. By using the definition of eveness of a function she got the latter to the form

$$\lim_{h\to 0}\frac{f(-x_0+h)-f(x_0)}{h} \qquad (*)$$

Anna realized that this should be $-f'(x_0)$. She explored the picture and also tried to do some formal manipulations, but she did not manage to solve the problem. Finally Anna discovered that changing the sign of h in the expression of $f'(-x_0)$ would solve the problem. Anna tried by using the picture to explore if this change was justified. She interpreted that the minus sign in front of h means that the approach to the point $-x_0$ (or the point $(-x_0, f(-x_0))$) happens from left and she tried on basis of the picture to find out if it was allowed to change the sign. However, Anna did not find any sufficient argument.

Ben: Ben appeared comfortable with the definitions of oddness and eveness of a function. Also he began by drawing a parabola as an example of a graph of a differentiable even function, drawing a tangent line first at the point x=1 then at the point x=-1 and concluding that f (-1) has to be -f (1). Then Ben began to make up a formal proof for the result. He wrote the formal definition of eveness in the general form f(a)=f(-a). He decided to prove the claim also in the general form and began to explore the relationship between the derivatives at the point x=a and at the point x=-a. Ben had difficulties to see how the proof should go. He wrote f (a) by using the definition of derivative and by using the definition of eveness he got the form

$$\lim_{h \to 0} \frac{f(-a-h) - f(-a)}{h}$$

Then he was not able to continue. The interviewer asked what would be the form he wanted. Ben said that the wanted form would be -f (-a) and he wrote it by using the definition of derivative:

$$-\lim_{h \to 0} \frac{f(-a+h) - f(-a)}{h}$$

Ben could not realize the connection between these expressions. Finally the interviewer suggested a change of variable to Ben and then after short thinking Ben realized that the problem will be solved by substituting j=-h.

RESULTS

Students had strong informal interpretations of the derivative and of the difference quotient, but not of the limiting process in the definition of derivative

Both students understood very well that visually derivative means steepness of the tangent line. This becomes clear especially from their responses to Task 4 where both students concluded visually that the values of the derivative on the opposite sides of but at the same distance from zero have to be opposite numbers. Their arguments were based on the positions of the tangent lines. Also in the responses to the other tasks it becomes clear that both students had a very strong mental connection between the concepts derivative and tangent .

The students also had a strong mental connection between the difference quotient and the secant. This is shown by their responses to Task 1. (Ben did not mention the word secant, but he had the idea of it.) Students explanations and drawings in their solutions show that the students understood (they were able to explain reasons) that the difference quotient in the definition of derivative means visually the slope of the secant line.

In their responses to Task 1 both students explained briefly the visual meaning of the limiting process in the definition of derivative. The only thing Ben told about this process was that the secant (a line in Ben s articulation) becomes tangent. Anna explained a little more:

”And now when we begin to take the limit, h will become smaller and so the secant will begin to turn and finally it will wrap to a tangent so that there will be only one point where it will touch the graph.”

Anna s response to Task 1 (see also the excerpt above) shows that she understood the situation in figure both when h was different from zero and when the limiting process was completed. (The graph which she used was curved and so the tangent had only one point of intersection with the graph.) But she had problems in understanding what happens during the limiting process. Her use of the word wrap perhaps means that she understood that the one of the points of intersection does not move in the limiting process. But it seems that Anna did not see what happens to the other intersection point. At the end of the response to Task 2 Anna stated that the approach from the right-hand of the point where the graph has a jump could be managed, but she did not see what would happen if the approach happens from the left-hand. However, she seemed to understand that also in this case one of the points of intersection of the secant and the graph is (1;4) (see Figure 1).

"I think that if the study was done at the right-side (she points to the right side of the point where the graph has a jump), then it could manage, but if we came from left, it cannot be the same, because this is here so much lower down (she points to the jump)."

Ben interpreted the limiting process of the difference quotients to mean visually the same as the limiting process of the derivative. This interpretation gave him a satisfying argument why the function in Figure 1 is not differentiable, but in the case of the function in Figure 2 it lead to a contradiction. This embarrassed Ben and he did not try at all to find a visual argument but he began to explore the situation formally.

Anna preferably used informal argumentation while Ben often relied on formal reasoning

Anna used quite often visual reasoning in her responses. Visual interpretations and visual conclusions seemed to have an essential role when Anna was convincing herself. A good example of this is her response to Task 4 where she convinced herself of the general result by analyzing the visual meaning of eveness. After it she was in fact not interested in the formal proof. In this phase she also did not see at all what the formal proof could be like. She knew that her claim was true and it would be possible to prove it by using the definition of derivative.

"I think it has to be possible to show this by using the difference quotient because the difference quotient or the limit is more or less essential here in the derivative."

Also in Task 3 Anna relied on visual reasoning. She did not change to formal reasoning even if she was not able to see what the graph is like or which one of the terms (x^4 or $\cos(1/x^3)$) determines the behavior of the function near zero. Instead she intuitively concluded that due to the oscillation of the term $\cos(1/x^3)$, the function was not continuous at zero.

Ben had - especially if compared to Anna - a quite strong tendency to explore things formally and to use formal arguments. Already after Task 1, when arguing why the absolute value function is not differentiable, Ben tried to plead to formal argument: he explained that the left-hand and right-hand limits are not equal. (He probably spoke about the left-hand and right-hand limits of the derivative, not of the difference quotient.) In Task 2 Anna s first argument was informal (not possible to draw a unique tangent line) but Ben s first argument was based on a formal reason (the existence of the limit). Also in Tasks 3 and 4 Ben used formal arguments more sensitively than Anna.

Anna utilized informal and formal reasoning simultaneously but Ben tended to hold them more separate

Anna tended to use informal and formal reasoning simultaneously. This comes out for example from her response to Task 4 when she tried to make a formal proof. Here Anna encountered the problem how to manipulate the expression (*) to the expression of -f (x$_0$). The problem was absolutely formal but despite of it Anna tried to find the solution by exploring the picture. It is difficult to say on the basis of the video how she tried to utilize the picture, because she spoke almost nothing. After deciding to change the sign of h she tried visually to find out if this was allowed. Anna s interpretation of the minus sign in front of h reveals that probably she intuitively thought h to be positive all the time and that usually approach happens always from the right. Anyway, Anna s response to Task 4 is a clear indication of Annas tendency to utilize visual interpretations in formal reasoning.

Ben tended to separate informal and formal reasoning. Often Ben began the solving process by exploring the situation visually and then he changed to a formal study, but during it he did not utilize visualization. When Ben encountered problems in informal reasoning he changed to formal methods. These features come out from his responsies to Tasks 2 and 3. In Task 4, Ben first concluded the answer to the question on basis of a picture, but after it, when making the formal proof, he did not use the picture at all. Even if he encountered problems he -unlike Anna- considered the situation only formally. The only situation where Ben changed from the formal reasoning to the informal one was at the end of his response to Task 3: after getting the expression of the difference quotient to the form $h^3\cos(1/h^3)$ he concluded the term h^3 to flatten it near zero. At the beginning of his response to this task, before beginning a formal study, he had made a corresponding informal conclusion. On the other hand, Ben seemed to have some strong visual representations and he was able to use them in a flexible way. For example, in Task 3 he could immediately draw the graphs of h^3 and $\cos(1/h^3)$ and conclude how the product of them would behave.

DISCUSSION

Both students understood very well the visual meaning of the derivative and the difference quotient but not of the limiting process of the difference quotient. Partly this similarity between the students can be explained by the impact of instruction: derivative as a slope of the tangent line and also the connection between the difference quotient and the secant line are usually considered in instruction, but the analysis of the real meaning of the limiting process may deserve less attention. But it is also noteworthy that the derivative and the difference quotient as concepts are static objects but the limiting process of the difference quotient is a dynamic process.

It is often more complicated to interpret informally processes than objects.

Pinto s (1998) research brought out that most students majoring in mathematics turn to informal reasoning at the beginning of their studies, because it is difficult for them to cope with the formal definitions and the formal theory (ibid. 294). The students taking part in this research had studied mathematics at university for a longer time and so they had more experience of formalism. Ben did not find formal investigation of things strange but Anna seemed to be little hesitant about it and so she more often turned to informal reasoning.

According to Stylianou (2002, 306) not enough is known about the interrelationship between visual and other strategies in the process of problem solving. This study shows too that this relationship is very complicated and more research on it is needed. But it comes also out in this study that when uzilizing informal interpretations and informal reasoning in mathematical problem solving it is important that the connections between the formal sentences and their informal interpretations are well understood. Then the informal reasoning can be a very usable method in mathematical problem solving like for example in Task 4 where both students found the general result by utilizing visualization. But erroneous interpretations and imperfect arguments in informal reasoning may lead to erroneous conclusions like what happened to Anna in Task 3.

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RESEARCH AND DEVELOPMENT OF UNIVERSITY LEVEL TEACHING: THE INTERACTION OF DIDACTICAL AND MATHEMATICAL ORGANISATIONS

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Abstract: *The relation between mathematics as a science, and the development and research on mathematics education practices, is of considerable importance for defining the objects and objectives of the latter as a scientific field of study. This paper aims to describe some specific aspects of this relation in the setting of university education, by proposing an analysis of it in terms of praxeologies.*

INTRODUCTION

Research in the didactics of mathematics focusing on university level teaching is an emerging area (see e.g. Holton, 2001). Currently, the most active themes include

- use of computer algebra systems and other technological tools (ibid., sec. 5);
- mathematics teacher education (ibid., sec. 7);
- certain curricular themes such as proof, problem solving and modelling.

The tertiary context cannot simply be viewed as a variant of primary and secondary education, with more advanced subject matter – it presents us with some specific and different phenomena. For instance, teachers are typically researchers themselves (in mathematics or another scientific field), and the students have specific professional or academic reasons to attend teaching. Students mainly attend mathematics teaching as a result of personal choices. And so, the position of the *researcher* is also different: he will be working with teachers who are simply colleagues with a relevant background in research (albeit typically not in didactics/education), and learning must be viewed with respect to students' choices and professional aims. Moreover, although the researcher must (as always!) be thoroughly acquainted with the subject matter taught, but he may not be an expert at the level of the teacher.

It seems thus that certain variables related, roughly speaking, to 'organisation' of teaching and research – including the *institutional position* of the researcher (cf. also Selden and Selden, 1999), the teacher and the students – must be taken into account in an analysis of the specific nature of such research, its impact on education and its appreciation by teachers (here, mathematicians). This paper is an attempt to propose and exemplify what I regard as a promising model for analysing these points. It is essentially based on ideas coming from the 'anthropological approach' in the didactics of mathematics (Chevallard, 1999). To motivate this specific model, I begin by presenting a global view of the main agents and activities to be considered in research on tertiary mathematics education. At the end, I outline some examples of how the model is helpful for the analysis of concrete contexts.

A GLOBAL (AND SOMEWHAT NAÏVE) VIEWPOINT

The classical activities of a university professor are *research* and *teaching*, the latter being somehow based on the former (however, that relation is by no means simple, and it is far from being well understood). This is true for any discipline, including *mathematics* and *didactics of mathematics*. Crossing these disciplines and activities we get a first picture of the relation between practice and research in the context of tertiary level mathematics education (fig. 1).

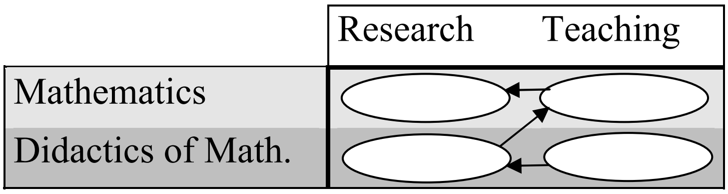

Fig. 1.

The horizontal arrows in the figure indicate that the *object* of teaching in a discipline are (at least based on) the *products* of research in the discipline; of course this is not true at the personal level, in the sense that the researchers' own research products are only rarely the objects of his teaching. The third arrow shows that the object of study of (tertiary) mathematics education is the *teaching* of mathematics (at this level, i.e. the teaching carried out by the mathematician). In principle, one could imagine a further level of didactical research, taking as its object *the teaching of the didactics of mathematics*. Although potentially interesting, it is not our subject here.

The crucial relationship between mathematics and its didactics is often viewed in terms of research fields (cf. e.g. Brousseau, 1997, 21-23). One point of fig. 1 is to show that research on tertiary level mathematics education adds an immediate relation among agents of these fields: the teaching of the mathematician becomes the object of study for the didactician. In the absence of further involvement of these two agents in each others' fields of practice, this creates a somewhat asymmetric relation between the two, with potential conflicts arising from insufficient insight and differences of interest. But it *also* makes tertiary level didactics a potential space for interaction and collaboration. However, in order to pinpoint the nature of these problems and potentials, we must say more about the contents (activities) of the 'oval spaces' in the figure. This is where the anthropological approach comes in.

THE ANTHROPOLOGICAL APPROACH TO THE DIDACTICS OF MATHEMATICS

The basic concept of the anthropological approach is that of *praxeology* (Chevallard, 1999, §1.2), a meta-theoretical construct used to model what is involved in mathematical and other human activities. A praxeology consists, abstractly, of a class of similar *tasks* which can be executed by applying a set of *techniques*, situated in and enabled by a larger system called *technology*, and with an overarching discourse of justification and regulation of practice (*theory*). The "anthropological" lies, so to speak, in the primacy of *tasks* understood as "requirements for human action to achieve rather precise goals". Tasks and techniques of a mathematical nature are clearly present much beyond the school context, and some analyses of mathematical 'culture' focusing on this broader perspective have considered associated values and ideologies which are associated to the development of mathematical technologies in general (e.g. Bishop, 1988). At any rate praxeologies are always *institutional constructions* and although didactics study mainly those enacted in a school-like context (where they are really re-constructions, cf. Chevallard, 1999, 241).

Returning to specific university contexts such as courses, lectures and seminars, the observable research and teaching activities of mathematics may both be analysed as two *systems of praxeologies*, with tight links between the components of each system, and with respect to the crucial relation between the two systems. We shall use the terms of Chevallard (1999) to refer to these systems: *mathematical organisations (MO)* and *didactical organisations (DO)*. It is clear that both MOs and DOs depend on the institutional context (and in fact, even MOs involved university teaching are not simply the MOs of mathematicians' research). Their description and analysis is of course not simple and their nature may be indicated only in general terms and by exploring select examples.

DOs are closely related to MOs, in the sense that a DO may be viewed *essentially as an answer to the question "How does one establish a MO* [for students, CW]*"* (Bosch and Gascòn, 2002, 35). The construction and implementation of didactical techniques and technologies may, conversely, necessitate modifications of the MO. Hence, the didactician is interested not only in MOs and DOs separately but also in the *reciprocal determination relation between MOs and DOs* (ibid., 32). Research in didactics concerns not DOs alone, but the entire system of DOs and MOs.

MOS OF THE UNIVERSITY CONTEXT: TRANSITIONS AND AUTONOMY

Some of the MOs students meet in university teaching are quite similar to what they have met in secondary school: types of tasks are easily identified with accompanying techniques; whether the student must also perceive the coherent technology or the theoretical discourse of justification depends on the context, and if so this may be a painful novelty. This is true for both of the main tracks of first year university teaching: concrete analysis and algebra. A student in a "mathematics for life scientists" course expressed this very clearly, when faced with the teachers' requests for providing justifying discourse in written homework (rather than just calculations):

> *But then… I like have to understand what is going on. I'd rather just use the formula.* (Notes from classroom observations, autumn 2003)

The justifying discourse is still not an end in itself. It appears, to some students, an unnecessary complication, if they understand the task and come up with the "good" solution, using appropriate methods. So *the first transition* in many university curricula is to *establish full fledged praxeologies* (including technology and theory) in contexts where task types and techniques are well known to students, or at least are of a familiar nature (concrete calculation and drawing, algorithms). At the same time, it becomes possible (and is commonly practiced) to extend the demands on students' *autonomy in the choice of techniques* (within a technology) and on *students ability to justify their choice* (often based on a theoretical view of the task).

The second step is more complex and varies according to the study programme's orientation. It has to do with *abstraction* and *professional perspectives* in various blends. Professional perspectives imply a fusion of praxeologies into semi-mathematical ones where the mathematical components become in a way subordinate to those of other disciplines (including education); for lack of space we refrain from a further discussion of this type of transition. Abstraction means that familiar, concrete task types disappear and are replaced with tasks situated in a realm previously perceived as technological or even theoretical. Often, linear algebra is where this occurs first. There is a big step for many students from solving linear systems of equations to understand why all finite bases of a vector space have the same number of elements. In analysis, operations on concrete functions are gradually replaced by arguments concerning abstract functions and function spaces. In general, tasks and techniques become more and more inseparable from theory and technology. Particularly in advanced courses, students' tasks are often directly concerned with perceiving and producing theoretical discourse such as proofs. Such tasks are often not easily identified with 'techniques' (let alone over-arching technology and theory) and may be said converge to those facing the researcher, essentially along the direction of *autonomy in production.* We thus have a *second transition* from MOs involving concrete and familiar types of tasks and techniques to MOs where tasks involve the production of full-fledged mathematical discourse (cf. also Tall, 1991, p. 20, describing a similar transition from a cognitive point of view). Of course, even in a first course on linear algebra, one may have elements of both types of transition. But the full completion of the second transition concerns mainly advanced students in programs of pure mathematics, and hence a much smaller public than the first one.

DOS OF THE UNIVERSITY CONTEXT: LEVELS AND CONSTRAINTS

In order to 'establish' MOs and in particular enable transitions for students, as explained in the previous sessions, the university and its teachers provide DOs in various formats and levels, ranging from entire study programs (e.g. 'bachelor in pure math.') over courses (e.g. 'Linear algebra 2') and its components (e.g. lectures) to specific tasks in a homework assignment. It is clear that only at the last level do we see direct links between tasks of MOs and elements of DOs, and the higher levels are ultimately organisational frameworks for the devolution (Brousseau, 1997, 230) of tasks to students and associated didactical moments (Chevallard, 1999, 249ff).

At each level, some constraints will be imposed from superior levels, e.g. study plans for a course or legal documents for a study program. However, the establishment and regulation of DOs comprised within the scope of a course leaves, in most cases and in principle, a great deal of autonomy to the responsible teacher(s) – often more so than in other contexts of education. To say 'in principle' is important here, because implicit constraints – discussed further below – are often the only immediate explanation for the surprising degree of homogeneity in formats and assignments among courses delivered by different teachers. The 'task' assigned to a course (or rather, its teachers) is typically a short list of topics 'to be covered', such as:

> *Mathematical analysis: metric spaces, continuity; Hilbert space; Fourier analysis, partial differential equations.* (Cited in Grønbæk and Winsløw, 2004)

To get from such a broad description to actual didactical designs would – in principle! – seem to leave a great deal of room for the teacher, even when taking into account the constraints given in terms of teaching and assessment formats etc. However, in practice, the teacher is also constrained by at least the following:

(1) Traditions to the effect that certain mathematical praxeologies are implied (for instance, in the above case, a cluster of items involving compactness is implicit). This is to some extent supported and made explicit by auxiliary texts, such as 'mainstream textbooks' and previous exam assignments;

(2) Expectations and capacity of co-teachers and students e.g. with respect to the nature of tasks and accompanying teaching measures

(3) General institutional constraints, e.g. implicitly restricting the rate of failure.

Still, the teacher must and does make a number of crucial decisions, locally as well as globally, in particular concerning *the work assigned to students*, its required *outcomes*, and forms of *evaluation* for this outcome. Here, the basic level concern *devolution of mathematical tasks* to students (in teaching as well as exams). This is where the transitions and levels of student autonomy can be worked on and controlled; and, as mentioned, it is where DOs and MOs interact most immediately.

The potential role of *assessment* in this matter can hardly be overestimated. Even ambitious students who are eager to learn will often try to focus on those tasks which they think are important for assessment. A common source of malfunctions in our context is a *lack of coherence and clarity* in the relation between these tasks and those emphasised and devoluted in teaching. The short life of study units and the pressure on students' time in general tend to reinforce this effect.

A particular problem in university teaching seems to be that professors – particularly in large introductory courses – don't pay much attention to the level of students work with the assigned tasks, possibly because of the difficulty of gaining proper and direct information about it. Professors occasionally observe that an exercise assigned "was too hard for most students" in the sense that "they couldn't do it". And as professors have little access to insight about students' actual work and difficulties with particular tasks, they tend to react by simply removing or trivialising those which result in persistent student failure. This can result in a disparity between the MOs involved in students' and professors' work: the lectures continue to 'deliver' the full praxeologies, but the students are only expected to master a limited range of task types and techniques, with little autonomy in linking them together. Superficially, such a degenerated situation may appear well functioning, although over time one cannot help to notice that "levels are dropping". Still, a study unit may appear to obey all of the constraints mentioned above. In such a situation, all of the problematic didactical 'effects' (Jourdain, Dienes, aging,..) described by Brousseau (1997) may be observed in abundance.

CO-DETERMINATION OF MOS AND DOS

There is thus a general *negative* version of the solidarity between DOs and MOs, arising from an institutionalisation – pertaining to the DO – of essential *dissimilarity* of the MOs worked on by students and teachers. A common format is that of sharp boundaries (in schedule, and often among teachers) between *lectures* and *exercise sessions*. The full praxeology is present only in the lectures, delivered by a researcher. In *exercise sessions*, students and assistants work on blocks of task types and closely associated techniques, assigned by the professor. In short, the DO entails a considerable reduction of the MO effectively worked on by students (except possibly for a minority who, by what appears to be a miracle, achieve the transitions and autonomy that may still be required in more advanced courses). As a student said during an interview: *You do what you need to do on each question and that's the end of it. You don't look for cross-references, like what's the use of gradients elsewhere.*

Conversely, with appropriate tools – belonging to the technologo-theoretical block of the DO – to set goals for, and to monitor, the progression of students with respect to MOs (transitions, autonomy), one may revise the techniques of DOs more directly with respect to eliminate the disparity between students' and teachers' work (cf. Grønbæk and Winsløw, 2004). This must be done while paying close attention to the interaction of students' and teachers' work with MOs at all levels (Winsløw, 2004), on pain of replacing one kind of trivialisation (just explained) with another one (full praxeologies, but too few new ones). In particular the *statement* of goal types – in terms of the levels of MOs – must become explicit, and less 'spontaneous' (Bosch and Gascòn, 2002, 24) – in order to gain control of this interaction (which, ultimately, remains of course constrained by a number of factors, including the effort teachers and students are willing to spend). To achieve this is both an endeavour of *local development* in the context of particular courses and programs – and, in a wider context of systematic frameworks and dissemination, a *research task.*

PRAXEOLOGIES OF DEVELOPMENT AND RESEARCH

Until now, we have considered the praxeologies involving university *teachers* and *students* of (mainly, pure) mathematics. We now come to the third agent – and class of praxeologies – personified by the *didactician of (tertiary) mathematics.* The relation between researcher (of didactics) and teacher (of mathematics) can be conceived differently, from what appears to be identity (as in Asiala et al., 1996) to a classical actor-observer relation. In between we have praxeologies which call for a certain distribution of tasks and responsibilities (as in Artigue, 1994 and Grønbæk and Winsløw, 2004). The rationality behind such a distribution is of a practical nature; but it is not (and should not be) a partition of the *terrain* in the sense of DOs and MOs (the didactician promoting certain general methods and epistemologies, and the mathematician defends what is perceived to be the needs for the MOs).

Instead, the roles must be determined with respect to definite *phases* – and tasks – in the praxeology. Roughly speaking, the three phases of *design*, *implementation* and *evaluation* must involve both in order to be functional and credible. The phase of *design* implies, in the terms of Artigue (1994), an *a priori analysis* of MOs, DOs, and various constraints, which can be considered a technologico-theoretical block that enables and guides the concrete choices of didactical techniques corresponding to the tasks. Simply giving the latter to the teacher seems to be inadequate for several rather obvious reasons; the whole analysis must be collaborative. For the *implementation phase*, the roles are more distinct: the task of the teacher is to enact and regulate the praxeology proposed (although never entirely determined!) by the design, while the researcher's main task is to gather data on the basis of a methodology (technology) in accordance with the a priori analysis (with its frameworks, questions and criteria). The two parts need not live this phase in separation, though; in fact, observations and on-going discussion regarding the implementation of the design may well contribute to improve it, especially due to the insight into students' perspectives that observations may contribute. Finally, the *evaluation and documentation phase* (perhaps just in preparation for a second 'cycle') – could usefully include the viewpoint of teachers, even if the overall responsibility is the researcher's. I now outline three examples of projects developed along these rough principles.

Example 1. Didactical engineering focused on CAS-use (more details in Solovej and Winsløw, 2001, Winsløw 2003). In this project, running on a major second semester course at the University of Copenhagen, the MO – roughly speaking, calculus in several variables and ordinary differential equations – was a priori taken for granted. The aim of the project was to give students more opportunity to reflect on, and discuss, overall aspects (technology, theory), while reducing the time spent on applying standard techniques for carrying out routine tasks. The idea was to modify the DO by introducing the use of a computer algebra system (*Maple*) in lectures as well as in two special "Maple classes" (for exercise sessions). More precisely, the strategy for introduction was to demonstrate how *Maple* can solve certain specific mathematical tasks (e.g. plotting a function of two variables, solve a linear differential equation symbolically). Concretely, this was done by providing students with a number of on-line work sheets where the use of pertinent commands were explained and exemplified. These uses were also shown in lectures and occasionally used for visualising examples. In other words, *Maple* was presented as a providing a set of *auxiliary techniques.* Students in the *Maple* classes were allowed to use these techniques in their work with exercises, at home, during class work, in mandatory project work, and at the final written exam (where the tasks could not be solved without considerable knowledge beyond *Maple* commands). In class sessions, one computer with projector was available, so that student homework could be discussed in plenary, and compared with manual methods. This way *Maple* techniques were supposed to be integrated in the *technology* mastered by the students. Occasionally, discussions could be lifted to a *theoretical* level (such as investigating variations of problems and solutions while leaving computations to *Maple*) that would otherwise be inaccessible in the time available. Problems and potentials with the set-up were carefully documented and analysed, and revised forms of the design have been used in the course since the pilot project ended (2001), as well as in other universities.

Example 2. Specific competency goals and thematic projects (more details in Grønbæk and Winsløw, 2004). The year after the course considered in Ex. 1, students of mathematics and theoretical physics take a course on real analysis, including metric spaces and Fourier analysis. The transition between the two is difficult, for the reasons mentioned in the section on MOs. An a priori analysis of these difficulties suggests a need for more explicitly focusing on the *types of tasks* students should be able to handle, and the coherence between the local level of specific tasks and techniques and the more global theoretical flow of the course. We decided to focus *exercise sessions* and the *written exam* on more local, specific tasks, while introducing a new work format – thematic projects – to help students work on more theoretical and integrated levels of the MO. The projects consisted in solving, in groups of 2-4 students, a series of rather open and challenging tasks related to the theme. These tasks would typically necessitate the use and choice of a variety of techniques, and were aimed at challenging and developing the command of a larger domain of theory (e.g., Hilbert space and Fourier series) or applications (e.g. the heat equation). And the projects were the basis for the *oral exam* associated to the course, replacing the requirement for students to explain a passage in the text book. A considerable freedom was allowed for students to interpret the project tasks according to their level of technological and theoretical ambitions. Students invested considerable efforts in the work on these projects, and we have much evidence to support the claim that the focus on *students' autonomous work with theoretical tasks* (including constructing proofs) both increased the level of 'deep' learning and reduced the failure rates considerably. This didactical design was crucially related to the analysis of the MO in terms of specific competency goals (essentially, explicit task types, cf. also Winsløw, 2004) and to the wish to attain these goals *both* in a local, technical sense and in a broader theoretical perspective with higher requirements for autonomous student work. The design was implemented, analysed and revised over two years (2002, 2003) and will be in use again in 2004.

Example 3. Organising a broad introductory course (in process). As a consequence of new study programmes in science at the University of Copenhagen, students of physics, chemistry and mathematics (and related fields such as statistics and nano-science) will take a common calculus course during the first 9 weeks of their studies. This course should thus cater to a variety of academic interests and needs; and it should also contribute to the initiation of the students to university studies. Based on an analysis of the problems related to the "first transition" (briefly discussed in the section on MOs) we have developed a DO which in a sense is an adapted combination of the ones discussed in Ex. 1 and 2. As in Ex. 2, there are exercise sessions focusing explicitly on elementary tasks and techniques that all of them will need to master, and these are assessed in two written tests placed in the middle and at the end of the course. While we do want students to experience the challenges of more autonomous tasks, the context does not enable the autonomy required by thematic projects. Instead, students will be given each week a 'mini-project' to be solved in groups within four hours, after an introductory lecture and followed by a discussion session. During the four hours, students have access to guidance from instructors, and part of the work is done in a computer lab, using *Maple* (techniques are introduced in the preceding lecture, cf. Ex. 1). Half the credits for this course will be given to students according to the grading of their reports resulting from their work on the 'mini-projects'. This design is clearly explicitly inspired by the classical structure of didactical situations (cf. Brousseau, 1997), with carefully designed devolution and institutionalisation. However, the 'school like' DO is to be gradually transcended by the MOs aimed at. And as students are organised in classes according to their study programme, the 'mini-projects' will be designed specifically for the academic orientation of students. This means that, for instance, chemistry students get to work with applications of techniques and technology in chemistry, while mathematics students will work on slightly more theoretical aspects of the MO. (A first run of this project will be completed in November 2004, and some of its general aspects have counterparts in projects carried out in a parallel statistics course.).

CONCLUSIONS

To maintain a spirit of collaboration and at the same time a mutual respect for what must be the responsibility of each part is not just a local *modus vivendi*. A praxeology of didactical research and development involves considerable institutional interests which, according to the position of the agents, may be at least partially in conflict. At the extreme one has the patterns of tribal conflicts between "mathematicians" and "educators" which thrive overtly in some parts of North America, particularly in the context of general education. University teaching of mathematics is more generally recognised as "mathematicians' territory", and an intrusion of what appears to be didactical colonisers may not be very well received. The institutional affiliation of the didactician (department, university) may be of some significance – but the most important is probably that collaboration is seen as meaningful and beneficial to both parties.

In my experience, the didactics of mathematics at university level can justify itself most immediately by the products of development it produces. Many mathematicians are well motivated – not only by external factors! – to improve the quality of their own teaching. But close to this need comes a justified expectation that these products are documented and in some sense 'reproducible'. And in order to make this possible – indeed, for development to be more than a private matter depending entirely on the circumstances, techniques must be grounded in didactical technology and theories, i.e. systematic research. Conversely, research can draw on data and opportunities made available from development projects.

Many (if not most) agents in tertiary didactical research have been (or are) active research mathematicians. But the active participation by mathematicians is both possible and necessary beyond this (cf. Selden and Selden, 2001). One reason is the practical circumstances for carrying out research and development projects. A more crucial reason is that the analysis of MOs is an important part of the venture which – especially in the university context – cannot be conceived as a lay anthropologist's (or educator's) excursion into mathematics. And there is no need for the points of view of didactical and mathematical (or target discipline's) research to be in conflict.

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MENTAL MODELS OF THE CONCEPT OF VECTOR SPACE

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This paper presents a pilot study on students' images of the concept of vector space. Three students have been asked to model a real world situation with the mathematical tool of a vector space in interviews and have revealed two very different priorities of looking at a vector space. The part of the interviews this paper deals with is concerned with the students' general notions of the character of a vector space. This is further considered in respect to the students' handling of abstraction. The difference in the students' conceptions is discussed in connection with different modes of thinking.

Keywords: linear algebra; abstraction; thinking styles

Introduction: focus of the paper

This paper is part of a pilot research study on the internal models that students develop of linear algebra in their first year of studies at a German university. In this study I am particularly interested in differences between students with a preference in functional or predicative thinking according to a definition given by Schwank.

Questions motivating the part of the investigation that I will present in this paper are: How do students' foci of conceiving structures influence their notions of the concept of vector space? How do students deal with the abstract concepts related to formal properties of a vector space and to encapsulation or reification?

I will first give an overview of the theoretical framework of the study by defining what I understand by functional and predicative thinking and by explaining what I mean by encapsulation and reification. Then I will give an outline of my research design and present students' conceptions of the concept of vector space, as they were revealed by my data.

Theoretical framework

As Dorier et al. (1999) have pointed out, modern linear algebra is a mathematical theory characterized by a high degree of abstraction and formalism. Therefore, in order to gain a better understanding of students' endeavor and progress in learning linear algebra, one of my focuses is to investigate their understanding of abstract constructions. The two types of abstraction I will consider in this paper are concerned with the mental convergences of processes to mathematical objects. They are closely related and may be used as mental tools to conceive the same mathematical concepts. One of them is the encapsulation of a process, which is the shift of viewpoint from the handling of a process to the conception of this process as a new entity, which is given the status of an abstract mathematical object and is treated as such. (See Dubinsky 1991). Dubinsky (1997) considers the setting of the dual space of a vector space as an example of a mathematical situation where encapsulated processes (here the elements of the dual space) will be de-encapsulated to regain the original processes (in this case linear mappings). The second kind of abstraction is reification of a mathematical action. This turns the focus from a mathematical action to its outcome. Here it is not the action itself that is regarded as a mental object, but it is substituted by its result. (See Sfard 1994). Gray et al. (1999) use the name "procept" for a notation, which can be read both as a description of a process and as the result, i.e. the reification, of the process. An example of such a notation is 2+3, which can be understood as a demand to add the two numbers 2 and 3 or as a representation of their sum 5. By the name "procept" they do not mean to identify action and outcome but they stress the possibility to switch between these two focuses. Reification and encapsulation both consider dynamic as well as static ideas, starting with a concept of action and ending with a new abstract object, and both can be reversed. An example of the close relationship and yet difference of underlying ideas is the following: The process of assigning each real number twice its value can be transformed into an object in different ways: It can be regarded as a function f: $R \rightarrow R$, $x \rightarrow 2x$. This function f is an encapsulation of the process of mapping. Here the abstract object "function f" represents a process. The outcome of the process is the set of all pairs of real numbers of the form (x, 2x). This reification of the mapping process is a description of the same function f. However, here the idea of a process is substituted by the idea of a static relationship.

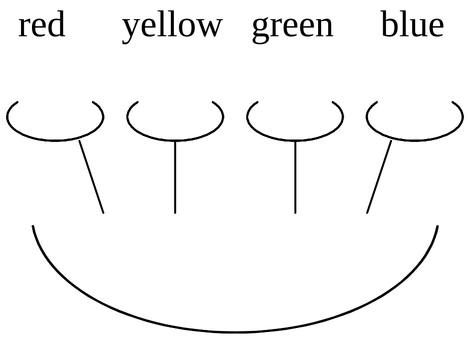
In her definition of functional and predicative modes of thinking Schwank (1999) distinguishes a preference to deal with structures by applying operations or actions to them and a preference to describe structures in terms of static properties or of relationships. As Schwank (e.g. 1999) has demonstrated in various papers these two mental preferences may result in very different mathematical conceptions and strategies. I am particularly interested in consequences of predicative and functional thinking in respect to their internal representations of the structure of vector spaces and to students' treatment of problems concerning reification or encapsulation.

Research methodology: a pilot study with interviews

Three students in their first year of studies at a German university were interviewed. In Germany, students attend university after 13 years at primary and secondary schools. Thus their first year at university corresponds to a sophomore year at college in most other countries. The three students attended a course on linear algebra. Here they learned an axiomatic definition of vector space and consequences from this definition such as the central theorems about the concepts of basis and dimension and they studied linear maps, which were dealt with both in an abstract definition and in a matrix representation based on a basis of the vector space. At high school the students had had some experience with the R^3 as a vector space of 3-tupels and as a description of linear geometry of the space.

The students were chosen for the interviews as the only volunteers. They were interviewed seperately and the first part of the interviews, which is dealt with in this paper, took about 10 minutes. Two of these students, B and S, revealed similar conceptions of vector space, but the third student, R, had quite different priorities. In this paper the answers of B and R will be discussed.

In the interviews the three students were confronted with a practical non mathematical situation. A machine for mixing colors was drawn on the blackbord:



There are four small boxes next to each other with tubes leading to a big box below them. Each small box is to contain one of the colors red, yellow, green, and blue and is labeled with the name of the color. The new color is to be mixed in the big container. A computer is to regulate the opening of the four tubes for producing a desired color. The students were asked to model this situation with a vector space. As the interviews proceeded, the students commented in particular on the two questions, what the characteristics of a vector space correspond to in the world of the color machine and how the concepts of linear dependence and linear independence can be explained with the color machine.

Neither in the course nor in the acompanying exercises the students had met a modeling task or been asked to apply their mathematical ideas to a real world situation.

There is no straightforward answer to this modeling task, because the color machine cannot produce negative quantities of colors. The machine can be modeled with a subset of R^4, where 4-tupels either correspond to colors or to the instructions given by the computer for producing the colors. These instructions can give the lengths of time each of the tubes is to be opened. This second idea has two advantages: To get the machine to work, computer instructions have to be given; if one of the given colors can be produced by the others (yellow and blue produce green), then different 4-tupels given as computer instructions would still mean different actions although they may result in the same color. In the first case the addition in R^4 corresponds to a mixing of produced colors. In the second case it corresponds to the carrying out of instructions, how much of each of the four beginning colors is to flow into the mixing box, one after the other. Multiplication by scalars would be a change of quantity of a color in the first case and a constant multiple of the opening times of the tubes in the second case. If the 4-tupels are to represent the produced colors, the mental act of reification is carried out, because the action of instructing the machine and producing colors is replaced by the result of this action. If the components of the 4-tupels mean instructions to the four tubes, then the whole tupel is the encapsulated process of the complete action.

The interviews were designed to serve as a pilot study of students' inner representations of vector spaces. Thus they did not follow a closely determined path of questions, but were open to a great variety of answers and ideas the students might utter. Their answers affected the proceeding of the interviews. As I wanted to be open to inner models of the concept of vector space which I did not anticipate, I started the interviews with a situation which was new to the students and which I hoped would not lead to an interpretation of vector spaces simply by remembrance. I did not stress the implicit modelling aspects, because my main interest is in the students' understanding of the mathematics involved with the concept of vector space. The open approach throughout the interview I considered necessary in order to find out about ideas of the students which might differ greatly from my own view of a vector space. A consequence of this decision is of course, that the analysis of the interviews takes a long time and cannot be done for a great number of students. Thus it does not lead to an overview or a classification of students' inner representations of vector spaces, but only gives an idea of the range of images that must be taken into consideration. Since the mathematical subject of vector spaces is both, complex in stucture and in cognitive challenges, it seems to be very difficult to design a research study which would give a statistically relevant overview of students' internal models. Sierpinska et al. (2001) tried such an analysis on students' theoretical thinking concerning basic concepts of linear algebra and concluded that the impact of the statistical results of the study may be small.

Samples and Results

The interview with student B

In the following protocol, the interviewer I explained the machine and the student B replied:

> I: We want a computer to tell the machine how to mix the colors. How could you model this by means of a vector space?
>
> B: I could imagine that you have this (she is pointing at the four color boxes in the first line) as a row or as a column vector. And you get a kind of subspace of what you can make of all of them. You get a kind of, like what you have with fields, you get a kind of ideal, that all vectors, which are generated by this one, are in the ideal.

So at first she interprets the four small containers, which are drawn in a row, as a vector. But then she associates this situation with the generating of substructures of algebraic structures. At this point of the interview it is possible that her explanation is concerned with the meaning of "vector" in the context of the color machine or with the concept of generating as a typical feature of a vector space. After the row vector (1 1 1 1) is written on the blackboard B is asked to explain what she means by the "ideal". She answers:

> B: By this ratio everything that is in there - . The color which is produced in here (she is pointing at the big box) is due to this ratio. No matter how much I take, the ratio will always be the same. If I take another ratio, I will get another color. With these four colors I can generate the complete spectrum of colors in here (she is pointing at the big box).

She is then asked, if she regards (1 1 1 1) or λ (1 1 1 1) as a vector, and answers that the multiple of (1 1 1 1) would be a typical vector.

The explanation reveals that she is only interested in the kinds of color that can be produced by the machine, but not in the quantities. The ratio of the four ingredients is the information determining a product. Two 4-tupels which differ only in a scalar multiple characterise the same color and are thus representations of the same "vector".

Throughout the interview she works with this mental idea of a vector without translating it into a precise definition of what she calls a vector.

Later on, when the interviewer asks her to explain the ideas of linear dependency and linear independency, B says:

> B: If you did this with several containers that you get, you have these ratios.
>
> I: What do you mean by "several containers"? The complete model?
>
> B: Yes, yes, several results, so to say. Different ratios. And if you have two, suppose you have three of these systems and you could generate the third by two of them.

Here B shows, that she does not consider how the ratio of the four basic colors can be varied, but views the complete drawing on the blackboard as a representation of all products of one kind of color differing only in quantity. For another kind of color she imagines a new drawing in analogy to the first! We see that for B the instruction for producing a new color is of no relevance.

B is describing the products on a very abstract level by explicitly identifying products of the same kind of color. In her modeling of the color machine as a vector space she uses reification on two levels: First, she replaces the production process by the product of the process. Second, she identifies the processes of generating different quantities of a color by labeling them with the ratio of the ingredients, which determines the type of resulting color.

The interview with student R

After the introduction of the machine the interview goes on:

> I: A computer is to regulate how much time the tubes are opened, that is to say, how much of each color is to come in. The question is, how can you model this by means of a vector space. This problem.

> R: I take for example vector x, then I would for example take r times the content of r, that is of red, plus s times yellow plus t times green plus y times blue.

and she writes down:

$$\vec{x} = r\begin{pmatrix} \\ \\ \\ \end{pmatrix} + s\begin{pmatrix} \\ \\ \\ \end{pmatrix} + t\begin{pmatrix} \\ \\ \\ \end{pmatrix} + y\begin{pmatrix} \\ \\ \\ \end{pmatrix}.$$

She further explains:

> R: But I would not really say so. This would be the simplest way for me, but I would not have, at high school, somehow you could invent some things, but I don't have them right now.

It is not clear, what R would like to invent here. At high school she operated with representations of planes which looked very much like the equation she is giving here, except that the column vectors had three components. When asked what vectors correspond to in the color machine, she says:

> R: $\vec{x}$ is the complete thing, that I want to get and r or s or y are the regulators determining the length of time to open things, so that for example red or yellow or green or blue comes in.

Translating the color machine to a vector space R clearly focuses the process of generating a new color, which corresponds to generating a vector by means of constructing a linear combination of other vectors. However, the correspondence she finds is incomplete, because to each coefficient she assigns the double meaning of representing both one of the small boxes and the length of time that box is to be opened. At the same time R does not determine what a vector corresponds to in the world of the color machine. Even when asked about this she only mentions, that $\vec{x}$ stands for the result, but she does not say what the empty column vectors represent. Now the interviewer hints, that the computer should give instructions in form of tupels:

> R: Yes, we would have, for example x would be three zero one zero or something like that.

> I: Yes. The computer would give information like that, would it? Like three zero one zero?

R: Yes, that is the complete color I want to get, for example. And then we would have, for example, red is zero one zero one, or something like that. And yellow is one one one one and green is zero zero. Oh, I could have done this simpler, could I? This was stupid.

I: Yes.

R: I could have taken vectors of a basis.

During this conversation R writes:

$$\begin{pmatrix}3\\0\\1\\0\end{pmatrix}=\vec{x}=r\begin{pmatrix}0\\1\\0\\1\end{pmatrix}+s\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\ \\ \end{pmatrix}+y\begin{pmatrix} \\ \\ \\ \end{pmatrix}.$$

When R replaces $\vec{x}$ by a column vector with four components and fills in numbers in the empty columns of her equality she comments that these columns each stand for one of the first four colors. She chooses arbitrary components and makes it very clear that the contents of these column vectors do not matter to her. The columns are any elements of R^4 and none of the coordinates has a meaning. What does matter is only the formal construction of the generation process which is described by the four coefficients. When R says that a basis would be a more convenient representation of the four starting colors, she probably is thinking of the canonical basis of R^4. In case she is thinking of "any basis", she could complete the choice she started with. She does not do so, but she now exchanges her right hand side column vectors with the canonical basis. With this new notation she is able to find the four instructions r = 3, s = 0, t = 1, and y = 0 for producing the color

$$\begin{pmatrix}3\\0\\1\\0\end{pmatrix}=\vec{x}=r\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+s\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+y\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

As we see by R's first choice, the column vectors are not constructed to represent the production processes, but simply serve to name the colors. The change of basis is done for convenience and only by accident do the components of the produced color now correspond to the four instructions.

R's interpretation of the color machine seems to be evoked by an internal model of a vector space which is determined by certain actions that are permitted in a vector space. She does not consider the character of the vectors and has great difficulties in interpreting the vectors when asked. As the interview proceeds R stresses the point of view that each of the components of the resulting vector gives the necessary information for the opening of one of the tubes. She avoids regarding the column vector as an object representing the complete production process: She does not carry out the encapsulation of the process.

Comparison of R's and B's mental images of vector space

The most striking difference of the two notions of vector space that are revealed in the interviews is the contrast between a functional view interested in processes and a predicative view interested in characteristics of objects. R regards a vector space as a principle of construction, where certain rules of construction and some constituents are given, and by these other objects called ‘vectors’ are generated. This point of view is one that stresses operations and processes. R does not mind much about the characteristics of the objects she uses for constructing linear combinations to generate other objects. At first she uses only empty forms, and later when she makes these forms concrete she regards them as column vectors with arbitrary components. For her, only the principle how ‘new’ vectors are generated is of interest. On the other hand, B concentrates on the elements of a vector space regarding the structure in which they are arranged. This structure is due to two operations which she does not define but which she uses implicitly when explaining the notion of linear dependence. She clearly emphasises static properties of the set. She starts her transferring of the characteristics of a vector space to the color machine by defining what objects are to correspond to the “vectors” and uses 4-tupels as meaningful names representing colors. She does not in any way allude to the process of generating colors or vectors.

In dealing with demands of abstraction, again R and B differ. R shows a way of thinking about vector space that is very similar to a formal definition by axioms which do not characterize the elements of a vector space but the rules of manipulating them. In stressing the character of the elements of the vector space B focuses on properties that may be important in examples of vector spaces but not in terms of a general definition or in terms of general theorems about vector spaces. She does not even mind to violate standard rules of operating with 4-tupels, when they are given new meanings in her model.

Concerning the challenge to grasp a process as an abstract mathematical object that can be dealt with in the same way as other mathematical objects, B and R have different strategies. R regards column vectors in her representation of colors either as names of colors or as four separate instructions which refer to the different tubes of the machine and which result in a certain color when all are carried out. Although the color may be labeled by the same column, in respect to the production process the column contains meaning only in its separate coordinates but not as a whole entity. Thus R does not carry out an objectification of the process. B reifies this process by replacing it by its product so immediately that she does not seem even to realize the process. The difference of behavior between R and B can be plausibly explained by their preference in their modes of thinking. Since B is interested in considering objects and their properties and regards processes as transitions of little interest in themselves, she focuses on the results of processes rather than the processes themselves. The identification of colors with the processes of their production is done in a spontaneous way. However, she does not see a problem in the fact that two different instructions may result in the same color. Obviously, she does not consider the production process at all; this may indicate that B hardly carries out the mental abstraction of reification, but simply avoids looking at the process. For R the two types of abstraction are difficult, because they demand on her to leave her point of interest and concentrate on an abstract object, which does not really concern her.

Conclusion

The behavior of R and B in these interviews shows that very different views of a vector space may be adopted by students even though they have attended the same course. It also implies that the measure of difficulty in learning certain mathematical concepts may be very different for students depending on their personal ways of mentally reconstructing concepts. In particular, this is true regarding preferences of functional or predicative thinking. The example presented in this paper shows that these two preferences pose difficulties at different points of understanding.

Although the two mental models presented here differ greatly, both reflect the character of vector spaces stressing different characteristics by taking different points of view. Therefore I do not estimate one of these attitudes superior to the other, but consider both of them valuable. Indeed I think that a thorough understanding of the concept of vector space would even allow a change of point of view according to the needs of a particular mathmatical situation. Thus it is desirable to be able to look at vector spaces both as sets of specific objects that are arranged in certain relationships and as a principle of construction that follows certain rules.

Further research is needed to find out in what the two students' models of the concept of vector space are typical consequences of functional or predicative thinking and in what they reflect other personal dispositions. A second challenge for research is to design didactical approaches that help students with the different preferences in their modes of thinking in constructing meaningful mental models that comply with these and with other personal dispositions. Since a teacher can neither predict the complex notions of each student nor evade all possible misconceptions, I suggest that such a learning environment must offer a rich range of meanings and of tasks that challenge students to evaluate, adapt, reconstruct and further develop their mental models on their own.

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