WORKING GROUP 4
Argumentation and proof

CONTENTS

ARGUMENTATION AND PROOF

Maria Alessandra Mariotti, *Università di Siena, Italy*

Christine Knipping, *Oldenburg Universtät, Germany*

Dietmar Küchemann, *University of London, United Kingdom*

Kirsti Nordstrom, *University of Stockholm, Sweden*

Topic Group 4 of CERME4 on "Argumentation and Proof " had 15 participants from different countries across Europe. During its sessions, no formal presentations of the 9 papers were made, but every author had a chance to present her/his main ideas. The discussion was organized around three main themes that emerged from the papers prepared for the Topic Group. This report is organized likewise, in that it presents an account of the arguments and comments made in the group in terms of the three themes rather than in the order in which the discussion occurred.

The three themes were

- The meaning of proof in mathematics education
- Comparing the teaching of proof at school
- Problem solving and Conjecturing.

Theme 1: The meaning of proof in mathematics education

The discussion on "The meaning of proof in mathematics education" was opened by David A. Reid who presented the working group with examples of proofs that raised the central question in his paper, "Is there any prototype of proof?". In line with the analysis in the paper, participants generally rejected certain arguments as mathematical proofs and accepted others, even though they were generally not able to define the characteristics of a prototype proof. Thus it seemed that when it came to judge arguments as proofs or non-proofs there was common ground, regardless of the different epistemological positions that participants might have held.

Whether there was common ground concerning teaching approaches towards proof is less clear. It is an interesting question, though, whether researchers' epistemologies have a more significant influence on the teaching approaches they espouse. This still needs further investigation. The "genetic approach to proof" presented by Hans Niels Jahnke clearly is an example of a coherent position in this respect. Starting from a specific and explicitly stated epistemological position, "to treat geometry in an introductory period as an empirical theory", Jahnke expounds a teaching approach that evolves from this epistemological position. First he points out that viewing mathematics introduced to novices as an empirical theory provides an explanation for the strong attraction to experimental verification felt by students when a conclusion is new and important to them. Second, Jahnke insists that therefore the meaning of proof should not be introduced as a replacement or contrast to measurement. Instead measurement and proofs should be combined into a "small theory". Jahnke's theory provides a nice explanation for a phenomenon reported by Küchemann and Hoyles. They found that only a small minority of their sample of high attaining school students could give a structural explanation for the divisibility of, say, 100! by 31, but even students in this group wanted their argument confirmed empirically. The relationship between empirical evidence and proof gains a new meaning in the "genetic teaching approach" that is consistent with the background epistemological position explicated by Jahnke. Jahnke's notion that there is a kind of dialectic or mutual support between intellectual proofs (to use a term of Balacheff's (1999), to identify the case when arguments are detached from action and experience) and recourse to data (pragmatic proofs) throws a new and refreshing light on their relationship and on the seemingly negative findings of Fischbein and Kedem (1982) and Vinner (1983)

Viviane Durand-Guerrier's contribution suggests that her epistemological position is not likely to be based on the same beliefs and principles as Jahnke's. Though she does not explicitly label the main ideas in her paper as an epistemological position, many of them can be read as such. As with Jahnke, she makes us aware that we need stronger theoretical backgrounds in order to better understand students' thinking and difficulties. However, the theoretical tools she refers to are distinctively different from the theoretical approach of Jahnke. Following Quine and others, Durand-Guerrier claims that predicate calculus, in particular "natural deduction" is a powerful tool to analyse proofs "in a didactic perspective". According to Durand-Guerrier logic can help us to understand students' difficulties with proofs, as a logical analysis makes us aware of the various properties of objects and their various nature that is silenced so often. Durand-Guerrier emphasizes that logic should also play a role in the teaching of proof, in the sense that predicate logic "provides an extension of the classical tools used in didactic of mathematics to analyse proofs". However, in her paper she is not propounding the idea that logic should be taught to students, neither is she making any hypothesis about logic being a powerful tool for students to understand proofs. Instead, her focus is on mathematics educators, teachers as well as researchers.

Theme 2: Comparing the teaching of proof at school

Despite the apparent similarities, the teaching of proof presents a great variability across different countries. Not many studies have been devoted to the comparison of the teaching of proof; the study carried out by Knipping (CERME3) and presented at CERME3, was based on classroom observation.. A different perspective is taken by Richard Cabassut who accomplished his analysis through the comparison of textbooks. In his paper, Cabassut uses examples of proofs found in French and German textbooks to analyse underlying differences in proof concepts and teaching approaches. His cross-cultural comparisons and analyses indicate that mathematical properties and concepts anchored in the curriculum can be one reason why specific proofs occur in textbooks of one country but not of another. But Cabassut also suggests that this is not the whole story, as certain mathematical properties are not explicitly referred to in a textbook's proofs, though the properties are explicitly mentioned in the corresponding curriculum. Therefore other factors must have motivated authors of school textbooks in their choice of proofs. However, we may still be some distance away from finding the tools to decipher those factors. For example, it seems that different functions of proofs, as a category for analysing textbook proofs, do not reveal underlying cultural differences in teaching approaches, at least comparing German and French text books. Also Cabassut's analysis of arguments, based on the Toulmin model, does not obviously uncover cultural differences in proof concepts. Two divergent conclusions can be drawn from this. i) There are no significant cultural differences between proofs in German and French textbooks. This conclusion would suggest that word-for-word translations of textbooks should be usable without any problem in the other country. Those who have taught in French or German schools might doubt this. ii) There are cultural differences, but to reveal them remains a challenge. International groups such as those meeting at CERME provide a possible avenue to a better understanding of cultural differences in mathematics education.

Kirsti Nordström in her contribution to the working group also takes a comparative approach to textbook analyses, but from a very different perspective. Her motivation for these analyses resides in the learner's perspective and the fact that textbooks and teachers guides influence the practices and choices teachers make in the classroom. Nordström rises the important question of how proof items in textbooks "relate to students' access to proofs". By examining two series of commonly used Swedish upper secondary school books, she found that different definitions for proof occurred in the same textbook and that the notion of proof was often left implicit or not defined in a mathematically appropriate and meaningful way. This is likely to lead to substantial difficulties for teachers and students and, as Nordström reminds us, the important question remains: "By what means is it possible to make proof and its different aspects visible to students at the upper-secondary level?"

The longitudinal study undertaken by Hoyles and Küchemann compares not across cultures and textbooks, but students' responses over time. Their large-scale survey using written questionnaires to investigate students' understanding was supplemented by case studies to identify teaching practices, which may have influenced these responses.

These various studies and the resulting group discussion underline the importance of using a variety of methods in our research to overcome the limitations of any single method, and the importance of considering teaching practices more globally, not just taking a single perspective, be it student results, curriculum documents or text books.

Theme 3: Problems and Conjecturing

Students' approaches to mathematical problems and students' production and validation of conjectures were discussed in the light of three papers contributed by members of the working group. Lourdes Figueiras & Jordi Deulofeu presented observations and analyses of first year university students' conjectures for Heron's Problem. Oleksiy Yevdokimov described the conjecturing processes of secondary school students about properties of special lines in a triangle. Consuelo Cañadas Santiago & Encarnación Castro Martínez presented an analysis of secondary students' (inductive) reasoning in the context of the problem of adding two even numbers.

One suggestion put forward in the ensuing discussion was that slight changes in the formulation of a task could radically alter the way it was solved and consequently the way students prove the correctness of their solution. For example, Figueiras & Deulofeu had posed Heron's Problem (in their university course for future primary student teachers) as follows: “Let s be a line and A and B two points at the same side of s. For which point P in s is AP+PB the shortest way joining A and B?”. Knipping had exposed a similar group of students at her university to Herons’s Problem in a slightly different way. In her formulation it was not assumed that A and B were points at the same side of the line s, but only that they were not *on* the line s. This slight difference seems to have mattered here: in particular, it allowed students to make productive use of their solution of the case where A and B are on opposite sides of line s, to solve the more elusive standard case. On the other hand, in a subsequent investigation by Küchemann, using wording similar to that of Figueiras, he found that exhorting students to work dynamically by inserting the sentence “Explore what happens to the distance AP + BP as P moves on S” seemed to make no difference to their approaches.

Comparative analyses like this may make it possible to understand students’ conjectures in the context of how tasks and problems are formulated and posed to students. It may also lead to a better understanding of the interrelation of tasks, conjectures and students’ engagement in mathematical reasoning. Yevdokimov’s work with open tasks such as “Find out as many properties of a bisector of a triangle as you can”, lead to an interesting working group discussion. Not all students were in every case able to prove the conjectures they came up with. This suggests that even if students find the process of conjecturing to be fruitful and satisfying they might still be discouraged by the task of proving. This runs counter to the commonly held hypothesis that the production of conjectures motivates students to want to construct proofs. Thus we have to be aware of the potential of tasks not only for producing conjectures, but also in respect of students’ aptitude for proving (or refuting) their conjectures.

This problématique is also provoked by the paper of Cañadas Santiago & Castro Martínez. The authors had observed students who engaged in justifying the result obtained when adding two even numbers and found that students would use “inductive reasoning” to justify their conjecture. Again, the difficult part for the students was not coming up with a conjecture, even for the general case, but in deriving a generally true justification. Promising research on this problématique has been done by some of our Italian colleagues who have introduced the notion of “cognitive unity”. This aims to describe the complex relationship between the process of producing a conjecture and proving it. A paper on the notion of cognitive unity was presented at CERME3 (Pedemonte, 2003). Applying the notion to the relatively simple task used by Santiago & Martínez suggests that if students form a conjecture on the basis of empirical evidence only, then they will try to construct a proof on the same basis. Küchemann’s current work with teachers, using similar simple tasks, suggests that in part students’ difficulties may be a matter of initiation In fact adopting deductive, as opposed to empirical, approaches may not be so natural as it could be expected and seems to require a specific didactic intervention.

As a complement to the notion of ‘proof prototype’ discussed in Theme 1, it is sometimes useful to think of ‘task prototypes’. Thus for example, Heron’s task in the form presented by Figuieras would seem to be especially useful for generating conjectures that turn out to be false (and hence need to be refuted). It is also a task where some students might fruitfully draw on experiences in physics (angles of incidence and reflection, or the notion of a weight supported by a string attached to the focii of an ellipse...). Consuelo Cañadas spoke of the classic task, “If n straight lines all intersect each other (inside a circle), how many regions do they make (inside the circle)?”. It is difficult (impossible?) to solve this task generically, i.e. simply by looking at one (general) case. On the other hand the task can

be approached 'incrementally' by looking at what happens when another line is added (the nth line produces 1+n–1 = n new regions), but this is relatively complex. By contrast, consider another classic task, "If n straight lines all intersect each other, how many points of intersection are there?". This can be solved generically (each line intersects the n-1 other lines, but this is counting each intersection twice as it is formed by two lines, so there are n(n–1)/2 points of intersection). Of course, as with the previous task, it can also be tackled incrementally or by induction from specific cases.
Küchemann has recently been using the following task in secondary schools: "Find the sum of three consecutive numbers. What do you notice? Prove your result." Students tend to notice many different things and tend to work empirically. Thus if they notice that the sum is 3 times the middle number, they tend to confirm this by means of examples. Of course, the task can also be approached structurally, along the lines of 'The first number is 1 less than the middle number and the third number is 1 more than the middle number; hence the sum is equivalent to 3 times the middle number". Küchemann would argue that students would benefit from becoming familiar with arguments of this sort, which are often not difficult to understand. However, not all tasks can be tackled in this way. Take this example: "Find the product of three consecutive numbers. When is the result a multiple of 24? Prove your finding.". Here it is fairly straightforward to find out and prove that the condition is satisfied when the first number is even; however, it is harder to realise that it is also satisfied when the middle number is a multiple of 8. Thus, in the school context at least, this a good example of a task where students might benefit from using an empirical approach, and indeed from generating lots of data, for example by using a spreadsheet. The argument here is that teachers and students would both benefit from not only knowing that there are different ways of solving problems and constructing proofs but that different tasks lend themselves to different approaches.

As a final remark rising from the group discussion we would like to stress the importance of grounding the notion of proof in problem solving context, that means developing a culture of arguments supporting one's own solution through the systematic use of open problems, where producing a conjecture accompanied by the arguments supporting it. Tasks design becomes crucial and as clearly emerged from the discussion both the individual and the social component of the proving process.
Generally speaking, it becomes interesting to identify contexts where mathematical knowledge might emerge from solving problems and validating solutions, according to different school levels and different curriculum requirements.
Never the less, overcoming the possible gap between arguments supporting a statements (for instance, empirical evidence based on a limited number of example, or authoritative reasons) and a mathematical proof (i.e. arguments acceptable in the mathematics community) remains a major difficulty requiring the specific intervention of the teacher, who as a cultural mediator introduces students to mathematics practices, and among others to proving.
Although any contribution to the working group directly addressed the issue of teachers training, the key role recognized to the teacher - both in designing of a problem solving environment and in guiding the evolution towards shared mathematically acceptable arguments – asks further investigation in this field.

References

Balacheff N. (1999) Apprendre la preuve. In: Sallantin J., Szczeciniarz J. J. (eds.) *Le concept de preuve à la lumière de l'intelligence artificielle* (pp.197-236). Paris: PUF.
Fischbeim E., Kedem I. (1982). Proof and certitude in the development of mathematical thinking. *Proceedings of the 6th PME Conference,* Antwerpen Belgium, 128-131.
Pedemonte, B. (2003) What kind of proof can be constructed following an Abductive argumentation?

Proceedings CERME3, Bellaria, Italy.
Vinner S. (1983) The notion of proof: Some aspects of students' views at the senior high level. *Proceedings of the 21st PME Conference,* Shoresh, Israel, 289-294.

List of the contributions

Theme1

- David A Reid

The meaning of proof in mathematics education

- Hans Niels Jahnke

A genetic approach to proof

- Viviane Durand-Guerrier

Natural deduction in Predicate Calculus A tool for analysing proof in a didactic perspective

Theme 2

- Richard Cabassut

Argumentation and proof in examples taken from French and German textbooks

- Kirsti Nordström & Clas Löfwall

Proof in Swedish upper secondary school mathematics textbooks - the issue of transparency

- Dietmar Küchemann &Celia Hoyles

Pupils' awareness of structure on two number/algebra questions

Theme 3

- Lourdes Figueiras & Jordi Deulofeu

Visualising and Conjecturing Solutions for Heron's Problem

- Oleksiy Yevdokimov

About a constructivist approach for stimulating students' thinking to produce conjectures and their proving in active learning of geometry

- Consuelo Cañadas Santiago & Encarnación Castro Martínez

Inductive reasoning in the justification of the result of adding two even numbers

ARGUMENTATION AND PROOF IN EXAMPLES TAKEN FROM FRENCH AND GERMAN TEXTBOOKS

Richard Cabassut, *IUFM d'Alsace, Didirem Paris 7, France*

Abstract: *A study of French and German curricula of secondary schools have shown that arguments of plausibility and arguments of necessity are both encouraged in mathematic teaching. We show that the mathematics textbooks used in these curricula give examples of argumentations and proofs involving both kinds of arguments. These examples illustrate the theoretical frame presented in this paper and chosen to explain the use and the combination of arguments which can be understood through the concept of the function of validation.*

We have shown in [Cabassut 2003, 2005] that plausible reasoning, pragmatic proofs, visual perception or induction are present in validations used in the mathematical curricula of secondary schools in France and Baden-Wurtemberg. We expose here the theoretical frame chosen to compare the validation of mathematic statements in France and in Germany.

1 The theoretical frame

1.1 Toulmin's theory of arguments

From [Toulmin 1958] we consider an argument as a three-part structure (data, warrant, claim). We apply a warrant to data to produce a claim. We call arguments of plausibility the "arguments in which the warrant entitles us to draw our conclusions only tentatively (qualifying it with a 'probably') subject to possible exceptions ('presumably') or conditionally ('provided that …')" [Toulmin 1958, 148]. Toulmin calls them 'probable arguments'. As probability has a strong mathematical connotation, we prefer the term 'argument of plausibility'. It's also a reference to the plausible reasoning of [Polya 1954] described under the name of abduction by [Peirce 1960, 5.189]: "The surprising fact C is observed, but if A were true, C would be a matter of course; hence, there is reason to suspect that A is true". We call arguments of necessity "the arguments in which the warrant entitles us to argue unequivocally to the conclusion" [Toulmin 1958, 148]. The 'modus ponens' is an example of argument of necessity: A is observed, and 'if A then C' is true, then C is necessarily true. We call 'validation' a reasoning that intends to assert, necessarily or plausibly, the truth of a statement. A 'proof' is a validation using only arguments of necessity and an 'argumentation' is a validation using arguments of plausibility and maybe arguments of necessity.

[Perelman, Olbrechts-Tyteca 1969] distinguishes different audiences to which the validation is addressed. The self as audience and a particular audience are subject to persuasion which is grounded on the specificity of the character of the audience. A universal audience is subject to conviction which is based on rationality. We can consider that these audiences or their rationalities are attached to the institutions in which the validation is established.

For [Balacheff 1991, 188-189] "argumentation and mathematical proof are not of the same nature: The aim of argumentation is to obtain the agreement of the partner of the interaction, but not in the first place to establish the truth of some statement. As a social behaviour it is an open process, in other words it allows the use of any kind of means; whereas, for mathematical proofs, we have to fit the requirement for the use of knowledge taken from a body of knowledge on which people (mathematician) agree". We can consider that in a mathematical institution where a statement is proved (for example in a journal of mathematics, or in a mathematical seminar), the people (the mathematicians) agree with the mathematical knowledge that only admits arguments of necessity. The arguments of necessity in a mathematical proof of a statement establish the necessity of the truth of this statement. When a mathematician uses an argument of plausibility to sustain a conjecture, or as a heuristic to look for the proof of an assertion, he doesn't make his assertion more true or more plausible from the point of view of mathematical theory. This mathematician can obtain the agreement from other mathematicians on the plausibility of his conjecture, but this agreement will be based on intuition, experimental methods or everyday mathematical practice using inferences taken from outside of mathematical theory. For example, plausible reasoning is not admitted in mathematical theory. In an other institutions, for example in biology classes, or in maths classes during the heuristic phase of problem-solving, or in a pupils' group discussing the proof of a statement, an argumentation that uses arguments of plausibility will establish the plausibility of the statement if the arguments of plausibility employed are part of the body of knowledge of this institution. In this case all the members of this institution agree with the argumentation. In mathematical institutions or other institutions, the partners' agreement means that arguments employed are in the body of knowledge shared by all the members of the institution.

Mathematical institutions have formal theories to prove statements. Other institutions often have informal theory to argue for statements. "Demonstrative reasoning has rigid standards, codified and clarified by logic (formal or demonstrative logic), which is the theory of demonstrative reasoning. The standards of plausible reasoning are fluid, and there is no theory of such reasoning that could be compared to demonstrative logic in clarity or would command comparable consensus" [Polya 1954].

To explain the combination of arguments of necessity and arguments of plausibility in a same validation we need to identify the different functions of a validation.

1.2 Functions of a validation

De Villiers proposed different functions of the proof in mathematics : "verification (concerned with the truth of a statement), explanation (providing insight into why it is true), systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems), discovery (the discovery or invention of new results), communication (the transmission of mathematical knowledge)" [De Villiers 1990, 18]. We extend these functions of the proof to the validation in the teaching of mathematics. For the function of verification we distinguish two functions: the function of plausibility verifies the plausibility of the truth of an assertion by means of arguments of plausibility; the function of proof verifies the necessity of the truth of an assertion by means of arguments of necessity.

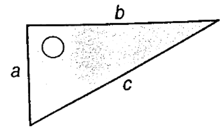
2 Examples from textbooks

2.1 mathematical properties available but not used

We consider a proof of Pythagora's theorem. We use Clarke's methodology looking at the differences for similar proofs. The first proof is found in a French textbook[1].

A. Découpage et constructions

1. Construire et découper, dans du carton, un triangle rectangle dont les côtés perpendiculaires mesurent par exemple :
a = 4 cm et b = 7,5 cm.
Ce triangle est une équerre.



2. À l'aide de cette équerre, réaliser les deux figures suivantes.

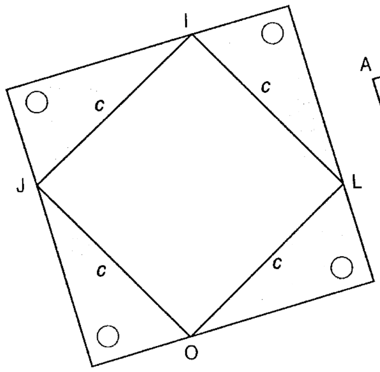


fig. 1

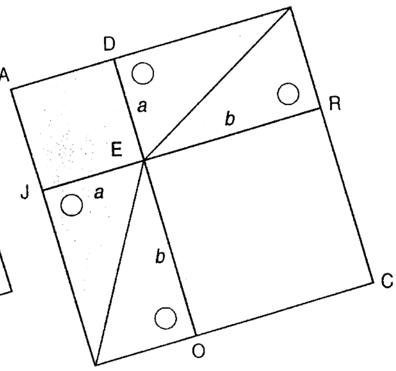


fig. 2

B. Observations, calculs et conclusion

1. En observant les figures 1 et 2, expliquer pourquoi :
aire(JOLI) = aire(JADE) + aire(OCRE).

2. Exprimer, en fonction de a, b ou c, les aires des carrés JOLI, JADE et OCRE.

3. Compléter par les lettres a, b ou c l'égalité suivante (qui résulte de ce qui précède) :

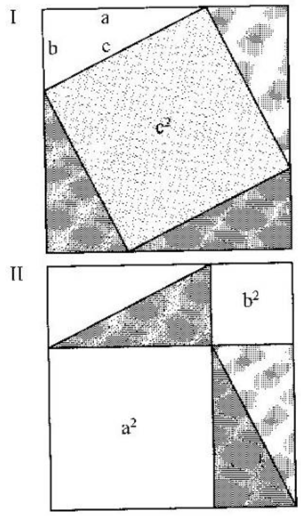
[......]^2 = [......]^2 + [......]^2

The proof is based on a puzzle technic[2] (pragmatic argument) where it is asserted without mathematical argument that the inner quadrangle JILO is a square. At this class level, it could be proved that JILO is a lozenge and that ∠JIL = 90° by consideration of angles. We assume that this fact is not proved because the main function of this proof is explanation. Pythagoras' theorem is explained as a theorem on the equality of areas: the area of the inner square equals the area of the two other squares which appears clearly in the puzzle technic. To prove that JILO is a square

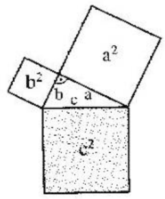
[1] The French textbook is : Le nouveau Pythagore class 4ème , 1998, Hatier, 165.
[2] [Knipping 2003, 84] observes classes where this proof is done. She mentions that the proof was known from the old Hindus.

would distract from the main explanation in terms of areas. We have found in Baden-Wurtemberg a textbook[3] from a similar grade level with the same kind of proof where it is not justified that the inner quadrangle is a square.

In jedem rechtwinkligen Dreieck nennt man die größte und stets dem rechten Winkel gegenüberliegende Seite die **Hypotenuse**.
Die beiden anderen, kleineren Seiten sind die beiden **Katheten** des rechtwinkligen Dreiecks.
In Fig. I wurden an den Ecken eines Quadrates vier kongruente Dreiecke abgeschnitten. Das verbleibende Viereck ist dann wiederum ein Quadrat mit dem Flächeninhalt c^2.
In Figur II wurden die abgeschnittenen Dreiecke anders in das ursprüngliche Quadrat eingeordnet. Die beiden verbleibenden Quadrate haben zusammen den Flächeninhalt $a^2 + b^2$. Es gilt also:
$a^2 + b^2 = c^2$.



Satz des PYTHAGORAS:
In jedem rechtwinkligen Dreieck haben die Quadrate über den Katheten zusammen den gleichen Flächeninhalt wie das Quadrat über der Hypotenuse.
$a^2 + b^2 = c^2$



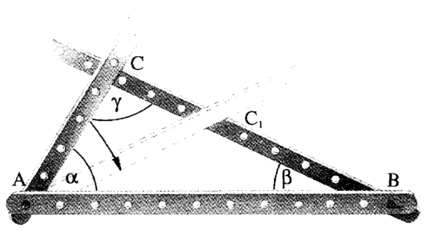
Both textbooks prefer a visual argument for which the inner quadrangle looks very plausibly like a square. The function of verification of the plausibility and the function of explanation make it so that a visual argument is preferred to a mathematical argument on angles. The German textbook used explicitly the isometric triangle property which is available; the French textbook used implicitly this property which is not available.

We have found other textbooks[4] where it is proven[5] that the inner quadrangle is a square with rigor and where the function of verification of the proof is stronger than in the previous textbooks.

[3] The German textbook is : Lambacher Schweizer class 9, Klett Verlag, 1997, 70.
[4] For example : Math 4[ème], Bordas, 1998, 203.
[5] Using properties of angles.

2.2 mathematical property not available

1 a) Welche Winkel des Dreiecks ABC ändern sich, wenn man den Stab AC nach rechts schwenkt und der Stab BC seine Richtung beibehält? Welcher Winkel wird größer, welcher kleiner? Vergleiche die Zunahme des einen Winkels und die Abnahme des anderen mit dem Winkelmesser.
b) Was lässt sich über die Summe $\alpha + \beta + \gamma$ sagen, wenn sich AC und BC mehr und mehr der Lage von AB nähern?



In Fig. 1 sind g und h parallel. In diesem Fall ist $\alpha + \delta = 180°$.
Ist – wie in Fig. 2 – h nicht parallel zu g, so entsteht ein Dreieck ABC. Der Winkel γ bei C hat gegenüber Fig. 1 um eine Winkelweite β abgenommen; gleichzeitig ist bei B ein neuer Winkel derselben Weite β entstanden (Wechselwinkel an Parallelen).
Jetzt gilt daher: $\alpha + \beta + \gamma = 180°$.

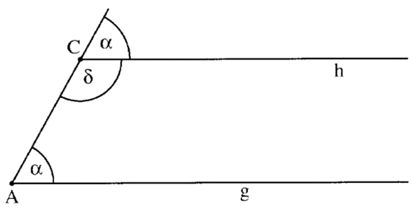


Fig.1

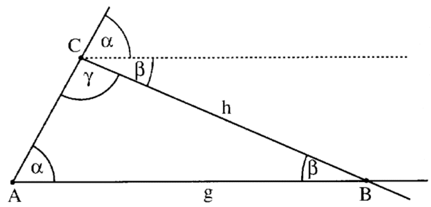


Fig.2

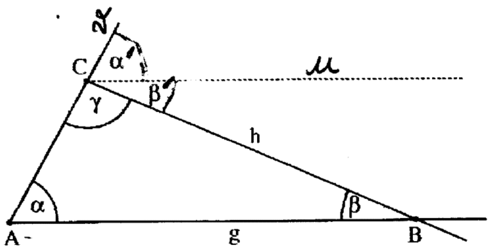
The proof of the property of the sum of angles in a triangle is illustrated in two textbooks. In both textbooks, the validation begins with an argumentation using pragmatic arguments (in the German textbook[6]: action with a Meccano game and measurement, in the French textbook[7]: cutting out, measurement), visual argument and mathematical arguments (calculation of the sum of angles). This previous validation develops two functions: the function of discovery of the property of the sum of angles in a triangle, and the function of the verification of the plausibility of the result. With the German Meccano technic, explanation of the proof is central: when the angle α decreases, the angle γ increases in a compensating way. The last German proof tries to develop the function of explanation, by determining the compensation β between α and γ, and the function of verification of the proof, by

---

[6] The German textbook is: Lambacher Schweizer class 7 (12-13 years old), Klett Verlag, 1994, 105.

[7] The French textbook is: “Nouveau transmath class 4ème” , 1997, Nathan, 225, class 5ème (12-13 years old).

using mathematical arguments of necessity. The use of mathematical argument of necessity seems to avoid the use of pragmatic arguments. In fact, there is a visual argument in figure 2 to assert that the angle β in C is equal to the alternate-interior angle β in B. Holland [2001, 56-57] shows that a formal proof without visual argument needs the technology of orientated angles: this technology is not available at this class level[8]. The proof with orientated angles without visual argument is the following:



step	data of the step	warrant of the step	claim of the step
1	hypothesis		ABC triangle with α=([AB),[AC)), β=([BC), [BA)), γ=([CA), [CB))
2	hypothesis		u is half straight line with C as origin, parallel to g, and with opposite direction to the half straight line [BA) with B as origin. A
3	hypothesis		v is the other half straight line of origin C completing the half straight line [CA)
4	hypothesis		α'= angle(u,v)
5	hypothesis		β'=angle([CB),u)
6	1, 2, 3, 4	definition of corresponding angles	α et α' are corresponding angles
7	1, 2, 5	definition of alternate-interior angles	β et β' are alternate-interior angles
8	2, 6	theorem of corresponding angles in case of parallelism	α=α'
9	2, 7	theorem of alternate-interior angles in case of parallelism	β=β'
10	3	definition of straight angle	([CA),v) = 180°
11	1, 2, 3, 4, 5	property of angles	([CA),v)= ([CA), [CB))+ ([CB),u) +(u,v) = α'+β'+γ
12	8, 9, 11	calculus	α+β+γ=180°

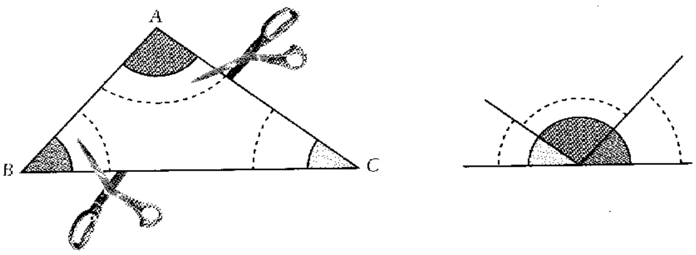
We can ask why the German textbook proposes another proof with a non mathematical argument (the visual argument revealed by Holland) following the

[8] class 7 for Germany and 5ème for France.

validations with Meccano or measurements proof which used also non mathematical arguments. On one hand, using visual arguments helps to develop the function of explanation and the function of verification of the proof, even if this last function is not completely accomplished in a formal way because of the visual argument; it could be considered as accomplished in a mathematics class where a visual argument could be, sometimes, used as an argument of necessity, for example in the cases in which orientation is involved. On the other hand, the function of systematisation is developed in the last proof because previous mathematical results[9] are used to establish, in a deductive way, a new result.

① En découpant

a. Trace sur papier blanc un triangle ABC comme celui dessiné ci-dessous.
b. Découpe chacun de ses angles, puis « regroupe »-les comme l'indique le dessin ci-dessous.



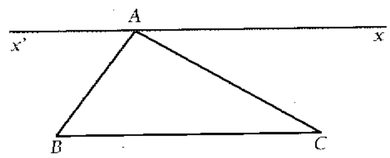
c. Quelle semble être la valeur de la somme des angles du triangle ?

② En mesurant avec ton rapporteur

a. Trace trois triangles.
b. Mesure les angles de chacun de ces triangles à l'aide d'un rapporteur, puis calcule la somme des angles de chaque triangle.
c. Quelle semble être la valeur de cette somme ?

③ Une démonstration à présent

La droite $(x'x)$ est parallèle à la droite (BC) et passe par A.



a. Compare les angles $\widehat{ABC}$ et $\widehat{BAx'}$, puis $\widehat{ACB}$ et $\widehat{CAx}$.
b. Explique alors pourquoi la somme des trois angles du triangle ABC est égale à 180°.

---

[9] Calculation rules, definition and properties of angles: corresponding, alternate-interior, supplementary.

3 Conclusion

We have seen that arguments of plausibility can be used to develop the functions of explanation and plausibility, or to replace a missing mathematical argument to help to develop the function of systematisation. We have seen that mathematical arguments can be used to develop the functions of proof, explanation or systematisation. In these examples both kind of arguments are combined in a same validation. In [Cabassut 2002, 2003, 2005] we show other examples[10] of these combinations. On one hand, there are non mathematical validations in others institutions[11]. Mathematical arguments can join non mathematical arguments to make a validation[12] in the institution that teaches mathematics: this new hybrid validation is called a didactic validation, and is created by transposing a validation used in a non-mathematical institution onto a validation used by institutions that teach mathematics. In this case, special functions are developed (function of explanation in the previous examples, function of discovery to prepare the limits in [Cabassut 2002]). On the other hand, the replacement[13] of some mathematical arguments (available or not) by non-mathematical arguments is a transposition of a mathematical proof onto a didactic validation where the functions of explanation, plausibility or proof, and systematisation can be developed. In this sense didactic validation is the double transposition of mathematical proof and of the validation of non-mathematical institutions. The combination of arguments of plausibility and mathematical arguments are generally not allowed in a context of assessment where the functions of communication, systematisation and proof are developed as shown in [Cabassut 2005]. The pupils have to understand the change in the didactical contract depending on what functions of the validation are developed and on what combinations of arguments are allowed. The main difficulty is that these two kinds of arguments refer to two different conceptions of truth.

[10] like formula of the circumference and area of a circle.
[11] like daily life or experimental sciences classes where pragmatic arguments can be sufficient.
[12] for example for the sum of angles of a triangle: measurement, use of Meccano or cutting.
[13] for example in the proof of Pythagora's theorem or in the last German and French proofs of the sum of angles of a triangle.

4 References

Balacheff N.: 1991, The benefits and limits of social interaction : the case of mathematical proof *Mathematical knowledge: its growth through teaching*, pp.175-192, Kluwer Academic Publishers, Alan J. Bishop et al. (eds).

Cabassut R.: 2002, Pourquoi démontrer ? un exemple allemand sur les aires et les volumes pour entrer dans le processus de preuve et d'explications, in revue *Repères* n°47, 17-39.

2003, Enseigner la démonstration en mathématiques c'est quoi ? pourquoi ? pour qui ? comment ? Eléments de réponses à partir de l'étude des programmes des premières années de l'enseignement secondaire en France et en Bade-Wurtemberg, in *bulletin de l' APMEP* n° 449, 757-770.

to be published in 2005, *Démonstration, raisonnement et validation dans l'enseignement secondaire des mathématiques en France et en Allemagne,* thèse, université Paris 7.

Clarke D.: 2004, Issues of Voice and Variations : The Problematics of International Comparative Research in Mathematics Education, paper submitted to discussion group 11: *International comparaisons in mathematics education (ICME 10)*, Copenhagen, Denmark, July 4-11.

De Villiers M.: 1990, The role and the function of proof in mathematics, *Pythagoras* n°24, 17-24.

Holland G.: 2001, *Geometrie in der Sekundarstufe,* Spektrum Akademisher Verlag, Heidelberg.

Knipping C.*: 2003, Beweisprozesse in der Unterrichtspraxis Vergleichende Analysen von Mathematikunterricht in Deutschland und Frankreich,* Verlag Franzbecker.

Peirce C.S.: 1960, *Collected papers*, Cambridge, MA: Harward University Press.

Perelman C., Olbrechts-Tyteca L.: 1969, *The New Rhetoric: A Treatise on Argumentation,* Notre Dame: University of Notre Dame Press.

Polya G.: 1954, *Mathematics and plausible reasoning*, Princeton University Press.

Toulmin S.E.: 1958, *The Uses of Arguments,* Cambridge University Press.

A PROPOSAL OF CATEGORISATION FOR ANALYSING INDUCTIVE REASONING

Mª Consuelo Cañadas Santiago, *Universidad de Zaragoza, Spain*
Encarnación Castro Martínez, *Universidad de Granada, Spain*

Abstract: *In this paper we present an analysis of the inductive reasoning of twelve Spanish Secondary students in a mathematical problem-solving context. Students were interviewed while they worked on two different tasks. Based on Polya´s steps and Reid's stages for a process of inductive reasoning, we propose a categorization of seven stages for analyzing this kind of reasoning. In this paper we present some results of a previous study of Cañadas (2002).*

Proof appears to be a real problem at different educational levels. On one hand, although Spanish pre-service teachers are accustomed to formal proof, they have difficulties in proof teaching (Cañadas, Nieto & Pizarro, 2001). On the other hand, Secondary students do not make as much progress as they are supposed to in their reasoning. One possible reason lies in the fact that they cannot suddenly acquire the necessary reasoning skills for developing formal proof. They need a period of time to transform their daily reasoning into formal one (Jones, 1996). Some studies show that Primary and Secondary students are able to formulate conjectures, examine and justify them if they start working from particular cases (Lampert, 1990; Healy & Hoyles, 1998). Some of these actions related to conjecturing are part of the inductive reasoning process. We are interested in analyzing the way Secondary students go from particular cases to generalization.

This paper consists of four main parts. First, we present the theoretical framework of the study, which includes our proposal of categorization for inductive reasoning. Second, we outline the methodology of our empirical study. Third, we present some sample data organized according to the stages of our categorization and finally, we discuss some results of the study.

THEORETICAL FRAMEWORK

Inductive reasoning

Inductive and deductive reasoning are the two traditional types of reasoning considered. We will focus on the first one although we are conscious that sometimes it is very difficult to separate them in students´ work. Inductive reasoning is very important from a scientific viewpoint because it allows us to obtain scientific knowledge (Pólya, 1967). In the particular case of mathematics teaching, Pólya indicates that inductive reasoning is a method of discovering properties from phenomena and of finding regularities in a logical way. We will consider that inductive reasoning in Mathematics Education is a reasoning process that begins with particular cases and produces a generalization from these cases.

There is a confusing term which involves proof and reasoning: mathematical induction (MI). It is a formal method of proof based more on deductive than on inductive reasoning. Some processes of inductive reasoning conclude with MI but this does not always occur. For example, we can not construct a correct proof by MI to justify that the angles in a triangle have a sum of 180º because the set of all triangles is not ordered.

From the curricular perspective, we find that at Secondary level and in the two courses before the university, mathematics has as the main aim developing a certain level of reasoning and abstraction. At the end of these studies, students must master operations of abstract thinking that allow them to understand conjecture formulation, observation of particular cases, experimentation, hypothesis validation, and to elaborate explanations and theories structured in some way, etc. (M.E.C., 1989, p. 74). These actions are related to inductive reasoning, as we will see.

Stages in inductive reasoning

Pólya (1967) indicates four steps of a process of inductive reasoning: observation of particular cases, conjecture formulation based on previous particular cases and conjecture verification with new particular cases. In this context of empirical induction from a finite number of discrete cases, Reid (2002) describes these stages: observation of a pattern, the conjecturing (with doubt) that this patterns applies generally, the testing of the conjecture, and the generalization of the conjecture. Based on these studies and taking into account our empirical work (Cañadas, 2002), we consider seven stages as describing the inductive reasoning process. In the following paragraphs, we explain these stages and we illustrate some of them in the context of the task of determining the maximum number of regions formed by *n* lines.

1. **Observation of particular cases**. The starting point is experiences with particular cases of the problem posed.
2. **Organization of particular cases**. The students´ responses are different when they are able to organize particular cases in some way. They use different strategies to systematize and facilitate the work with particular cases.
3. **Search and prediction of patterns**. Observing particular cases (organized or not), we can think about the next, unknown case. In this sense, students are thinking about a possible pattern just for the cases they are observing. They are not thinking about applying the pattern to all cases.
4. **Conjecture formulation**. A conjecture is a statement based on empirical facts, which has not been validated. This "conjecture formulation" is like Reid's "conjecturing (with doubt)" which means making a statement about all possible cases, based on particular ones, but with an element of doubt. A clear example for this is when some students claim: "I think that you get the double of the number of straight lines" but they are not sure about that because they are thinking of what happens in the first two particular cases.
5. **Conjecture validation**. When students formulate a conjecture with doubt, they are convinced about the truth of their conjecture for those specific cases but not for other ones. At this stage, they try to validate their conjectures for new specific cases but not in general. In our example, they might validate their conjecture by drawing more than two lines.
6. **Conjecture generalization**. Mathematics patterns are related to a general rule, not only to some cases. Based on a conjecture which is true for some particular cases, and having validated such conjecture for new cases (conjecture validation), students might hypothesize that the conjecture is true in general.
7. **General conjectures justification**. The first step on the way to confirm or reject a general conjecture is validating it with particular cases. But this is not enough to justify a generalisation. It is necessary to give reasons that explain the conjecture with the intention of convincing another person that the generalization is justified. At this point, a formal proof can provide the final justification that guarantees the veracity of the conjecture.

These stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels necessarily occur, there are a lot of factors involved (as will be discussed below).

METHODOLOGY

The problem posed

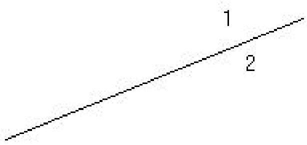
Cañadas (2002) chose two tasks to analyze inductive reasoning in a group of twelve Spanish Secondary students. Here we have chosen one of the tasks considered in that study to illustrate some of the results of the large set of data gathered. This task was proposed to the students in an interview context in the following way:

"What is the maximum number of regions you can get in the plane if you draw of straight lines?"

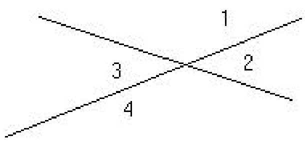
We notice here some specific characteristics of this problem that are considered for our analysis. The correct answer was unknown by the students so it is a problem for them. We attend that students recognize the general pattern. For this reason, we do not mention the number of lines they have to draw. Students must notice the functional relationship between the number of lines and the number of regions. This relationship relates to a second degree polynomial, which appears in Spanish Secondary curriculum.

Another important aspect was the way of representation. Although we posed the task in a verbal way, one of our interests is to analyze in which type of representations the students express their conjectures and their own justifications.

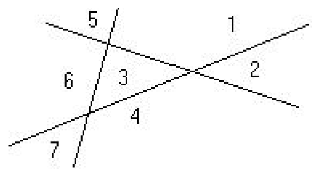
The correct answer was unknown to the students so it can be considered a problem for them. All the students started by drawing straight lines when they heard the problem. We show in what follows a correct (not the unique) way to solve this problem.



1 straight line
2 regions



2 straight lines
4 regions



3 straight lines
7 regions

Continuing with particular cases, we can get the following data that we organize in a table:

Num. Straight lines	Num. Regions
1	2
2	4
3	7
4	11
5	16
…	…

With these data, we observe that with the first straight line, we get two regions; when we draw two lines, we get four regions (two more than in the previous case); when we draw three lines, we get seven regions (three more than in the previous case); when we draw four lines, we get eleven regions (four more that in the previous case), etc. Generalizing this pattern, we can say that when we draw the n^{th} line, we get n new regions. In this sense, we can write the number of regions as $2+\sum_{i=2}^{n} i$. Developing this expression and calling a_n to the number of regions, we get:

$$a_n = 2+\sum_{i=2}^{n} i = 2+\left[\frac{n(n+1)}{2}-1\right] = \frac{n(n+1)}{2}+1$$

After students´ conjecture formulation, they were asked to justify their conjectures. At Secondary, we did not expect for a formal proof. Our aim was to analyze if students can develop a particular way to justify their own conjectures and what characteristics these ways of justification have.

Interviews

We used individual interviews to observe the students' reasoning. The interviewer was one of the researchers and her aim was to propose the problem to the students and ask them questions in order that they explain what they were doing. She had an interview plan which allowed her to guide students though questioning so that we could observe their reasoning from particular cases to generalization.

Students

Secondary is an adequate level to investigate what happens to inductive reasoning in this kind of problem from content and cognitive viewpoints. In Primary, mathematics is strongly connected to empirical facts and didactical materials. Spanish curriculum claims for Secondary that empirical-inductive reasoning must be reinforced in parallel to the use of deductive reasoning. In this sense, we observe that there is an evolution from the basic level of mathematics, based on empirical work and inductive processes, to high level mathematics, based mainly on mathematical structures and relationships among them.

We interviewed twelve Spanish students (six girls and six boys) from the four years of school before going to university (14-18 years old). We choose three students from each year with different academic results in order to obtain a wide variety of responses.

DATA COLLECTION AND ANALYSIS

We collected data in three complementary ways: the interviews were recorded on audio tape, we gave worksheets to the students so that they could write their work (if they wanted to) and the interviewer took notes during and after each interview about relevant details that could not be recorded on the tape.

We analyze these data in a qualitative way using Nud*ist revision (N4). This program allowed us to see the data in a structured way and to discover details, patterns and relations that would be more complicated to discover by hand.

For presenting and analyzing the data, we symbolize the students as 1, 2, 3 and 4 depending on the year to which s/he belongs to. A, B or C indicates high, medium or low academic results. For example, 3A is a third year student whose academic results are higher than her/his classmates.

In our analysis, we notice that not all the stages in inductive reasoning are necessarily present in all tasks and not all students show the same stages for the same task. However, there are two relevant characteristics because they appear in all the stages:

- Spontaneity. We analyze if students were able to advance in their reasoning by themselves or, on the contrary, they need or even required the interviewer intervention.
- Representation mode. We posed the problem in the verbal language. However, we consider four possible ways to express their reasoning related to this problem: verbal, arithmetic, geometric and algebraic.

Moreover, we noticed some general characteristics that facilitate the advance of one stage to the next one. These characteristics allow us to compare the work of these students in each stage but they can not be considered as sub-stages because there is no order relation among them. We summarize this in table 1:

Table 1. Stages and characteristics

Stages	Characteristics
Observation of particular cases	Number of particular cases Type of particular cases Systematic way
Organization of particular cases	Tables
Search and prediction of patterns	Based on…
Conjecture formulation	Use of school knowledge
Conjecture validation	Based on…
Conjecture generalization	Characterization of even numbers
General conjecture justification	Justification necessity Based on particular cases General case

RESULTS

We will comment our results using the categorization described above and the related characteristics. We will summarize some of the main results in two tables. In table 2 we present the responses related to the first two stages which are related to particular cases. In table 3 we summarize results mainly related to conjectures.

Table2. Particular cases

Student	Spontaneity	Number	Systematic	Organization
1C		4	x	
1B	x	5		x
1A	x	4		x
2C		3		x
2B		4		
2A	x	4	x	x
3C	x	5		x
3B	x	8	x	
3A		5		x
4C		4		
4B	x	5	x	x
4A	x	3	x	

- Observation of particular cases. As we can see in table 2, seven students turned to particular cases in a spontaneous way, without any interviewer intervention. Finally (in the rest of the cases with the interviewer's suggestion), all of them considered a number of particular cases higher than three, so their work was similar in this respect. This common fact is relevant because they considered (some of them, explicitly) that the more particular cases they considered, the easier it would be to obtain a pattern. They had made a translation from verbal representation to the graphical one.

One difference in work with particular cases that had influence on the outcome of the problem solution was the systematic way of drawing the straight lines. 1C, 2A, 3B, 4B and 4A took into account the order of particular cases, starting by drawing one line and they continued with successive cases. In other cases, the students tried with different number of lines in no order. For example, 4C drew three lines and the next particular case he considered was drawing eight lines.

- Organization of particular cases. Organizing particular cases can make it easier to observe a pattern. All of the students at this stage were able to express the particular cases in an arithmetic representation. As we can see in table 2, seven of the students organized their particular cases in tables or equivalent ways like lists of numbers.

- Search and prediction of patterns. The first pattern we observed in the students' reasoning was in graphical representation, when most of them decided that they had to draw the new line for a particular case, intersecting as many previous lines as they could.
We observed that it is very difficult to formulate a conjecture for students who have not organised the data.

- Conjecture formulation. All of the students base their reasoning on the particular cases they have considered and as it is very difficult for them to obtain new particular cases, they refer to their knowledge about linear relationships. In this sense, we observed students who thought that they would get one more region than the number of lines they drew; students who thought that they would get twice as many regions; and students who tried to establish a proportional relationship. They were not convinced of these conjectures. They formulated them with some doubt because they could not be sure they were is true in general.

- Conjecture validation. In the previous stage, they simultaneously try to validate the conjectures with particular cases. It is the method they use to be more convinced about their own conjectures. Some students used particular cases to accept the conjecture. And for other students, these cases allowed them to notice that they had formulated an invalid conjecture. For example, when 1A thought that he would get twice as many regions, he claimed:

> 1A: yes. With two lines, four regions. Then with three lines, I must get six… but… no because I get seven.

All of the students formulated a conjecture but in many cases, these conjectures were not correct. They observed whether their conjecture were true for the particular cases they considered before. The following reaction of these students´ next was to verify their initial conjectures with all the particular cases they were able to draw. When they had a conjecture that was valid for them, they formulated a conjecture for the general case.

Table 3. Conjectures

Student	Infin.	Simple relation	Linear			Recurr.	No recurr. Generaliz.
			n+1	2n	Rule of 3		
1C		x		x			
1B		x		x			
1A		x	x	x	x		
2C	x	x		x	x	x	
2B	x	x					
2A	x	x	x			x	x
3C		x		x		x	
3B	x			x		x	
3A	x	x		x			
4C	x	x		x	x	x	
4B	x			x		x	
4A	x	x		x		x	

- Conjecture generalization. Eight students did not recognize the functional relationship between the number of straight lines and the number of regions at the beginning of their reasoning, although they had been working with some particular cases. These students claimed that, as they can draw as many lines as they want, they can get an infinite number of regions. The interviewer tried to guide them towards the objective of the problem.

The second general conjecture was considered by all the students except 3B and 4B. These ten students noticed that the more lines they draw, the more regions they get. This fact is indicated in table 3 as "simple relationship".

All the students except 1A, 2B and 3A used algebraic language to express their conjecture for the general case. They clearly kept in mind the link between this kind of language and the generalization process.

Seven students noticed the recurrence relationship, observing the difference between two consecutive particular cases, although not all of them expressed it in the same way. Most of them, although they tried to use algebraic language, were able to express the recurrence in arithmetic or verbal language. In general, they had difficulties expressing the recurrence relationship in algebraic terms. For example, we observed student 3B, who feels completely confused in this sense:

> 3B: uff! [...] the difference between this case and the next one, we consider "z". Then "z" will be the difference and "x" is the number of lines. "y" is another number of regions and "a" is another number of regions. Then "z" will be... will be… ufff, I don't know.... […]. Let's see, here we have a set of regions, they will be "x", "y" and "h". Then the differences between "x" and "y" will be "z". The difference between "y" and "h" will be "z+1". The difference between "h" and "j" will be "z+2".

Other students noticed that the recurrence relationship is useful on some occasions but not always. For example, 4B claims:

> [...] I don't know how to express numbers.. there is something here that not... I don't know how to write that because if I use the number of lines that we had in the previous case, then I need to know the other number... and that number is what we had in the previous too plus…

But although some students recognize the limitations of the recurrence relationship, just one of them (2A) detected a different way to generalize the functional relationship.

- General conjecture justification. No student recognized the necessity of justifying their result on their own. They saw the result as an evident consequence from particular cases, without needing any additional justification to be convinced of its truth.

DISCUSSION

We show a categorization of seven stages for describing the inductive reasoning process. These have arisen from concrete tasks but they have a general character that permit us to use them for other problems. Every stage admits some characteristics that depend on a number of factors, such as the kind of task or the students involved. We have presented them in a general way, so they might be applied for other problems, too.

Inductive reasoning appeared implicitly or explicitly in the work of all the Secondary students interviewed. Students turned to particular cases when they tried to get a general pattern, so we can conclude that inductive reasoning appears naturally at these educational levels. These students show a tendency to take an empirical approach rather than to work with mathematical structure. We are now collecting new Spanish data that suggests the same idea, which is in contrast with the curricular expectations of students at the end of Secondary level.

The seven students who justified the general conjecture found a mathematical pattern from particular cases obtained from the characterization used in their justifications. This confirms that searching for patterns is a relevant and necessary step in inductive reasoning process in lower University levels and this kind of work can provide students a way to get the generalization process in a more significant form. In this sense, if students do not have habits of mind needed in order to discover a mathematical structure from particular cases, it will be very difficult for them to work with mathematical structures significantly in later courses.

Many students considered the conjecture obvious on the basis of particular cases and did not think a general conjecture justification necessary to validate their statements.

Spanish students at these levels are still not accustomed to proving their conjectures. In some way, students have validated their conjectures with some particular cases and some of them recognize the need to prove the general expression to be convinced of its truth. This kind of activity can be used to introduce students to justification tasks.

We have found two main points where students have difficulties. The first point was in finding the pattern. Even when they organized some particular cases of the underlying pattern, they did not know how to obtain the pattern. They considered the linear pattern but did not try the quadratic function as a possible pattern, in spite of their work with it in mathematics classes. The other difficult point was when they had found the pattern – the recurrence relation in most cases - and they did not know how to express the pattern.

Algebraic language appeared in all cases when the students tried to express the generality of the pattern. So generalization activities can be considered as a way to introduce algebra, as Mason pointed out (Mason, 1996).

In general, students do not feel sure with their own work and they need – even require - the interviewer's intervention.

We did not notice significant differences among students' reasoning in the different courses of Secondary level. It happened in the same way with students with different academic results belonging to the same year. We only detected some differences in the way they expressed their argumentations.

Acknowledgments
I wish to thank David Reid for his remarks on a previous version of this paper.

References

Boletín Oficial del Estado. (2003). *Real Decreto, 831/2003 y 832/2003*. Madrid: Ministerio de Educación y Ciencia.

Cañadas, M. C. (2002). *Razonamiento inductivo puesto de manifiesto por alumnos de secundaria*. Granada: Universidad de Granada.

Cañadas, M. C., Nieto, E. & Pizarro, A. (2001). El valor de la demostración en la educación secundaria. In J. Berenguer, B. Cobo y J. M. Navas (Eds.). *Proceedings of Investigación en el aula de matemáticas: retos de la educación matemática del siglo XXI, España*, 167-172.

Healy, L. & Hoyles, C. (1998). *Justifying and Proving in School Mathematics. (Technical Report on The Nationwide Survey)*. Institute of Education, University of London, February, 1998.

Jones, L. (1996). A Developmental approach to proof. *Proceedings of the 8th International Congress on Mathematical Education*, 109-154.

Küchemann, D. & Hoyles, C. (2002). Students´ understanding of a logical implication and its converse. In A. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th International Group for the Psychology of Mathematics Education, Norwich UK*, Vol. III., 241-248.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29-36.

Mason, J. (1996). Expressing Generality and Roots of Algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to Algebra. Perspectives for Research and Teaching*, pp. 65-86. Dordrecht/Boston/London: Kluwer.

Pólya, G. (1967). *Le découverte des mathématiques*. París: DUNOD.

Reid, D. (2002). Conjectures and refutations in Grade 5 Mathematics. *Journal for Research in Mathematics Education 33*(1), 5-29.

NATURAL DEDUCTION IN PREDICATE CALCULUS A TOOL FOR ANALYSING PROOF IN A DIDACTIC PERSPECTIVE

Viviane Durand-Guerrier, *IUFM de Lyon & LIRDHIST, Université Lyon 1, France*

Abstract: *In this paper, we intend to provide theoretical arguments for the importance of taking account in quantification matters while analysing proofs in a didactic perspective, not only at tertiary level, where various research are still available, but also at secondary level and we argue that natural deduction in predicate calculus is a relevant logical reference for this purpose. Following Quine, we emphasize on an example the interest of formalizing mathematical statements in Predicate Calculus in a purpose of conceptual clarification. In a second part of the paper, we give some short insights about the theory of quantification before exposing the system of Copi for natural deduction. The last section is devoted to analysing a proof using the logical tools offered by natural deduction in predicate calculus.*

I. About he importance of quantification in elementary geometry

It is widely recognized that in tertiary mathematical education, quantification matters are central and source of strong difficulties, even for gifted students (Dubinsky & Yparaki, 2000, Selden & Selden, 1995, Epp 2004, Chellougui, 2003) ; but it seems that, in secondary mathematical education, a low interest is paid on quantification matters.

Concerning the French context, in most research in didactic of mathematics, the logical tools used to analyse proofs are referring mainly to propositional calculus. This is the case for Toulmin's model, used for example by Pedemonte (2003) or Hoyles & Küchman (2003). This is also the case for Duval's analysis (Duval, 1991, 1995) widely used in French research about argumentation and proof. In these frames, the focus is on the fundamental inference Modus Ponens. Toulmin (1993) introduces the notion of *warrant*, which is a statement supporting the claim as a consequence of the data. Duval (1995) speaks of "Enoncé-tiers", a statement in the form "if *p*, then *q*", namely a theorem that is already known. Giving such a theorem, once it is stated that, in the considered situation, the conditions required in the antecedent are fulfilled, the conclusion is to be detached. In such frames, the bricks for analysing reasoning are clearly propositions; that means linguistic entities likely to be either true or false. Their main interest is their simplicity and their wide spectrum of application. However, as we have shown it on various examples (Durand-Guerrier 2003a, 2003b), there are a lot of proofs, even in elementary mathematics, that do not fall under this type of analysis. Indeed, quantifications matters are usually involved in mathematical proofs, and not all of them can be absorbed in propositional calculus The ones with which no problem may arise are those classical proofs that require a universal conditional theorem with exactly one free variable: "$\forall x\ T(x) \Rightarrow F(x)$". But of course, even in a Geometry course taught in middle school, you can find statements involving two or more variables and eventually both existential and universal quantifiers. We will now illustrate this point with an example. Let us consider the following statement: "for all three points that do not lie on the same line, there exists a unique circle on which these three points are lying". In order to formalise this statement, three one-place predicates: *P* (to be a point) ; *L* (to be a line) ; *C* (to be a circle) ; a two-place predicate : R (to lie on), and six variables *x*, *y*, *z* *t*, *u*, v are required to provide the following formula :

$$(1)\ \Big(\forall x \forall y \forall z P(x) \wedge P(y) \wedge P(z) \wedge \neg\big(\exists t\big(L(t) \wedge R(x,t) \wedge R(y,t) \wedge R(z,t)\big)\big)\Big) \Rightarrow$$
$$\exists u\Big(C(u) \wedge R(x,u) \wedge R(y,u) \wedge R(z,u) \wedge \Big(\forall v\big(\big(C(v) \wedge R(x,v) \wedge R(t,v) \wedge R(z,v)\big) \Rightarrow v = u\big)\Big)\Big)$$

Obviously, this is generally hidden in the mathematic class. The main reason is that most often the figure provides the information expressed in the statement, so that it is no use to have in mind all the condition while solving a classical problem relying on a particular figure. However, if students are supposed to be able to overcome the figure information to deal with general statements, it is worthwhile to keep in mind the complexity of the formula. The syntax of formula (1) reminds us that behind our ordinary simple gestures in mathematics, are hidden many constraints to be taking in account. In this particular case, it becomes apparent when you have to reason in general cases : either it is necessary to control that the considered points do not lie on a same line, or it is necessary to distinguish the particular cases from the general ones. More over, the last part of the formula gives a path to proving uniqueness in such a case. Considering precisely the conditions lead to explicit the difference between a line, a general curve, and a circle, and to emphasize the fact that in Euclidian Geometry "to lie on the same line" for two points is a general result, while for three points or more it is a property that might be satisfied or not, and to continue with the fact that you have the same result replacing "to lie on the same line" by "to lie on the same circle" and "two points" by "three points". Finally, a question such as "what is it for four points" could arise. More generally, it emphasizes the fact that behind symbols, there are objects of various nature and with various properties This is in keeping with Quine' s point of view who claims that the formalisation in Predicate Calculus contribute to conceptual clarification.

II. Difficulties with statements involving quantifiers

We have given in the previous paragraph some arguments that focusing on quantifiers may be useful for teaching mathematics, even at secondary school. But, since the late seventies, after the disaster of the so called " Réforme des mathématiques modernes" in France, logical considerations have been thrown out of programs and progressively from mathematics classroom. In a French famous book addressed to mathematical prospective teachers, Glaeser (1973) notes that quantification matters and consideration about symbol's logical status overcome the purpose of its book, due to the subtlety of distinction between individual and variable.

A way to escape these questions is to work with generic elements, so that quantifiers and variables disappear to benefit of individuals. This is the main way we do in secondary mathematics class in France, following the Euclidian tradition. In this respect, the theoretical frame proposed by Duval (1995) to analyse proofs fit with this ordinary practise. However, as we have shown elsewhere, this may lead on the one hand to deep misunderstanding in statements, particularly those involving implication (Durand-Guerrier 2003a) or negation (Durand-Guerrier & Ben Kilani, 2004), and on the other hand to invalid proofs (Durand-Guerrier & Arsac, 2003). In the last case, it happens that even tertiary mathematical teachers do not recognize the logical error due to an incorrect handling of quantifiers. It can also occur that a well-founded doubt about validity appears while the proof is actually valid (Arsac & Durand-Guerrier, 2000). Our own research corroborates results from Selden & Selden (1995) who claim that unpacking the logic of a mathematical statement is necessary for advanced mathematics, and provide empirical data showing that doing this is very difficult for most students. Chellougui (2004) shows clearly, on the example of the notion of supremum, that difficulties with quantification matters at university are underestimated in French textbooks and, in a Tunisian context rather similar in this respect with the French one, in courses provided to students. Bloch (2000) had already shown that difficulties with quantification matters could arise when dealing with the notion of upper bound. As a consequence, far from contributing to conceptual clarification, formalisation in predicate calculus leads to opacity for definitions and erratic use in proofs. Nevertheless, it appears, in semi-directive interviews that, as soon as there are no more quantifiers involved, students recover their ability to prove.

III. Presentation of the natural deduction in *Symbolic Logic* by Copi.

III.1. Some brief insights about the theory of quantification

Although Aristotle was already deeply concerned with quantification matters, it takes a very long time before a sound quantification theory could be established. According with most authors, Frege (1879) is the first logician who established it on a secure foundation ; he was followed by Russel (1903) and many others who developed it all along the first half-twentieth century. Quine (1902-2002), the famous American logician and philosopher defended all along his life the importance and the relevance of predicate calculus system and its methods for sciences in general and mathematics in particular (Quine, 1950, 1987). In his widely used textbook of formal modern logic (Quine, 1950, fourth edition 1982), two parts apart from four are devoted to quantification - Part 2: General terms and quantifiers; Part 3: General theory of quantification -. A large treatment of proof for logical validity is provided, referring in particular to what he calls the main method, consisting in proof for inconsistency, namely *reductio ad absurdum,* and the dual method for direct proof for logically valid implication. As an introduction of the main method, he wrote :

" We turn now to a proof procedure that will be found to be complete: adequate to establishing the validity of any quantificational schema and hence also to establishing any implication[1] and any inconsistency." (Quine, 1982, p.190)

It is important to point this at the very beginning of the exposition because, in this method, a rule relying on a logically invalid quantificational schema is introduced; and so it is for the dual of the main method for direct proof that he presents further and which inherits of the completeness of the main method. The direct method is "of a type that is known as natural deduction and stems, in its broadest outlines, from Gentzen and Jaskowski (1934)" (ibid. p.244). Many versions of natural deduction can be found in the literature; all of them follow the same purpose: to remain as near as possible of the way mathematicians reason. What we learn from history is that quantification is much more complex than it is generally thought and that the main purpose of the promoters of predicate calculus is the control of validity, especially in mathematic. In this respect, although these methods have been elaborated for controlling validity in Predicate calculus itself, and opposite with other methods for validity, natural deduction systems provide relevant tools for controlling validity of mathematical proofs[2].

II1.2. General features of natural deduction systems

The main interest of natural deduction in a didactic perspective lies in rules for elimination and introduction of both propositional connectors and quantifiers. Elimination for implication is the well-known inference rule called Modus Ponens. Introduction of implication is called by Quine "Condizionalisation" :

"[The rule of conditionalization] consisted in showing that whenever one statement could be deduced from another in the concerned system, the conditional formed of the two statements could also be proved as a theorem by the original rules of that system"(Quine, 1982, p.244)

It is easy to recognize here the classical way to prove a conditional statement in mathematics, in case both antecedent and consequent are propositions. It is related with the theorem of deduction established separately by Herbrand and Tarski and it plays a fundamental role in mathematical proofs. However, it is rather remarkable that the importance of this rule is generally not emphasize in didactic research about reasoning, the focus being preferably pointed on Modus Ponens. The same phenomenon is to be seen concerning the rules for elimination and introduction of quantifiers that are nearly never considered in mathematics classroom, even at the university, being replaced most often by informal reasoning rules (Durand-Guerrier & Arsac 2003)[3]. There are four rules concerning quantifiers ; two of them are relying

[1] In this textbook, Quine considers that the term "implication" must be kept for the logically valid conditional (for some development, see Durand-Guerrier, 2003, pp.9-10)

[2] For examples, see Arsac & Durand-Guerrier (2000); Durand-Guerrier & Arsac (2003) and Durand-Guerrier & Arsac to appear in *Educational Studies in Mathematics*.

[3] And for a reference in English, Durand-Guerrier & Arsac to appear in *Educational Studies in Mathematics*

on logically invalid quantificational schema so that restrictions are required to insure validity. The main differences in the various systems concern these restrictions. Quine version is very complete and brightly exposed, but it is rather technical and spread in several different chapters. For this reason, we prefer presenting the version of Copi (1954, second edition 1965) that is exposed in a more compact way, and thus most easy to summarise.

III.3. Quantification rules in the Symbolic Logic by Copi (1954, 1965)

Symbolic logic by Copi is a first textbook written to serve to undergraduate and graduate students, in which, as it is the case in Quine (1950), quantification matters are widely developed. We refer for our presentation to the second edition (Copi, 1965), in which the quantification rules are presented twice. In this paper, we present only the preliminary version, that we will complete by some specific restrictions needed for preserving validity.

The first rule of inference concerns elimination of the universal quantifier ; it is called Universal Instantiation (U.I.) and states that :

" (...) any substitution instance of a propositional function can validly be inferred from the universal quantification. We can express this rule symbolically

$\frac{(x)\Phi x}{\therefore \Phi v}$ (where v is any individual symbol)" (Copi, 1982, p.50)

This rule relies on the valid schema $(x)\Phi(x) \Rightarrow \Phi(y)$ and expresses that "*what is true for all is true for any*".

The second rule is the dual of the first one and concerns the introduction of the universal quantifier ; it is called Universal Generalisation (U.G.) and states that :

"(...) the universal quantification of a proposition can validly be inferred from a substitution instance with respect with the symbol y. Our second expression for this quantification rule is

$\frac{\Phi y}{\therefore (x)\Phi x}$ (where y denotes any arbitrarily selected individual)" (ibid., p.51)

This rule relies on the invalid schema $\Phi(y) \Rightarrow (x)\Phi(x)$. For this reason, it necessitates a restriction; you must be sure that no assumption other than the property expressed by Φ has been done. Obviously, it is build "by analogy with a fairly standard mathematical practice"(ibid., p.50)

The third rule concerns the introduction of the existential quantifier; it is called Existential Generalisation (E.G.) and states that :

"(...) the existential quantification of a propositional function can be validly inferred from any substitution instance of that propositional function ; (...) Its symbolic formulation is :

$\frac{\Phi v}{\therefore \exists x \Phi x}$ (where v is any individual symbol)" (ibid., p.52)

The fourth rule is the more delicate to use. It concerns the elimination of the existential quantifier and states that :

"(..) from the existential quantification of a propositional function we may validly infer the truth of its substitution instance with respect to an individual constant which has no prior occurrence in that context. The new rule may be written as

$$\frac{\exists x \Phi x}{\therefore \Phi v}$$ (where v is an individual constant having no prior occurrence in the context)

(Ibid. p. 52).

As the second rule, this one relies on an invalid schema, namely $\exists x \Phi(x) \Rightarrow \Phi(y)$. The restriction is here very strong and many errors are due to forgetting it, particularly when, in a proof, an existential instantiation follows a universal instantiation[4]. We have shown (Durand-Guerrier & Arsac, 2003, 2005 to appear) that this restriction is closely related with the dependence rule. A consequence of these restrictions is that you must not only have a control for each step of the proof, but also have a control on the global proof. In particular, it is not possible to make a universal generalisation on an individual introduced by an existential instantiation.

IV. Analyse of a geometrical proof

We come back now to the theorem introduced in the first paragraph: "For all three points that do not lie on the same line, there exists a unique circle on which these three points are lying".

A fairly classical proof of this theorem for pupils grade eight can be written in the following form.

Proof: Let A, B and C be any three points not relying on the same line. Let us consider Δ_1 the mediator of the line segment [AB] and Δ_2 the mediator of the line segment [AC]. As the lines (AB) and (AC) are not parallel, then the two mediators are secant; O denotes their intercept. As O is on the mediator of [AB], OA=OB; for an analogous reason, OA=OC. Conclusion, B and C are on the circle, say Γ, whose centre is O, and whose radius is OA. As a circle is perfectly determined by its centre and its radius, this circle is unique.

IV.1. First analyse of the structure of the proof

First of all, it is to notice that the first assumption is a universal instantiation of the antecedent of conditional that remains implicit in the given formulation of the theorem. Indeed, the formulation is given with a bound quantifier, that restricts the scope of the universal quantifier to those elements that satisfy the required property. But of course, it is to prove a conditional, namely: "For all three points, if they do not lie on the same line, then there exist a unique circle on which these three points are lying".

[4] See for example Bagni (2005)

The general structure of the theorem to prove is (1) '$\forall x \forall y \forall z (\Phi(x,y,z) \Rightarrow \Psi(x,y,z))$'. and the macro structure of the proof compounds three steps :

1.*To apply Universal Instantiation* to the given statement to get the propositional conditional (2) '$\Phi(a,b,c) \Rightarrow \Psi(a,b,c)$'.This first step remains implicit as it is generally the case in mathematics. The corresponding statement will be "Given three points A, B and C, if they do not lie on the same line, then there exists a unique circle on which these three points are lying".

2.*To prove by "conditionalization"* the statement derived by universal instantiation. In order to do this, an auxiliary premise is introduced. It is a derivation in the proof. Following, for example, Hofstader (1979), we indicate its debut and its end by two brackets :

[$\Phi(a,b,c)$
Mathematical treatment
$\Psi(a,b,c)$]

(2) $\Phi(a,b,c) \Rightarrow \Psi(a,b,c)$ *Introduction of implication*

The brackets indicate that the statement on the first line (the antecedent of the conditional to prove) is introduced as an auxiliary premise, i.e. in this context, it is not assumed as a true statement. As a consequence, the statement on the last line before the bracket (the consequent of the conditional to prove) is not a true statement in the context. So, according with us, writing "conclusion" before this last statement is not relevant. Indeed, the proof is not over! The conclusion of this step is the conditional statement (2) formed by the statements on the first line and the last line in this order.

3.*To apply Universal Generalisation* to statement (2) in order to infer statement (1), that is the statement to prove.

In France, in mathematics classroom, it is rather common to consider only the step 2. More over, in exercises, generally, only the part of the proof that we have written in brackets is considered, without making clear that an auxiliary premise is considered. As an example: (*A,B,C,D*) is a convex quadrilateral ; *I, J, K, L* are respectively the midpoints of its sides [*AB*], [*BC*], [*CD*], [*DA*]. Prove that *(I,J,K,L)* is a parallelogram. Once more, what is to prove is the conditional statement. May be you could think that the status of all these statements are obvious, but it is likely that it is not the case for some students. We may wonder if the nearly exclusive focus on the very core of the proof, where indeed the mathematical treatment is done, could be a didactical obstacle for an adequate appropriation of the specificity of mathematical reasoning, that means proving general results by mean of hypothetical-deductive method. Frege (1971) said that mathematicians often avoid to distinguish between proving $A \Rightarrow B$

and proving B by Modus Ponens on $A \Rightarrow B$ and A, and emphasized the importance of this distinction. This question of determining if this practise is a possible didactic obstacle remains open and requires further empirical research. Some argument supporting this thesis is thus provided by Mathematical Induction. Mathematical Induction is introduced in France at grade eleven; it is typically a method of proof that requires clarifying both *conditionalization* and *Universal Generalisation* and it is well known that many students feel strong difficulties to capture the signification of this method.

As it must be clear, the macrostructure that we describe in this paragraph is rather independent of the content of the considered statement. We considered a very general level of syntax, shared by most of mathematical statement, in order to focus to the incompleteness of the proof usually provided in class. However, as we have seen in the first paragraph, the deep structure of the statement is more complex. And the simplicity of the proof we provide is deceptive. Indeed, there are many elements involved in the proof that do not appear, especially those theorems which assert existence under conditions. To illustrate this point of view, we provide in next paragraph an analyse of a short excerpt of the core of the considered proof in Copi's system.

IV.2. Mathematical and logical features in the core of the proof.

We consider in this paragraph a small abstract of the proof in order to examine on the one hand hidden existential instantiation and on the other hand how logical and mathematical considerations work together. At this stage of the proof, we are in the core of the proof in which the auxiliary premise '*AP1 : Points A, B, C do not lie on a same line*' has been introduced, and we consider two intermediate conclusions ;

'*C1: A, B, C are three different points*', which is an immediate consequent from AP1, and '*C2 : There exists exactly one point lying on both mediators of [AB] and [AC]* '.
While analysing the proof with Copi's system[5], it appears that it is necessary to express premises under the general form, which is not necessary the case in the standard format. In our cases, the four following premises are required.

P1: For all point M, N and P, if M and N are different, P lies on the mediator of [MN],if and only if PM=PN ;

P2: For all points M, N and P, if M and N are different and if P lies on the mediator of [MN], then P is different from M and N ;

P3: For all two points M and N, if M and N are different, then there exist a unique circle whose centre is M and whose radius is MN ;

P4: For all M, N and P, if M is different from N, and if MP = MN, then P lies on the circle whose centre is M and whose radius is MN.

An immediate remark is that in all these statements interplay implication, universal quantification and for some of them existential quantification. This is a very general case in Geometry, but generally, it remains implicit because most of the information

[5] We give in appendix the analyse with Copi's system of this excerpt

is visible on the figure and consequently only part of the statement is explicit. For example *P3* is replaced by '*P lies on the mediator of MN if and only if PM = PN*', so that the condition that points M and N ought to be different disappears. As every teacher knows, while dealing with general proof, it is very common that students forget particular cases or existence's conditions. Taking care of these particular cases and existence's conditions requires that the focus be moved from propositions to objects. More precisely, if the mathematical effective arguments are indeed expressed through propositions, it is nevertheless necessary, to apply them in a sound manner, to keep control about the handling of quantifiers.

V. Conclusion

In this paper we try to show in which respect natural deduction in predicate calculus provides tools for analysing proof by taking in account quantification matters. By the exigency of introducing every symbol used in the proof, distinguishing dummy variables and individual symbols, natural deduction in predicate calculus offers a possibility to control the validity of proof and in certain cases point out implicit that might lead to incorrect proof. By focusing on interplay between propositional connectors and quantification, it provides an extension of the classical tools used in didactic of mathematics to analyse proofs. According with us, this enlightens the fact that many things are silenced in the standard manner, especially those considerations concerning the existence of the objects that are introduced. We have shown that it is relevant for analysing proof in tertiary mathematical education. We make the hypothesis that it is also the case in secondary mathematical education, and we thought that we give in this paper some insights to open a path for further research considering this question. More generally, these considerations show clearly that for us, the logical validity is a prominent criteria for analysing proofs in a didactic perspective, prior from the focus on *proving to convince* or *proving to explain*[6].

REFERENCES

Arsac G., Durand-Guerrier, V. : 2000, 'Logique et raisonnement mathématiques. Variabilité des exigences de rigueur dans les démonstrations mettant en jeu des énoncés existentiels' in Assude T.., Grugeon B. (Éd *)* *Actes du Séminaire National de Didactique des Mathématiques* (pp 55-83), IREM de Paris VII.

Bagni, G.T. : 2005, Quantificatori esistenziali : simboli logici & linguaggio nella pratica didactica, *L'Educazione Matematica,* Serie VIII- Vol 1/2

Chellougui, F.: 2003, 'Approche didactique de la quantification dans la classe de mathématiques dans l'enseignement tunisien', *Petit X* n°61, 11-34.

Chellougui, F.: 2004, *L'utilisation des quantificateurs universel et existentiel en première année universitaire entre l'explicite et l'implicite*, Doctoral dissertation, Université Lyon 1, France.

Copi, I. M.: 1954, *Symbolic Logic*, second first edition 1965, Macmillan Company, NewYork.

[6] Although we do not underestimate these aspects.

Dubinsky, E. and Yiparaki, O. : 2000, On students understanding of AE and EA quantification. *Research in Collegiate Mathematics Education IV, CBMS Issues in Mathematics Education,* 8, 2000, 239-289. American Mathematical Society : Providence.

Epp, S.(2003) The role of logic in teaching proof in the *Mathematical Association of America, Monthly 110.*

Durand-Guerrier,V. : 2003a, 'Which notion of implication is the right one ? From logical considerations to a didactic perspective' in *Educational Studies in Mathematics,* 53, 5-34.

Durand-Guerrier, V. 2003b, Logic and reasoning from a didactical point of view. A model theoretic approach electronic proceedings *of CERME 3* :
http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG4/TG4_list.html

Durand-Guerrier, V., Arsac, G. : 2003, Méthodes de raisonnement et leurs modélisations logiques. Le cas de l'analyse. Quelles implications didactiques ? *Recherches en Didactique des Mathématiques,* 23/3, 295-342.

Durand-Guerrier, V , Ben Kilani, I. : 2004 Négation grammatical versus négation logique dans l'apprentissage des mathématiques. exemple dans l'enseignement secondaire tunisien. *Cahiers du Français Contemporain,* n° 9, 29-55.

Durand-Guerrier, V., Arsac, G. : An Epistemological and didactic study of a specific calculus reasoning rule, to appear in *EducationalSstudies in Mathematics*

Duval, R. : 1991, 'Structure du raisonnement déductif et apprentissage de la démonstration', *Educational Studies in Mathematics*, 22, 233-261.

Duval R. : 1995, *Sémiosis et pensée humaine*, Peter Lang, Bern, Switzerland.

Frege, G.: 1879, *Begriffschrift*, Nebert, Halle, Germany

Frege, G.: 1971, 'les recherches logiques" in Ecrits logiques et philosophiques, Editions du Seuil, Paris, pp.170-234

Gentzen G.:1935, Untersuchungen über das logische Schliessen in *Math. Zeitschr*, 39.

Glaeser G.:1973, *Mathématiques pour l'élève professeur*. Hermann, Paris.

Gochet P.,, Gribomont P.: 1990, *Logique. Méthodes pour l'informatique fondamentale. Vol. 1.* Hermès, Paris.

Hofstadter, D.: 1979, *Gödel, Escher, Bach, an Eternal Golden Braid.* Basic Books, Inc., Publishers, New-York.

Hoyles, C., Küchemann, D.: 2002 'Students' understanding of logical implication', Educational Studies in Mathematics, 51,3, 193-223.

Jaskowski, S.: 1934, 'On the rules of supposition in formal logic', *Studia Logica* n°1.

Pedemonte, B.: 2003 Etude didactique et cognitive des rapports de l'argumentation et d e la démonstration dans l'apprentissage des mathématiques, in Durand-guerrier, V. & Tisseron, C. (eds) *Actes du Séminaire National de Didactique des Mathématiques, Année 2002, ARDM et IREM de Paris 7,* 153-173.

Quine, W.V.O. 1950, *Methods of logic*, fourth edition 1982. Holy, Rinehart & Winston, New-York.

Quine, W.V.O.: 1987, *Quiditties. An Intermittently Philosophical Dictionary*, The Belknap Press, Cambridge, Unites States.

Russell, B.: 1903, *Principles of Mathematics.* Cambridge University Press, Cambridge, United Kingdom.

Selden, J. and Selden, A.: 1995, 'Unpacking the logic of mathematical statements', in *Educational Studies in Mathematics*, 29, 123-151.

Appendix

Analyse with Copi's system of an excerpt of the proof studied in section V.

The first seven statements are the premises. Let us remind that AP is the auxiliary premise, antecedent of the conditional to prove, and that C1 and C2are intermediate conclusions, while P1, P2, P3, P4 are theorems.

(1) AP. Points A, B, C do not lie on a same line

(2) C1. A, B, C are three different points

(3) C2. There exists exactly one point lying on both mediators of [AB] and [AC]'

(4) P1. For all point M, N and P, if M and N are different, P lies on the mediator of [MN],if and only if PM=PN ;

(5) P2. For all points M, N and P, if M and N are different and if P lies on the mediator of [MN], then P is different from M and N ;

(6) P3. For all points M and N, if M and N are different, then there exist a unique circle whose centre is M and whose radius is MN ;

(7) P4. For all M, N and P, if M is different from N, and if MP = MN, then P lies on the circle whose centre is M and whose radius is MN.

(8) O lies on both mediators of [AB] and [AC]	*Existential Instantiation on (3)*
(9) O lies on the mediator of [AB]	*Separation on (8)*
(10) if A and B are different, then O lies on the mediator of [AB] if and only if OA = OB	*Universal Instantiation on (4)*
(11) A and B are different	*Separation on (2)*
(12) O lies on the mediator of [AB] if and only if OA=AB	*Modus Ponens on* (10) & (11)
(13) OA=OB	*Modus Ponens on (9) & (12)*
(14) O is different from A and from B	*Universal instantiation followed by Modus Ponens on (9), (11) & (5),*
(15) O is different from A	*Separation on (10)*
(16) There exist a unique circle whose centre is 0 and whose radius is OA.	*Universal instantiation followed by Modus Ponens on (15) &(6)*
(17) Γ is the circle whose centre is 0 and whose radius is OA	*Existential instantiation on (11)*
(18) B lies on the circle Γ whose centre is 0 and whose radius is OA	*Modus Ponens on (11), (13) &* (7)
(19) C lies on the circle Γ whose centre is o and whose radius is OA	*Substitution from (9) to (18)*
(20) A lies on the circle Γ whose centre is O and whose radius is OA	*Substitution in (13), (15), (18).*
*(*21) A, B and C are lying on circle Γ	*Conjunction on (19), (20), (21)*
(22) There exists a circle on which points A, B and C are lying	*Existential generalisation*

VISUALISING AND CONJECTURING SOLUTIONS FOR HERON'S PROBLEM

Lourdes Figueiras, *Universitat Autònoma de Barcelona, Spain*

Jordi Deulofeu, *Universitat Autònoma de Barcelona, Spain*

Abstract: We analyze with a practical teaching porpouse some aspects of using visual constructions during problem solving processes. In particular, we analyze some geometric constructions made to conjecture the solution to Heron's problem, and obtain two different categories of visual representations. The paper includes explicit contributions of some participants of group on proof at CERME IV, specially trying to make connections between research and teacher practice.

1 Introduction

Visualization has been a main topic in mathematics education research for the last decade, and many different theoretical perspectives and practical resources have been developed [16] [12] [17] [8]. Especially in the field of geometry many contributions have been made, emphasizing the connection between symbolic and visual representations of mathematical knowledge [10], [1].

Attending the specific moment of formulating a conjecture, many investigations have focused on the role of drawings within geometrical activity, and inquired into the intimate relationship between drawings and concepts [14].

The main theoretical ideas that we will consider here as a framework are the following:

1. Visualization is a concept connecting both symbolic and iconic representations, which are linked by the problem solver in a process of interaction. This view of visualization is characterised by a continuous change between these representations [10]. In the school context, the teachers often gives support to this change by acting as an external mediator between iconic and symbolic thinking.
2. Among the functions attributed to visualization, we focus here on its power as a builder of mathematical ideas, and we consider it necessary for heuristic reasoning [12], and thus of producing conjectures. The symbiosis between concepts and geometric figures stimulates new directions of thought, but there are logical or conceptual constraints which control the process [7].

3. Both, visualization and conjecturing have an important contextual component, and any observation is made within a particular context, so that its applications in other situations, surely vary [12], [5]. It is not our objective to produce some general classification of conjectures, able to be applied for every problem or any situation, but to analyze what in a very concrete setting occurs to build bridges between practice and research. For theoretical works supporting this link between teaching and research, see [2, 3, 13].

As we analyzed in this paper the kind of conjectures arsing from the discussion of Heron's problem, we pose it now, and comment briefly its solution:

Heron proofs in his *Catroptica* that the light rays cover a minimal distance supposed that incidence angle and reflection angle are equal [9]. A modern statement of the problem, more commonly used, is the following:

Let s be a line and A and B two points at the same side of s. For which point P in s is $AP + PB$ the shortest way joining A and B? [4]

The most usual geometrical construction used to solve the problem is the same proposed by Heron:

Let C be the point symmetric to A with respect to the line s, so that the segment $\overline{AC}$ is perpendicular to s. The line joining B and C intersects s in P, which is the point we are looking for. (see Figure 1).

We analyzed here which kind of geometrical ideas were used to conjecture the solution to Heron's problem, when it was posed to future primary teachers.

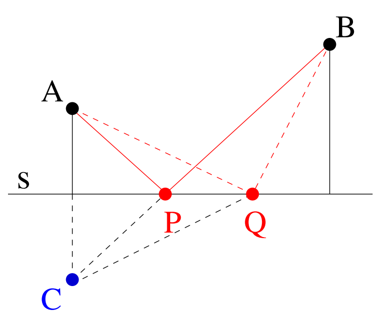


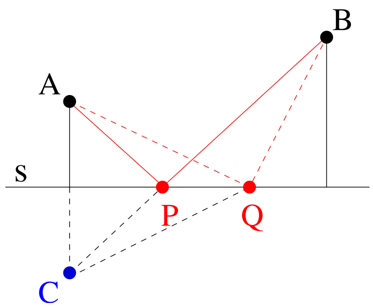
Figure 1:

2 Drawing of visual diagrams to conjecture the solution

Heron's problem was proposed in a primary teacher training course on mathematics. It was their first year at the university, and there were approximately fifty people in attendance. The problem was proposed to the students, who discuss the solution in groups. Afterwards, different approaches were discussed in the whole class.

When trying to solve the problem, five conjectures arose, some of them depending on the refutation of others. One of the groups explained conjecture 1 in Table 1 and presented it as a solution of the problem. Afterwards, the whole class were invited to share diverse approaches to the problem, and other conjectures were discussed. These conjectures are summarized in Table 1.

Table 1: Geometric constructions to visualize a solution of Heron's problem.

<table>
<tr><th>Conjecture</th><th></th><th>Proof or counterexample</th></tr>
<tr><td>1. Construction suggested by Heron. P is the intersection between s and the line joining B and the symmetric point to A with respect to the line s</td><td>
</td><td>For any other point $Q \in s$, it must be proved that $\overline{AQB} > \overline{APB}$. Since

$$\begin{aligned} AP &= CP\,, \\ AQ &= CQ\,, \\ AP + PB &= CB\,, \quad \text{and} \\ AQ + QB &= CQ + QB\,, \end{aligned}$$

we deduce from the triangle inequality for $\triangle CBQ$ that

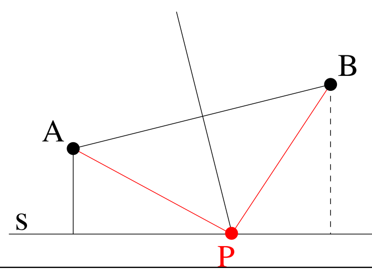
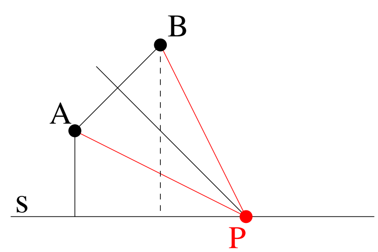
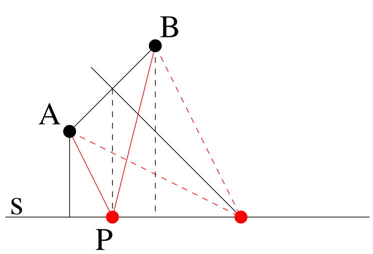
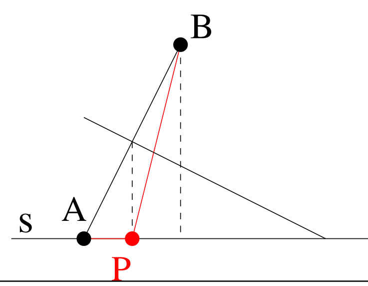
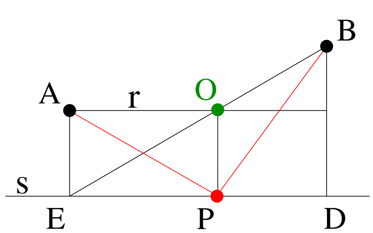
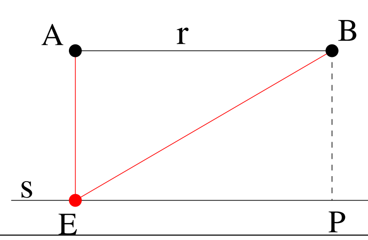
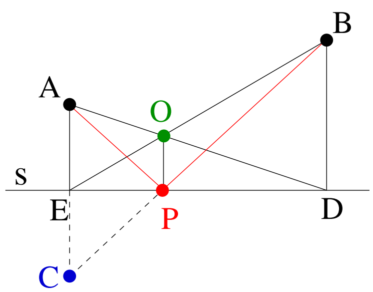
$$\begin{aligned} AP + PB &= CB \\ < CQ + QB &= AQ + QB \end{aligned}$$</td></tr>
<tr><td>2. P is the intersection between s and the median of $\overline{AB}$.</td><td>
</td><td>
</td></tr>
<tr><td>3. P is the intersection between s and its perpendicular through the mid-point of $\overline{AB}$.</td><td>
</td><td>
</td></tr>
<tr><td>4. r is the perpendicular to $\overline{BD}$ through A; O the intersection of r and $\overline{BE}$; and P the intersection between s and its perpendicular through O.</td><td>
</td><td>
</td></tr>
</table>

Table 1: Geometric constructions to visualize a solution of Heron's problem.

Conjecture		Proof or counterexample
5. O is the intersection between $\overline{AD}$ and $\overline{BE}$. P is the intersection between s and its perpendicular through O.		This conjecture gives a correct solution to the problem. Since $\frac{AE}{OP} = \frac{ED}{PD}$ y $\frac{BD}{OP} = \frac{ED}{EP}$, we deduce that $AE \cdot PD = BD \cdot EP$ $AE : EP = BD : PD$ Thus $\triangle AEP$ and $\triangle BDP$ are similar, and $\angle APE = \angle BPD$.

All of them look for the point P by intersecting lines drawn from the line s and the given points A and B. All of the students assumed implicitly that P must lie between the points E y D obtained by intersecting s and the lines perpendicular to s through A and B, respectively.

Attending the use of conceptual procedures in the constructions of visual diagrams to conjecture a solution, these have been firstly classified into two categories:

C1. Constructions based on previous knowledge -conceptual, iconic or procedural- or on the refutation of a previous conjecture.

The first three conjectures in Table 1 follow from this kind of construction. The first one reduces the problem to the consideration of joining two points by a straight line. In this case, the concept of minimal path is embedded in both the problem and the diagram, and those who made this conjecture had the insight to apply their knowledge that the shortest distance between two points is a straight line, but to do so is not trivial, as other knowledge, related to the preservation of distance under reflection, must come into play [15].

The second one determines P as the point of intersection between the perpendicular bisector of $\overline{AB}$ and s. The consideration of P as lying on the perpendicular bisector between A and B allows one to *balance* the distance between the two drawn segments. Students implicitly assign to its points the property of being at the same distance to the extremes of the segment, and this makes this conjecture a good one to solve the problem, illustrating an interesting confusion that occurs in solving such problems: equal distances are often confused with shortest distances.

The refutation of this conjecture, using a new construction where P does not lie on the segment, induces a new conjecture. This new conjecture depends on the previous one, which affirms that P lies on $\overline{AB}$ (Conjecture 3), and could be based on the fact that the mid point of AB is conserved under vertical projection. This new construction permits them to minimize one part (AP) of the total path APB, and is, according to [14], an attempt to control the process of producing a more and more satisfactory drawing. But the solution is also easily refuted by a counter example in which the horizontal distance between A and B is small compared to the vertical distance between them.

The two remaining conjectures were considered independent from the later two, as they arose in different groups after they said they have no idea how to find P.

C2. Spontaneous constructions

Constructions of this kind apparently do not follow any procedure or conceptual strategy, except to obtain a point P between the extremes E y D (Conjectures 4 and 5) [1]. They assume that we expect from them a geometrical diagram leading to P, and produce it by simply making lines starting from the given points, but they are not able to explain why they are doing it so. Although both drawings in conjectures 4 and 5 are created in the same "reckless" way, the fourth is very easy to refute with a simple counterexample; the last conjecture, however, leads to a correct solution for the problem, and generated a long discussion involving the whole class.

When trying to refute the last conjecture by a counter example, all new diagrams proved useless. The students then started to check that *in all cases*, the point P and the one obtained using Conjecture 1 are the same. But the problem is that this new diagram does not permit them to visualize the point P produced by conjecture 5 as a solution of the problem, and even once its validity have been proved, the logical argument involving similar triangles is not significative enough.

3 Didactical approach to conjecture

The analysis of the situation described above led us to four questions addressing what in our view are important demands for research on conjecture, arising directly from the teaching practice:

1. *Conjecturing and proving*

   For the students working on this problem, the use of geometrical drawings to solve the problem is easy, as opposed to giving a proof. Once they have guessed Conjecture 1 (Table 1), the conflict is not that they do not know *how* to prove it, but to understand that they *need* a proof. The construction is considered in itself as a solution of the problem and no symbolical proof is required. For many students, this construction remains a proof, at least until they construct a more elaborated concept [11]. But this is not the case when considering the fifth diagram.

   The formulation of the conjecture, or *empirical* solution [10], needs to offer a key for a second phase involving the *meaning* of a proof in order to be effective. As a consequence, some research linking types of conjectures and proof is needed, emphasizing the more systematic and deductive side of mathematics, and having in mind that

[1] To see the counterexample in 4, consider the locus of all points X in the plane such that $AX+BX = AE+EB$, which is an ellipse through E and D with focal points in A and B. The segment ED is entirely contained in that ellipse, and therefore $AE + ED > AY + BY$ for any point Y in ED, different from E and D.

usually students, and many teachers, look at Mathematics from a more inductive and experimental point of view. Making explicit some nexus between problems, conjectures and proof could generate a proper way to make proof significative for the students.

2. *Types and elements of conjectures*

   Conjecture 5 offers them no clues about its validity, as the first one did. While in the first case the construction is enough for the students to visualize the solution of the problem ("there is nothing to prove –said one of the students–, because the shortest path between two points is a straight line"), this is not the case with the fifth diagram. A priori, spontaneous constructions offer no clue to find the solution. According to [10], even though the use of different representations is a key to progress in problem solving, geometrical representations and their continuous use do not yield, by themselves, the process towards the solution. This last construction instills uncertainty in the students, as there are no elements that enable them to recognize the problem. This generally happens with spontaneous constructions. Students should analyze their geometrical constructions in terms of the given problem in order to be able to use them for solving the problem. Some investigations argued that when producing a drawing, students try to reach harmony between figural and conceptual aspects [14]. However, this does not seem to be so in the case analysed here, as there are some spontaneous constructions which lack this harmony.

   For our practical purpose of analyzing this particular experience from the perspective of teaching practice, the distinction between spontaneous and non-spontaneous has been adequate and productive enough. This is not the case when we want to address a more complete frame to conduct practical experiences involving guessing. This point calls then for research a) characterizing types of conjectures, and b) identifying through categories the cognitive processes involved in each type. This call addresses the more inductive and experimental side of Mathematics, and it is specially relevant for teaching practice, as the students should also learn about effective guessing.

3. *Creating knowledge by linking conjectures*

   Training of visualization or the use of geometrical diagrams to conjecture the solution of a problem should consider the analysis of spontaneous constructions, as these arose frequently during our investigation. The act of visualising and producing new diagrams has an important contextual reference. Here, it is the continuous creation and refutation of conjectures which promotes the creation of new ones. It is in this sense that we are not able to extend these conclusions to other groups and individuals.

   On the one hand, the lack of conceptual knowledge prevents the students from knowing if they are really constructing a satisfactory drawing or not. On the other, the students also need strategies of interpreting their spontaneous constructions in terms of the statement of the problem in order to understand the role of the drawing in the problem solving process.

   At CERME 4 in Sant Feliu, David Reid discussed the problem with us in the working group on proof, and offered us the following possible link between Conjectures 2 to 5 in an hypothetical situation:

   "The diagram created in Conjecture 2 (Table 1) in a context of reasoning suggests the next conjecture (conjecture 3). The problem it reveals is that the point P should stay on the segment ED that is the projection of AB onto s. This constraint, combined with the earlier consideration of trying to equalise distances, suggests that the midpoint of ED, or alternately the projection of the midpoint of AB onto s, is the point P. This new construction also accounts for the known special case. If A is on s, however, a new counterexample is produced (counterexample to Conjecture 3). This diagram shows clearly the correct solution in another specific case: When A is on s then the shortest path is AB itself. The emphasis shifts to defining P is such a way that it can be seen as a continuous transformation from this initial situation. As A moves up, P must move to the right. The segment EB provides a mechanism to produce this motion: A is projected horizontally onto EB to the point O and then O is projected vertically onto s to the point P (Conjecture 4). Again a counter-example is not hard to find (see Table), because this construction does not work in the original special case, when EA=DB.

Combining the two special cases (and the mirror image of the second) suggests Conjecture 5: O is the intersection between AD and BE. P is the intersection between s and its perpendicular through O" [15].

Then, it seems plausible to think that a sequence of conjectures, each based on a single special case –or two special cases, as in the last one– a conjecture is produced that turns out to be correct and it provides a few clues for the proving process, although does not produce an instant proof (as with conjecture 1). How to bring this process of linking conjectures nearer to the students seems to be very important in order to improve their knowledge about guessing, as well as to let the teachers know about the importance of making explicit these plausible links. This would help to conceptualize an otherwise *spontaneous* and often meaningless conjecture.

4. *Problems and conjectures*

Heron's problem has been especially interesting to analyse because it requires an approach via a geometrical construction to conjecture its solution. The problem is also rich enough to give rise to many conjectures. When the students are able to create and interpret properly those constructions, they are often in a good position to conjecture the right solution or to downright solve the problem properly. But not all problems have the same potential to enable the students guessing, as not all problems are equally interesting to learn about proving.

From a very concrete perspective of an in-service teacher, an expository material including commented examples of problems to make understandable types and elements of conjectures would be desirable. Such a material should give them keys to evaluate problems from this point of view, understand and re-create their practice, as well as permit them to question research on the basis of their own practice.

References

[1] Arcavi, A. (2003): The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215–241.

[2] Bartolini Bussi, M.G. (1998): Theoretical and empirical approaches to classroom interaction. En: Biehler, R. Scholz, R.W., Strässer, R. y Winkelmann, B. (Eds.)

Didactics of mathematics as a scientific discipline, 121–132. Kluwer Academic Publishers, Dordrecht.

[3] Bishop, A.J. (1998): Research and practioners. En: Kilpatrick, J. y Sierpinska, A. (Eds.) *Mathematics education as a research domain: a search for identity*, 33-45. Kluwer Academic Publishers, Dordrecht.

[4] Courant, R. and Robbins, R. (1941): What is mathematics? Oxford University Press, New York.

[5] Cunningham, E. (1994): Some strategies for using visualization in mathematics teaching. *Zentralblatt für Didaktik der Mathemathik*, 26, 83–85.

[6] Dreyfus, T. (1994): Imagery and reasoning in mathematics and mathematics education. En: Robitaille, D., Wheeler, D. y Kieran, C. *Selected lectures from the 7th ICME*. Université laval, Quebec.

[7] Fischbein, E. (1993): The theory of figural concepts. *Educational Studies in Mathematics*, 24, 139–162.

[8] de Guzmán, M. (1996): El rincón de la pizarra. Ensayos de visualización en análisis matemático. Pirámide, Madrid.

[9] Heath, T.L. (1921): A history of greek mathematics. Vols. I y II. Dover, New York.

[10] Kadunz, G. and Strässer, R.(2001): Visualisation in geometry: multiple linked representations. *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, 3, 201–208.

[11] Ibañes, M. y Ortega, T. (2001): Pruebas visuales en trigonometría. *AULA*, 10, 103–116.

[12] Kautschitsch, H. (1994): Neue Anschaulichkeit durch “neue” Medien. *Zentralblatt für Didaktik der Mathematik*, 26, 79–82.

[13] Malara, N.A. y Zan, R. (2002): The problematic relationship between theory and practice. En: Lyn, D. (Ed.) *Handbook of international research in mathematics education*. Mahwah, New Jersey.

[14] Maracci, M. (2001): The formulation of a conjecture: the role of drawings. *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, 3, 335–342.

[15] Reid, D. (2005): Personal communication at CERME V. Sant feliu de Guíxols, Girona.

[16] Zimmermann, W. y Cunningham, S. (Eds.) (1991): *Visualization in teaching and learninig mathematics*. Mathematical Association of America, Notes, 19.

[17] Peters, V. (Ed.) (1992): Analysis: Visualization in mathematics and didactics of mathematics, part 1. *Zentralblatt für Didaktik der Mathematik*, 26, 77–92

[18] Peters, V. (Ed.) (1992): Analysis: Visualization in mathematics and didactics of mathematics, part 2. *Zentralblatt für Didaktik der Mathematik*, 26, 109–132

A GENETIC APPROACH TO PROOF

Hans Niels Jahnke, *Universität Duisburg-Essen, Germany*

Abstract: *The paper proposes the metaphor of a 'Theoretical Physicist' for explaining the thinking of students in proof situations and for designing a genetic approach to proof. In regard to students' cognitions, a formative and an established phase in the (personal) development of a theory are distinguished. In the genetic approach to proof which is sketched in this paper, geometry is, during an introductory phase, treated as an empirical theory.*

The Metaphor of a "Theoretical Physicist"

Quite a few students of the secondary and even tertiary levels tend to confirm mathematical statements by appealing to measurements or special examples. They consider them as sufficient argument though they already have had experiences with mathematical proofs and, according to their teachers, should know that the general validity of a mathematical statement cannot be established in this way (see for example Healy & Hoyles, to appear; Coe & Ruthven, 1994). Some papers report on students asking for additional testing of a statement by means of an example after its proof has been treated in the classroom and and in spite of the fact that the students claim to have understood it. (Fishbein, 1982, 16). Some students believe that a proof is valid only for the special diagram which is attached to it (Hefendehl-Hebeker, 1995).

In the present paper I will search for explanations of these phenomena and try to show that this type of thinking occurs with a certain necessity and is a consequence of a meaningful behaviour. In a second step some proposals will be made for dealing with these ideas of the students in a constructive way and for leading them step by step to mathematical argumentation proper. I call this a 'genetic approach to proof'.

Frequently, difficulties of students with proof are considered as indicating a lack of logical insight (see f.e. Stein, 1990). Therefore, some textbooks introduce the topic 'proof' by exposing logical ideas as, for example, the difference between a statement and its converse and the distinction between a statement which is true for some cases and a generally valid one. In contrast to this, the present paper departs from the hypothesis that ‘mathematical proof’ is above all a *problem of epistemological meaning* (Hanna & Jahnke 1993, 433f). The role and function of mathematical proof will not be explained to the students by referring exclusively to mathematics and to purely mathematical examples. Rather, the contribution of mathematical proof to human cognition in general and to human understanding of the surrounding world has to be exposed. Put in another way, I will consider mathematical proof not through the eyes of a pure mathematician but from the point of view of a *theoretical physicist*. The term ‘theoretical physicist’ is meant metaphorically and designates a person who develops and evaluates mathematical deductions in order to better understand the world in which he lives.

Using this metaphor we bring ourselves in a position beyond the established division of labour which separates mathematics from other disciplines. Especially, we get a fruitful alienation of our view on the relation of mathematical proof and special cases. While in pure mathematics we believe in the general validity of a theorem established by a mathematical proof, the situation in physics is different. Physicists will not accept a conclusion of a theory derived by a mathematical argument without experimental verification. If a conclusion is new and important, a physicist will develop an experimental test of it. In this sense, the metaphor of a ‘theoretical physicist’ might help to analyse the thinking of a pupil who prefers empirical arguments.

Proof and Measurement in Mathematics Teaching

Students are frequently asked to measure the angles of a triangle, for example. After they find that their sum is always nearly equal to 180 degrees, they are told that measurement can establish this fact only for individual cases and that they will have to prove it if they really want to be sure that it is true for all triangles. This explanation may have been obvious at the time of Plato and Euclid, but it must seem unsatisfactory at a time when experiment and measurement are considered the foundation of scientific methodology. To avoid contradiction, the students are, of course, told that the triangles they draw are fundamentally different in nature from the triangles that play a role in geometrical theorems - the latter are ideal or theoretical entities, whereas the former are empirical. As valid as this distinction may be, teachers themselves still assume that one can be sure about the sum of the angles of empirical triangles, without any further reflection, after the respective theorem for ideal triangles has been proved, and they cannot help conveying this conviction to their students.

In physics lessons, however, the message is exactly the other way round. One would never seriously entertain the idea that a natural law might be established by a theoretical proof, and one would insist that all such laws are founded upon experiment and measurement. (see Hanna & Jahnke, 1996, 892 f).

Therefore, in teaching beginners an intellectually honest way is to take side by the physicist and to say that the angle sum theorem is true because of empirical measurements. Only at a later stage, one should expose the idea of a purely mathematical theory separated from reality.

In the next section we will explain the consequences of this position and the meaning of mathematical proof in empirical theories.

Proof and Measurement in Physics

Theories of physics are systems of statements/theorems whose quantitative consequences are expressed in natural laws. Natural laws are mathematical equations between measurable magnitudes, for example the law of gravity or the dependence of the product of pressure and volume on (absolute) temperature in ideal gases. The ensemble of natural laws contained in a theory can be derived from a few fundamental assumptions by *mathematical proofs*. Within a certain domain of tolerance the validity of the natural laws is corroborated or refuted by experiments/measurements.

An isolated natural law which is not linked to other natural laws can only be corroborated by experiments which are directly related to it. In contrast to this, a natural law within a theoretical network is corroborated not only by direct measurements but by all the measurements of all the natural laws belonging to the network. Thus, a mathematical proof deriving a statement (natural law) of the theory does not provide absolute certainty to this statement, but it will considerably *enhance* its certainty since the corroboration does not only come from an isolated set of direct measurements but from all measurements related to the theory. Such a theoretical network of statements and measurements connected by mathematical proofs is the safest form of knowledge at our disposal, though this does not change its, in principle, preliminary character.

In the development of science the firmness of a theory becomes especially visible in such moments when a mathematical deduction allows a not obvious prediction which afterwards is corroborated by measurements. The ability of a theory to make such predictions is the most important criterion of its firmness and fruitfulness.

Applied to our example of the angle sum theorem this means that its proof will not give absolute certainty to the theorem, but it will enhance its certainty since it connects the theorem with other geometrical theorems which can also be corroborated by measurements. Thus, the theorem is not only tested by measuring the sum of the angles of a triangle, but also by measuring corresponding angles, alternate angles or sums of angles in 4-, 5-, 6-gons. In a long school career the students come to know so many statements which are connected with the angle sum theorem and empirically testable that they understand why Euclidean geometry is the eldest and empirically best corroborated theory we have and why, for a long time, mathematicians and philosophers attributed absolute certainty to it.

To fully understand the problem of justification we have to discuss a further idea which is especially stressed by *holistic* philosophies of science. They point to the fact, that the methods for measuring the magnitudes in a natural law suppose, as a rule, the validity of the theory containing this law. For example, it is not possible to measure the magnitudes force (F) and mass (m) appearing in Newton's law $F = m \cdot a$ without supposing in some way that this law is already valid (see Jahnke, 1978). Philosophers of science say that the observational language is 'theory loaded'. P. Duhem (1908/1978) was among the first who pointed to the importance of this fact.

As a consequence holistic philosophies of science from P. Duhem to J. D. Sneed (1971) claim that *theories as a whole* have to be corroborated or refuted. If we accept this claim, and I think there is no reasonable alternative to this, then we have to draw the conclusion that a theory has to be *jugded* whether it is *successful*. In the process of assessment many criteria and points of view play a role which, in part, are beyond the limits of explicit reflection. All in all this assessment is a matter of judgement and, thus, a *pragmatic decision*. The ultimate reason for accepting a law is not of a logical, but of a *pragmatic nature*.

Assessing a theory as being valid or acceptable implies a statement about its *future*. Scientists express their *expectation* that it will be possible to derive further phenomena and applications from the theory at hand which will be corroborated by experiment. However, scientists are always conscious of the possibility that new phenomena might be discovered which falsify the theory in its present form and require to modify some laws or even to introduce new parameters. In principle, scientific theories are open to revision.

I would like to call this fundamental fact the *circle of justification*. In regard to the angle sum theorem we have to conclude that a logical proof alone cannot provide certainty to it. A logical proof reduces a theorem only to other theorems as for example to the theorem about alternate angles at intercepted parallels. The latter, however, has no higher degree of certainty than the former. The statement "In future, I will consider this theorem as valid" requires an assessment of the whole situation independent and different of the mathematical proof. This assessment takes into account results of measurements, other theorems, considerations of plausibility. Such phases of assessment are not part of the teaching of 7th graders to whom the angle sum theorem is conveyed. The underlying concept of proof which is determined by a purely logical view excludes this type of reflection.

A judgement about a theory as valid and successful, however, can only be made in an *advanced* and *late* period of its development. Consequently, we have to distinguish between two phases, that before and that after this judgement. We call the former the *formative* phase of a theory, the latter its *established* phase. In the formative phase a theory is considered as one among several possibilities of explaining a certain area of phenomena. It is not clear which the adequate basic concepts are and the theory is taken as hypothetical. In the established phase the theory has been judged as being valid and successful and is taken as the only legitimate explanation of the phenomena to which it is related. Scientists agree upon definitions of the basic concepts, the explanations of the theory are considered as safe, the theory has been transformed into a "system of theorems which can be derived from a few axioms".

The Different Functions of Proof in the Formative and Established Phases of a Theory

Mathematical proofs have different functions in the formative and established phases. When Newton published his *Philosophiae naturalis principia mathematica* Kepler's laws of planetary motion were well-tested empirical statements whereas Newton's law of gravity was an uncertain and to a high degree contestable hypothesis. Newton's proof, thus, couldn't have the function to establish the truth of Kepler's laws. Rather, it was the other way round. The fact that Kepler's laws could be derived from the law of gravity was the decisive argument in favour of the latter. To use a term of I. Lakatos Newton's proof didn't effect a flow of truth from the assumptions to the conclusion, but, vice versa, from the conclusion (Kepler's ellipses) to the assumption (the law of gravity). In the last regard, Newton's proof was a 'proof' of the law of gravity.

However, the situation was even more complicated and this sharpens our stance. At Newton's time astronomers were well aware that Kepler's laws did not exactly describe the movements of the planets. Therefore, astronomer Cassini proposed certain ovals (4^{th} degree curves) instead of ellipses as paths of the planets. Some astronomers followed him. In this situation Newton's proof was a strong argument in favour of Kepler's ellipses and, thus, there was also a flow of truth from the assumptions to the conclusion.

However, in order to arrive at this conclusion the scientists at that time had to *evaluate the situation as a whole*. It was the *fit,* or, to say it more emphatically, the *harmony* between Kepler's laws and the law of gravity which served as the decisive argument to accept both of them at the same time as the adequate and 'right' theory of planetary motion. In a second step the deviations of a planet from the elliptic path were explained as perturbations by the other planets, in order to get an empirically satisfactory theory.

Therefore, in the formative phase of a theory proofs do not effect an uniquely directed flow of truth since there are no accepted foundations. What a proof means in a concrete situation is subject to a judgement which has to take into account the whole theoretical and empirical context.

The Behaviour of Proof Novices

Proof novices are considered in this paper as students in their personal formative phase of a geometric theory.

Semi-formal interviews with 6 students of grade 7 which took place after a first introduction into geometrical proof showed some interesting results. 4 students were ready to consider the possibility that there are triangles with an angle sum different of 180° degrees. Rather, the low achievers denied such a possibility (see Balacheff (1991) for similar experiences).

Being asked to prove the angle sum theorem none of the interviewed students gave the Euclidean proof though this proof had been treated in class and was required in a written test a week ago. However, five students were able to reproduce the Euclidean proof after they had been shown the figure related to this proof. Instead of the Euclidean proof four students derived the angle sum theorem from the fact that an exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

These findings show: the students move tentatively in a network, but the network is still lacking structure. They are still in the formative phase and behave accordingly.

Another example from teaching is as follows. After a proof of the fact that the perpendicular bisectors of the sides of a triangle meet in one point a pupil remarked: “You can’t say that this is true for every triangle, there are still other triangles [pupil points to the drawing]”. The teacher answered: “We have only used properties valid in every triangle” and continued with another subject (Hefendehl-Hebeker 1995; see also Williams, 1979, and Balacheff, 1988, for the observation that, frequently, students think a proof valid only for the triangle shown in the drawing). Clearly, the pupil in this episode is in the formative phase of a theory. He is not sure whether there has not been used some hidden or not explicitly mentioned feature of a triangle by referring to the drawing. Also, there could be some parameters involved which are not known for the moment but which might influence the situation. Whether a drawing is special or general and whether one uses a general drawing in a special way is a matter of an *evaluation* which presupposes experience with proofs in Euclidean geometry, but is not part of the proof and not at all a matter of pure logic. Thus, the conclusion that the proof is valid for all triangles presupposes an evaluation of the whole situation which involves pragmatic aspects. On the other hand, the teacher is in the established phase of the theory and argues from his point of view. To him it is obvious that the parameters used in Euclidean geometry (length, angle) determine the situation completely, and he has a sense, built up by experience, which aspects of a figure are general and which are not. However, his inability to realise that he is arguing from a point of view completely different to that of his pupil causes a serious problem of teaching and of understanding the nature of proof in general.

The Genetic Approach

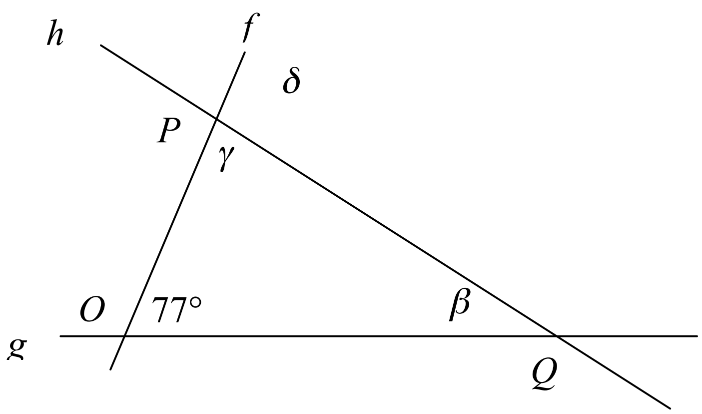
The point of departure for introducing pupils to mathematical argumentation is the question ‘why’. A regularity or pattern is observed and the question arises what makes things the way they are. Answering such questions can be considered like performing a *thought experiment*. The basis of argumentation and the chain of conclusions have to be developed simultaneously. Such activities can already occur at the elementary level (for examples see Wittmann & Müller, 1990)

After such a 'culture of why questions' has been developed over several years of mathematics teaching there will come a time when proof is explicitly taught and the term 'proof' explicitly mentioned to the students. In Germany, this is usually the case in grade 7 in the course of geometry teaching. One of the first proofs is that of the angle sum theorem for triangles.

Usually the theorem is proved in the classical Euclidean way and reduced to the alternate angle theorem. The latter is 'proved' by means of transformation geometry. Mathematically, this is not terribly honest since the axiom of parallelism is tacitly used and not mentioned to the students. Obviously, the intention of text book authors and teachers is to convey to the students the idea that proof leads to absolute certainty in contrast to measurements which are not precise and valid only for single cases.

In contrast to this, I would suggest that –at this stage!- geometry is presented as an *empirical theory.* Then the meaning of proof is not to replace measurement by something more certain, but to relate measurements and proofs in a theory. Proofs help to measure in an intelligent way (Winter, 1983; Jahnke, 1978).

To give an example I would propose to present the alternate angle theorem neither as a theorem proven nor as an evident truth but as an *uncertain empirical hypothesis*. Doing this we introduce the idea of modelling and the possibility of non-Euclidean geometries into teaching in a way adequate to pupils of the 7th grade. Consider the following work-sheet.



> The straight line *h* rotates counter-clockwise around vertex *P*.
> For some positions of *h* measure the angles β, γ and δ. Tabulate and write down your observations.

Students will observe: (1) Q travels more and more to the right. (2) γ increases, β decreases. (3) Losses of β are exactly compensated by gains of γ. (4) There is at least one position of *h* with $\delta = 77°$. (5) In this position *h* and *g* will not intersect. (6) To this position corresponds that *Q* has travelled to the infinite and that $\beta = 0°$.

It is plausible that the losses of β are exactly compensated by gains of γ also in the infinite (there shouldn't be a jump). But we cannot know this and we cannot test this directly. But we can prove: if exact compensation takes place then the angle sum of a triangle is exactly 180°.

To this idealised chain of observations and arguments the teacher could add the remarks that Euclid wasn't sure about exact compensation in the infinite and therefore added this statement as a hypothesis to geometry and that Gauss tried to test this hypothesis by measuring angles in very large triangles.

Having reached this starting point further theorems involving angles in triangles and general polygons might be proved and empirically tested. Theorems which are proved before any measurement has been done are like predictions in physics. In this way students come to study a 'small theory', and they get an idea of what a theory is, how measurements and proofs are related and why it is better to have a theory than not to have one.

In subsequent teaching further small empirical theories should be developed as it is done in the project "Arguments from physics in mathematical proofs" (see Hanna & Jahnke 2002) where for example axioms from static are the starting point for a theory.

In a final phase theories are constructed for their own sake. These theories are no longer considered to be empirical theories.

Bibliography

Balacheff, N. :1988. Aspects of Proof in Pupils' Practice of School Mathematics. In: D. Pimm(ed.), *Mathematics, Teachers and Children*.. London, Hodder & Stoughton**:** 216-238.

Balacheff, N. :1991. Construction et anlyse d'une situation didactique. Le cas de 'la somme des angles d'un triangle'. *Journal für Mathematik-Didaktik* **12**(2/3): 199-264.

Coe, R. & Ruthven, K.:1994. Proof practices and constructs of advanced mathematics students. *British Educational Research Journal* 2(1): 41-53.

Duhem, P. :1908/1978. *Ziel und Struktur der physikalischen Theorien*. Reprint of the first German edition: Felix Meiner Verlag, Hamburg, 1978

Healy, L. & C. Hoyles (to appear). Curriculum Change and Geometrical Reasoning. In: P. Boero (ed.), *Theorems in School*. Dordrecht, Kluwer

Fishbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics* **3**(2): 9-24.

Hanna, G. & H. N. Jahnke (1993). Proof and Application. *Educational Studies in Mathematics* **24**(4): 421 - 438.

Hanna, G. & H. N. Jahnke (1996). Proof and proving. In: A. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (eds.), *International handbook of mathematics education*. Dordrecht, Kluwer**:** 877 - 908.

Hanna, G. and H. N. Jahnke (2002). Arguments from physics in mathematical proofs: An educational perspective. *For the learning of mathematics* **22**(3): 38-45.

Hefendehl-Hebeker, L. (1995). Mathematik lernen für die Schule? Mathematische Semesterberichte 42: 33-52.

Jahnke, H. N. (1978). *Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik - Beweisen als didaktisches Problem*. Bielefeld, Materialien und Studien des IDM, 10.

Sneed, J. D. (1971). *The logical structure of mathematical physics* . Dordrecht: Reidel.

Stein, M. (1990). Zur Einführung in das Beweisen. *Praxis der Mathematik* **32**(1): 14-26.

Williams, E. (1979). *An Investigation of Senior High School Students' Understanding of the Nature of Mathematical Proof. Unpublished doctoral dissertation*. Edmonton, University of Alberta.

Winter, H. (1983). Zur Problematik des Beweisbedürfnisses. *Journal für Mathematik-Didaktik* **1**: 59-95.

Wittmann, E. C. & Müller, G. N. (1990). *Handbuch produktiver Rechenübungen, Bd. 1 und 2*. Stuttgart, Klett.

PUPILS' AWARENESS OF STRUCTURE ON TWO NUMBER/ALGEBRA QUESTIONS

Dietmar Küchemann, *University of London, United Kingdom*

Celia Hoyles, *University of London, United Kingdom*

Abstract: *A large sample of high attaining pupils were given a written proof test in Yr (age 13.5 years) and similar tests in Yrs 9 and 10. We look at their responses to two number/algebra questions which were designed to assess whether pupils used empirical or structural reasoning. We found that the use of structural reasoning increased over the years, albeit at a modest rate, but that the use of empirical reasoning, in the form of inappropriate number pattern spotting or through the desire to perform rather than analyse a calculation, was still widespread. However, we also identified a more advanced use of empirical reasoning, namely to check the validity of a structural argument.*

The analysis presented here forms part of The Longitudinal Proof Project (Hoyles and Küchemann: http://www.ioe.ac.uk/proof), which looked at pupils' mathematical reasoning over time. Data were collected through annual surveying of high-attaining pupils from randomly selected schools within nine geographically diverse English regions. Pupils were given a written proof test at the end of Yr 8 (age 13.5 years) and given similar tests at the end of Yrs 9 and 10. Overall, 1512 pupils from 54 schools took all three tests. The tests comprised items in number/algebra and in geometry, some in open response format and some multiple choice. We discuss here two open response questions in number/algebra, A1 and A4.

There is a considerable body of research to suggest that school pupils tend to argue at an empirical level rather than on the basis of mathematical structure (Bell, 1976; Balacheff, 1988; Coe and Ruthven, 1994; Bills and Rowland, 1999). Of course, inductive reasoning can be important and fruitful in mathematics (Polya, 1954) but, in the UK at least, the emphasis on generating data and looking for number patterns, even at the upper end of secondary school, has often seemed to be at the detriment of looking for structure.

Item A1 involves a pattern of white tiles and grey tiles and was devised to see whether pupils could make a 'far generalisation' (Stacey, 1989) on the basis of the pattern's structure or whether they would resort to a 'function' strategy or 'whole-object scaling' (ibid) based on spurious number patterns. Brown et al (2002), in their work with student teachers, found that the students quite often chose to "perform computations when reasoning about computations would suffice". Question A4 explores this tendency. It is based on a suggestion made by Ruthven (1995) and concerns divisibility, but where the dividend is too large for computation to be a viable strategy.

Pupils' responses to A1: changes in number pattern spotting from Yr 8 to Yr 10

Item A1 (which was familiar to English pupils) was asked in Yrs 8 and 10 (a different but parallel item was used in Yr 9). It is a standard number/algebra item involving a tile pattern, and was designed to test whether pupils could discern and describe a structure (assessed by carrying out and explaining a numerical calculation).

The Yr 8 version is shown in Fig. 1. We had deliberately built numerical distractors into the item, in the form of simple, but irrelevant, relationships between the numbers of white tiles mentioned in the item (namely, 6 × 10 = 60) and between the number of white and grey tiles in the given configuration (6 × 3 = 18).

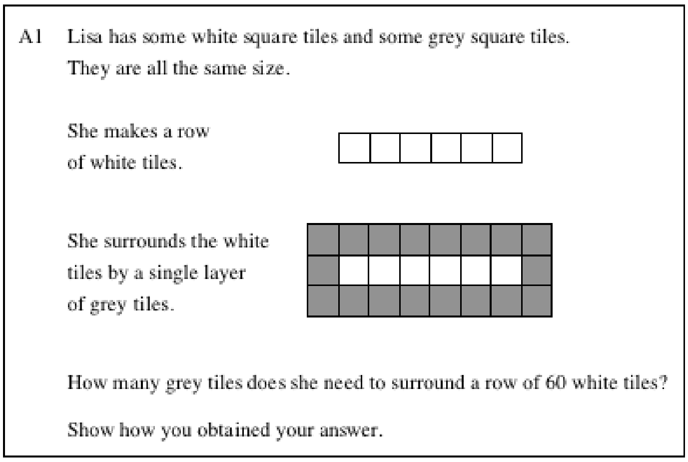


Figure 1: Item A1 (Yr 8 version)

Responses to question A1 were coded into 5 broad categories, or codes:

Code 1: Spotting number patterns, no structure;
Code 2: Some recognition of structure (incomplete or draws & counts);
Code 3: Recognition and use of structure: specific (correct answer, eg showing 60+60+3+3);
Code 4: Recognition and use of structure: general (correct answer and general rule, eg ×2, +6)
Code 5: Recognition and use of structure: general, with use of variables (correct answer and general rule, with naming of variables in words or letters).

Analysis of the code frequencies revealed that a substantial minority of pupils had fallen for our distractors, in that they had used inappropriate ‘number pattern spotting’ strategies (code 1), such as ‘There are 10 times as many white tiles so there will be 10 times as many grey tiles’ (scaling approach), or ‘There are 3 times as many grey tiles as white tiles’ (function approach). The scaling approach was far more popular than the function approach, but both give the result 180 (10 × 18 or 3 × 60), and altogether 35% of the total sample gave such responses in Yr 8. This fell to 21% in Yr 9 but stayed at 21% in Yr 10.

Complementing the changes in frequency of pattern spotting responses, the frequency of correct responses (codes 3, 4 and 5) went up from 47% in Yr 8 to 68% in Yr 9 but only to 70% in Yr 10. However, this small increase from Yr 9 to Yr 10 masks a substantial rise in the use of variables (expressed in words or with letters) in pupils’ explanations (code 5 responses), from 16% in Yr 9 to 26% in Yr 10 (starting from just 9% in Yr 8).

Longitudinal data showed that the improvement in pupils’ responses from Yr 8 to 10 was far from smooth. Regarding the number pattern spotting responses in particular, while only about half of the pupils who gave such a response in Yr 8 or Yr 9, did so again the following year, we also found that of those who gave such a response in Yr

9 or Yr 10, about half had *not* done so the previous year, ie their responses had 'regressed'. This suggests, that for some pupils at least, there is an element of chance about their responses: rather than being wedded to a particular way of construing such tile patterns (with some going for the underlying structure and some for a superficial number pattern) they seemed to hit upon one way on one occasion and another way on another occasion. An examination of individual scripts also shows that some pupils flipped between approaches on a given occasion. On the other hand, while 80% of those who gave a correct structural response in Yr 8 (codes 3, 4 or 5) also did so in Yr 10, only 54% of those giving a pattern spotting response in Yr 8 gave a correct structural response in Yr 10.

The use of algebra

The Yr 9 and 10 versions of question A1 had an added part, A1b, where students are asked to express the relationship between the numbers of tiles in algebra. It was worded as follows:

A1b: Write an expression for the number of grey tiles needed to surround a row of n white tiles.

We were interested in whether students were able to express any relationship they discerned in the tiling pattern in algebra, and indeed whether this was consistent with explanations of structure given in words or numbers. We have noted in a previous study (Healy & Hoyles, 2000) that Yr 10 students rarely used algebra as a language with which to described mathematical structure, even though they accorded high status to algebraic 'proofs'.

A1b asks pupils to map the number of white tiles onto grey, ie it requires a function approach ($n \rightarrow 2n + 6$). Thus we were curious to see whether this would force a rethink on pupils who had used a scaling number pattern approach (18 grey tiles ×10 = 180 grey tiles). In the event, this turned out to be the case for over half these pupils. Thus for example, of the 197 pupils who gave a scaling number pattern response to A1 in Yr 10, 43% gave the response '$3n$' (or equivalent) to A1b, ie they switched to a function response which still fitted their (incorrect) answer of 180 but no longer fitted the method they had used to obtain it. A further 13% of these pupils gave the correct response of '$2n + 6$' (or equivalent) to A1b, ie abandoned their pattern spotting approach entirely. On the other hand, and not surprisingly, few of the 110 pupils giving a function number pattern response to A1 switched for A1b: 87% gave the response '$3n$', which fits their pattern spotting answer and approach, with only 1% switching to a correct response. Interestingly too, 93% of those who had given a correct, but specific and non-algebraic response (code 3) to A1 in Y10 (which is all that the item requires), gave a correct algebraic response to A1b. We found similar response patterns in Yr 9.

As part of our research, we undertook case studies of some of the project schools (in fact those schools that had performed particularly well in the proof tests), which enabled us to interview some individual pupils after they had completed some or all of the tests, so we could ask them to reflect on the way they had answered in different years. We consider here three such pupils, MS, EC and JG, each of whom had given a number pattern spotting response to A1 in Yr 8 and/or Yr 10 (and sometimes also in Yr 9).

Pattern spotters

MS gave a pattern spotting response in Yr 8 and Yr 10 (and also in Yr 9); however in Y8 he used the '×10' strategy while in Yr 10 he used '×3' (his Yr 10 response to A1b, '3*n*', was consistent with this). He was interviewed a few days after taking the Yr 10 test so had little difficulty remembering his Yr 10 response:

MS: ...there's 6 there and 18…altogether times it by 3, then I thought it would be the same if you wanted to find out how many grey tiles would be in 60 so I timesed by 3…

He also seemed to have little difficulty in 'making sense' again of his Yr 8 response but seemed unperturbed by the fact that it was different (albeit leading to the same answer 180). His approach seems to have been fairly spontaneous and unreflective in both years.

It so happened we had also interviewed MS a year previously, a day after he had taken the Yr 9 Test. In this extract the interviewer asks him about his Yr 8 strategy of multiplying by 10, and again, there is no sign of any doubt in the validity of the method used:

I: Ok, and what makes you feel that that's the right way to do it?
MS: Because I found it the easiest way to get to 60 white tiles.
I: It's a nice quick way to get from 6 to 60 but, umm, how do you know it's the right way to get from 6 to 60 ?
MS: I just know, the sum of 6 times 10 is 60.

Pupil EC gave a '×10' pattern spotting response in Yr 8 (and Yr 9) but gave a correct answer in Yr 10 based on the geometric structure. He was interviewed about a week after taking the Yr 10 test and asked to compare his Yr 8 and 10 responses.

I: Which one do you think is the right one?
EC: I think this one is.
I: The Year 8 one.
EC: Yeah, kind of the first instinct I had.
I: You go by instinct.
EC: Yeah, I think, sort of, in the majority of the time the first instinct is right, so, I think maybe that one looks right.

EC is then asked to explain his Yr 10 response in more detail, which he seems able to do quite well. However, this is not enough to change his mind about the relative merits of his Yr 8 and 10 responses:

I: ... So, you ended up here in Year 10 with this double thing and then add six.
EC: Yeah.
I: So, how did that come to you, I mean, why would you have done that?
EC: I think, because we needed 60 and there was six along each row, each of the white things, so that means 12, so I just thought that doubling it, and then there's three left over, so I just, plussed three on one, so, I'm not really sure.
I: Okay. I mean, that sounds sensible enough, so, the trouble is, we've still got these two different answers. So are you going to stick with your instinct, your Year 8 instinct?

EC: I think so, yes.

Now, it is just possible that EC's statement above that "I just thought that doubling it", is evidence of an *empirical* generalisation (of the fact that 12 is double 6), rather than of genuine insight into the structure of the tile pattern; also he seems to have difficulty keeping track of all the structural elements, in that he mentions one "three left over" but not the other. Nonetheless, having demonstrated that he can make some use of the geometric structure to find the correct number of tiles and that he can describe crucial features of the structure to the interviewer, it seems surprising that he was so ready to abandon this potentially insightful approach for the sake of simplicity. Perhaps at this stage, through a lack of experience or guidance, EC does not have the meta knowledge needed to classify his different responses in an appropriate way and to recognise their positive or negative qualities. This is partly borne out by his responses to A1b, which are correct in Yr 9 and Yr 10, even though in Yr 9 he gave a scaling ('×10') pattern spotting response to the first part of A1.

JG had a similar set of responses to EC, in that she gave a '×10' number pattern response in Yr 8 and a structural response in Yr 10 ("60 times 2 is 120, 3 times 2 is 6, 120 plus 6 is 126"). She was interviewed about a week after taking the Yr 10 test. At first she could make no sense of her Yr 8 response ("I have no idea why I wrote that in Yr 8"), though she comes up with an interpretation eventually:

I: I mean, say someone else had done it, not you…could you sort of try and figure out why on earth they did it?
JG: (Long pause) No.
I: No, you can't see any logic in it?
JG: Well…yeah. I can now. It's because there's 6 there so, I figured 6 times 10 would be the 60 that they were talking about in the question, and so I just had to times the amount around the outside by the same number. Oh yeah.

However, unlike EC, she prefers her structural Yr 10 response which to her makes more sense:

I: Can you say a bit more why it makes more sense?
JG: I don't know…just a couple more years' practice of finding patterns and stuff.
I: So how does this answer sort of fit the pattern, the Year 10 answer fit the pattern better?
JG: Well because, I didn't just times the ones round the outside…by the same number as the ones on the inside…I worked out sort of a rule for it, rather than just a rule for that, that number.
I: Right. How did you get the rule for the Year 10 answer?
JG: Well, the three at each end won't change, it's a single row of tiles say…you just use the top…the grey tiles above and below the white…
I: Right, okay…

Her answers here are interesting. First she justifies her preference for the structural response with an external reason ("more years' practice"), which has nothing to do with the quality of the actual response and which is certainly no more valid than EC's quest for simplicity. However, she is able to describe the structure itself very nicely ("three at each end... ...grey tiles above and below...") and she does in fact say something, albeit in a rather cryptic way, about the quality of this explanation, namely about it being *general*: thus she found "sort of a rule" in contrast to "just a rule for that ... number". Notice also the statement about multiplying "the ones round the outside”, which potentially provides a test for this number pattern spotting strategy, since the outcome would be a set of grey tiles that no longer fits snugly around the white tiles. Further, she gives correct responses to A1b (in Yr 9 and Yr 10) which shows she is able to express the structure using algebra. All in all, we seem here to be witnessing the beginnings of a meta knowledge about structural explanations, even though the concepts and/or language may not yet be well-formed.

We do not know how representative these few interviewees are, but their responses do suggest that, for some pupils at least, the simplicity of number pattern responses may have a stronger appeal than the insight that might be gained from taking a structural approach. It also points to possible discontinuities between modeling with numbers, narrative descriptions of these models and modeling with algebra. JG did have insight derived from a structural approach, though her attempts to describe the characteristics of her response were still quite limited. This is a phenomenon we have found elsewhere and which may well be widespread even amongst the highest of our high attaining pupils, since they will generally have had little experience of producing mathematical explanations and reflecting upon them.

Pupils’ responses to A4: the tendency to calculate

We found a strong tendency for students to work at an empirical level on all our test items. In the case of A1, this was manifested by the pattern spotting responses discussed above, and by (the small number of) students who generated further data (e.g. for 7, 8 and 9 white tiles) and who used this to induce a rule. The same tendency was very strong in responses to question A4 (shown abbreviated in Fig. 2, below).

The question was used in Yrs 8 and 9 (but not in Yr 10). In part *a* students are asked about the divisibility of 5! by 3 and in part *c* about the divisibility of 100! by 31 (or 50! by 19).

Yr 8 version of A4

A4 a) 4! means $4 \times 3 \times 2 \times 1$.
5! means $5 \times 4 \times 3 \times 2 \times 1$.

Is 5! exactly divisible by 3 ?

Explain your answer.

b) What does 100! mean?

c) Is 100! exactly divisible by 31 ?

Explain your answer.

Yr 9 version of A4

A4 a) 4! means $4 \times 3 \times 2 \times 1$.
5! means $5 \times 4 \times 3 \times 2 \times 1$.

Is 5! exactly divisible by 3 ?

Explain your answer.

b) What does 50! mean?

c) Is 50! exactly divisible by 19 ?

Explain your answer.

Figure 2: Yr 8 and Yr 9 versions of question A4

Most students could correctly determine the divisibility of 5! by 3 (76% in Yr 8, 83% in Yr 9), but the overwhelming majority did so by evaluating 5! and by performing the calculation 120 ÷3, with only 2% of the sample in Yr 8 and only 6% in Yr 9 basing their argument on the notion that 3 is a *given* factor of 5!. Students were not allowed calculators, so that the latter kind of argument was essential to answer part c) correctly, and in the event 3% of the sample did so in Yr 8, and 9% in Yr 9. Most students wanted to evaluate the factorial, and had no viable alternative strategy. Some students wrote statements like “That would take years to work out and if there is some short cut I don't know it". Some evaluated 100! as being 2400, on the basis that 100 = 20 × 5 so 100! = 20 × 5!, and so answered 'No'; others answered 'No' because 100! is even and 31 is odd or a prime.

We initially interpreted our interviews as suggesting that students *needed* to calculate and were insecure about using number relationships even when they apparently understood them. This would perhaps helps explain the seemingly astonishing fact that nearly half (26 of 55) of the students who did answer part *c* successfully in Yr 8 'regressed' in Yr 9. However, we have modified our views on this: Jahnke (2005) uses the metaphor of 'theoretical physicist' to describe pupils' behaviour as they learn to engage in mathematical proof. From this perspective, pupils' recourse to empirical evidence can be seen as a perfectly rational attempt to test the validity of a proof argument, rather than as a rejection or lack of appreciation of such arguments.

We illustrate this propensity to calculate on A4, by looking at our interviews with two students, MH and AM. From their written responses (summarised in Table 1, below) it appears that both students made considerable progress from Yr 8 to Yr 9, and both would seem to have a clear understanding of the 'divisibility principle' by Yr 9, as evidenced by the responses to part *c*. However, our interviews with them, which took place the day after they had taken the Yr 9 test, suggest their understanding is not quite so secure.

Yr 8	**Yr 9**
Pupil MH A4a: "Yes"; calculates 120 ÷ 3	**Pupil MH A4a**: "Yes"; calculates 120 ÷ 3
Pupil MH A4c: Attempts various calculations, eg of 10!, but gives up. Answers "No"	**Pupil MH A4c**: "Yes. If you times it by a certain number you will be able to divide by it"
Pupil AM A4a: "Yes"; calculates 120 ÷ 3	**Pupil AM A4a**: "Yes. The number has been multiplied by 3, so it must be divisible by 3"
Pupil AM A4c: Leaves blank	**Pupil AM A4c**: "Yes. The number has been multiplied by 19, so it must be divisible by 19"

Table 4: Pupil MH's and AM's responses to A4a and A4c in Yr 8 and Yr 9

In our interview with MH, he is first asked to say a bit more about his Yr 9 explanation to A4c. He replies with this rather nice way of describing the inverse relationship between multiplying and dividing by 19:

MH: Well if you times in from 50 down to 1 you times it by 19 somewhere, so if you want to divide it by 19 you'd just get the same answer as if not timesing it by 19 and just leaving it out altogether.

The interviewer then asks MH about the fact that he had changed his Yr 9 answer to part c) from No to Yes:

I: How did it hit you because interestingly you put 'no' first and then changed it to 'yes'. Was that a quick change, did you quickly see it or did you sit there and think?
MH: Sitting there and think about it 'cos then I figured out that it was divisible by 19 if you can times it by 19, so I changed it.
I: Did you, did it help, did you look back at part *a* ?
MH: Yes.
I: You did?

MH: Yes, I noticed that part *a*, everything you times in by it you can divide by, can be divided by 5, 4, 3, 2 and 1.
I: Now when you say you noticed that, is that because you know your tables well enough to know that 5 goes into 120 and four goes into it, or did you do it in your head and actually worked it out?
MH: In my head and worked it out.
I: You did, well did you work out every one?
MH: Yes.
I: What, you divided 120 by all these numbers?
MH: Yes.
I: Really, that's quite a lot, a lot to do. So do you think, are you then arguing that because it works for 5 you reckon it ought to work for 50 -
MH: - Yeh, it should do.
I: So your feeling is that all the numbers you multiply by you can divide by. Is that simply because it happens to work here, or is there some deeper reason for it, is it kind of...?
MH: I just feel that it works, and you know that it works in the first five, and when you get higher and higher and you times by any of the numbers that you can divide it by, and say in the 50-exclamation-mark you can divide something like 99, you're not going to be sure it can work or not, 'cept if you use calculator, which we wasn't allowed to.

This sequence, and especially the last sentence, might suggest that MH's awareness and acceptance of the inverse relationship between multiplication and division, is not entirely secure, and is based on an empirical generalisation (of the divisibility of 5! by 5, 4, 3, 2 and 1) rather than on genuine insight. However this is a harsh interpretation since to determine that, say, 50! is divisible by 19 one also needs to realise that the terms following ×19 in 50! (ie ×18, ×17, etc) do not affect the divisiblity.

AM's set of written responses is similar to that of MH, except that in Yr 9 he gives an explanation based on divisibility in part a) as well as part c), rather than one based on evaluating 5!. Thus, in part a) he had written "The number has been multiplied by 3, so it must be divisible by 3". In the interview, he is asked how he arrived at this explanation:

AM: I think I was just thinking about it. I was thinking 5! would be 5 × 4 × 3 × 2 ×1 so if it's been timesed by three you can almost certainly divide it by 3 and I was just thinking sort of that was that. Also I was saying because it's only then been timesed by 2 and by 1, because 1's obviously not going to change and 2's just going to double it. So you're just going to be able to divide it by 3.

This is interesting on several counts. First, though AM does not talk of calculating 5! his explanation is still very much grounded in the steps of the potential calculation. It is not enough that × 3 is one element in a string of factors, he carefully checks that the subsequent factors, × 2 and × 1, won't affect the divisibility by 3. This also nicely illustrates the fact that a notion like 'divisibility' involves a whole nexus of ideas (see Brown et al, 2002), including some understanding of the associative and commutative properties of multiplication. Last, AM says he is "almost certain" but not entirely that 5! is divisible by 3. The interviewer explores this further:

I: You said just now it's almost certainly divisible by 3, you're hesitating slightly.
AM: I think it is divisible by 3, I think at the time I wasn't completely certain.
I: What yesterday, but you're certain today?
AM: Yes, after all that I'm almost certain that's good.
I: Why is there still this edge of doubt? You say *almost* certain.

AM: Don't know. The thing is, I am certain, but not quite... I can't see why, I can see slightly why it works but not entirely, I haven't thought 'suppose you had a bigger number, would it…'

I: You have with part *c* made the same conclusion because you said the number's been multiplied by 19 so it must be divisible by 19, you've got the word 'must' interestingly, not 'it might be', 'almost certainly will', so you do seem to think that a bigger number would work. So say we had a thousand factorial and we wanted to know whether 128 went into it?

AM: Yeh, it would be divisible by 128... as far as I know (*laughs*), but it's very hard to be certain.

Thus it would seem that for AM and for MH too, their lack of certainty about the 'divisibility principle' (the argument that if you multiply by a certain number the result is divisible by that number) is not because they do not understand the basic argument or appreciate its power, but because of some awareness that other features of the situation (eg that in 100!, the term × 31 is followed by a long string of other terms) may render the principle invalid.

Conclusion

The propensity to calculate occurred on many of our items. For example, in Yr 10 we asked pupils to prove the statement that "When you add any 2 odd numbers, your answer is always even". The most common approach was a purely empirical one (given by 31% of the sample), whereby pupils simply demonstrated that the statement was true for one of more pair of odd numbers. This was followed in popularity by an exhaustive approach (19% of the sample) concerned with the units digits of odd an even numbers. This can lead to a valid proof but is still essentially empirical in that it is concerned with surface features of odd and even numbers rather than underlying structure. On the other hand, a substantial minority did attempt proofs containing some element of structure, expressed in narrative (18%) visual (6%) or algebraic (5%) form.

The propensity occurred on many of our items. For example, in Yr 10 we asked pupils to prove the statement that "When you add any 2 odd numbers, your answer is always even". The most common approach was a purely empirical one (given by 31% of the sample), whereby pupils simply demonstrated that the statement was true for one of more pair of odd numbers. This was followed in popularity by an exhaustive approach (19% of the sample) concerned with the units digits of odd an even numbers. This can lead to a valid proof but is still essentially empirical in that it is concerned with surface features of odd and even numbers rather than underlying structure. On the other hand, a substantial minority did attempt proofs containing some element of structure, expressed in narrative (18%) visual (6%) or algebraic (5%) form.

The tendency to adopt an empirical approach is very strong in English schools and beyond, which might at least partly be explained by particular classroom approaches (see for example Morgan, 1997). Our view is that the situation does not have to be like this – we can change pupils' habits of mind (Cuoco et al, 1996). Our findings lends some support to this, since our pupils did make some, albeit modest, progress in the use of structural reasoning, and the data suggest that switching strategies (even between incorrect strategies) might be helpful in catalyzing a new perspective on a problem. In our current work (Proof Materials Project) we are working with teachers to see whether this change of habit can be put into effect more widely, in particular by making pupils more aware of different kinds of proof strategies and explanations.

Acknowledgement

We gratefully acknowledge the support of the Economic and Social Research Council (ESRC), Project number R000237777.

References

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm (ed.), *Mathematics, Teachers and Children*, Hodder and Stoughton, London, pp. 216-235.

Bell, A. W.: 1976, 'A study of pupils' proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7, 23-40.

Bills, L. and Rowland, T.: 1999, 'Examples, generalisation and proof', in L. Brown (ed.), *Making Meanings in Mathematics*. Refereed Proceedings of the British Society for Research into Learning Mathematics, QED Publications, York, pp. 103-116.

Brown, A., Thomas, K. and Tolias, G.: 2002, 'Conceptions of divisibility: success and understanding', in S. R. Campbell and R. Zazkis (eds.), *Learning and Teaching Number Theory*, Ablex Publishing, Connecticut, pp. 41-82.

Coe, R., and Ruthven, K.: 1994, 'Proof practices and constructs of advanced mathematics students', *British Educational Research Journal* 20(1), 41-53.

Cuoco, A., Goldenberg, E. P. and Mark, J.: 1996, 'Habits of Mind: an organizing principle for mathematics curricula', *J. Mathematical Behavior* 15, 375 - 402.

Healy, L. and Hoyles, C.: 2000, 'A study of proof conceptions in algebra', *Journal for Research in Mathematics Education*, 31 (4), 396-428.

Jahnke, H. N.: 2005, 'A genetic approach to proof'. Paper presented at the *Fourth Congress of the European Society for Research in Mathematics Education* (CERME 4), Februrary 2005, Sant Feliu de Guixols, Spain.

Morgan, C.: 1997, 'The institutionalisation of open-ended investigation: some lessons from the UK experience', in E. Pehkonen (ed.), *Use of Open-Ended Problems in Mathematics Classroom*, Department of Teacher Education, Helsinki, Research Report, 176, 49-62.

Polya, G.: 1954, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*, OUP, London.

Ruthven, K: 1995, 'The unbearable mightness of being', in L Healy (ed), *Justifying and Proving in Mathematics: Draft Activities*. Unpublished manuscript, Institute of Education, London.

Stacey, K.: 1989, 'Finding and using patterns in linear generalising problems', *Educational Studies in Mathematics* 20, 147-164.

PROOF IN SWEDISH UPPER SECONDARY SCHOOL MATHEMATICS TEXTBOOKS - THE ISSUE OF TRANSPARENCY

Kirsti Nordström, *Stockholm University, Sweden*

Clas Löfwall, *Stockholm University, Sweden*

Abstract: *We have investigated proof in two sets of commonly used Swedish upper secondary school mathematics textbooks. The frequency of proof items is low in each mathematical topic, even in the domain of geometry where pupils traditionally have learned proof. We explore the proof items with respect to different aspects of proof and discuss how they relate to students' access to proof. We show with some examples how proof often exists invisible in the textbooks and discuss the difficulty of giving a correct definition of proof at upper secondary school level.*

1. Introduction

The role of proof in the Swedish schools has diminished during the last twenty years, a development, which is similar to that of many other countries (Hanna, 1995; Waring, 2001).

The Swedish curriculum for upper secondary school does not clearly state the aims of introducing the students to proofs and proving activities. Only the main goals are stated. "The school in its teaching of mathematics should aim to ensure that pupils develop their ability to follow and reason mathematically, as well as present their thought orally and in writing." (Skolverket, 2002, p. 60) Local schools and teachers have the possibility of applying these goals in their own way. However, one of the Criteria for 'Pass' (lowest mark of a three-level grading scale: Pass, Pass with distinction, Pass with special distinction) for any of the five courses A-E, into which upper secondary school mathematics is divided, is that "pupils differentiate between guesses and assumptions from given facts, as well as deductions and proof" (pp. 60-66). Furthermore, one of the 'Criteria for Pass with special distinction' is that "pupils participate in mathematical discussions and provide mathematical proof, both orally and in writing" (pp. 60-66)

Most of the university entrants, who responded to the survey conducted by Nordström (2002), stated that they had had little experience about proofs in upper secondary school, especially in form of own practice. The aim of this textbook study is to complement the survey in order to create a more varied picture of students' school background concerning their experiences about proof.

Researchers have paid relatively little attention to the role of the textbooks in the teaching and learning of proof (Hanna and de Bruyn, 1999). “In particular, there has been almost no examination of the actual occurrence in mathematics textbooks of proofs, discussions of proof, and exercises requiring the construction of proofs.” (p. 180) Many studies show that textbooks and teacher guides influence the teaching practices and the choices of the contents being treated in the lessons. This is particularly true for mathematics (Englund, 1999).

The first impression when skimming through some Swedish upper secondary school textbooks is that the frequency of proofs and proving tasks is minimal. However, we wanted to systematically explore the occurrences of proofs and proving tasks in different mathematical domains. Our aim was also to analyse the nature and level of the proofs included in the textbooks and the manner in which the proofs were presented to the reader. The main question is how the textbooks enhance the students’ access to proof and in what way the textbooks help students to develop their understanding of different aspects of proof.

2. Theoretical frame

Our approach to the issue is socio-cultural. According to the socio-cultural theory, learning is an aspect of interrelated historical, cultural, institutional and communicative process (Renshaw, 2002)

2.1. Proof as an artefact

The notion of transparency of artefacts as Lave and Wenger (1991) define it is central for the study. We consider proof, in a very similar way as Adler (1999) considers talk when studying multilingual mathematics classrooms, as a *resource for mathematical learning*. Then proof needs to be both seen (be visible) and to be used and seen through (be invisible) to provide access to mathematical learning. We argue that Lave and Wenger’s concept of transparency captures this dual function of proof as a learning resource in mathematics.

“…the notion of transparency, taken very broadly, is a way of organising activities that makes their meaning visible,…” (Lave & Wenger, 1991, 102).

Access to artefacts in the community both through their use and through understanding their significance is crucial. We study how different aspects of proof are visible/invisible in the textbooks.

2.2. Conceptual frame

We use the conceptual frame created by Nordström (2004) in the analysis of the data. The aspects in the frame are *Inductive/ Deductive approaches, Conviction/Explanation*, *Formality, level of rigour and the language*, and *Aesthetics*. We explore if the different aspects of proof are visible in the textbooks and discuss how they relate to students’ access to proof. For an account of these aspects see Nordström (2004).

2.3. A review of relevant research

We found only two studies dealing explicitly with proof in textbooks. Hanna and de Bruyn (1999) investigate the frequency of proof items in year 12 (aged17-18) advanced- level mathematics in Ontario. They found that only in the topic of geometry, textbooks do a reasonable job of providing opportunities to learn proof and a range of proof types was presented. No visual proofs, heuristic arguments or the presentation of correctly or partially completed proofs were used for critique by the student.

Grenier (2000) studies the status and role of proof in 11-16 years pupils' curricula and schoolbooks in France. She finds a discrepancy between the curriculum goals and the contents in the textbooks. She remarks that even if the curriculum gives possibilities to employ new ways of treating proof, the proof practice seems to live primarily in the field of geometry and under two forms: 1) Simple deductive reasoning (with a small number of steps), based on the use of statements or theorems from the theoretical part of the course. 2) Writing proofs, with more attention given to language and syntax rather than logic and semantics.

Of course, there are a lot of studies where proof is considered in textbooks but not as a main issue (e.g. Bremler 2003). Bremler studies derivative in the textbooks during 1967-2002. His thesis is of interest for us because it concerns the Swedish upper secondary school textbooks. He briefly describes some changes in the treatment of proof and states that after 1994 (when a new curriculum was implemented) there is no proof of the general rule for derivative of a polynomial in any of the textbooks. The number of textbooks that neither show nor mention the proof has doubled since 1994. However, Bremler (2003) points out that some of the textbooks have preserved the proof but as an exercise with hints.

3. Methodology

The upper secondary school mathematics in Sweden is divided into five courses from A to E. We chose two commonly used set of textbooks, covering the courses from A to E, *Liber Pyramid* and *Matematik 3000*. The topics in both sets are: Algebra, Geometry, Statistics and probability, Functions and calculus, Exponents and logarithms, Trigonometry and Complex numbers. The textbooks are divided in chapters and sections within chapters. The sections contain explanatory text and examples, followed by exercises (with hints and solutions). Most of the exercises are divided into three levels, which correspond to the different marks in a three-level grading scale: Pass, Pass with distinction, Pass with special distinction.

When designing the rational of the study one of our aims was to examine the frequency of proof items in the textbooks as Hanna and de Bruyn (1999) did in their study on Ontarian textbooks. We examined the textbooks for the frequency of occurrence of various kinds of proofs and of items related to constructing a proof. Very soon though, we realised that the occurrence of proof or discussion on proof was very low compared to the Ontarian textbooks. Furthermore, proofs were seldom made explicit in the explanatory sections. It was not reasonable to use the quantitative method which Hanna and de Bruyn (1999) developed in their study. Therefore, we decided to work mainly with qualitative content analysis. We counted the percentage of the proving tasks, studied the textbooks, chapter by chapter and picked the items that have some significance for learning of proof. Afterwards, we related these items to the research questions, earlier research on proof in the textbooks and to the conceptual frame created by Nordström (2004). During the exploration we created some posteriori subcategories and checked the textbooks again with respect to them.

4. Results

We start with reporting to what extend proving tasks occur in different mathematical domains and go on with the different aspects of proof. All these results have some significance to students' access to proof and the notion of transparency, to which we particularly denote the last section of the chapter.

4.1. Frequency of different kinds of proving tasks

The space given to proving tasks is minimal compared to practical applications and routine exercises (about 2%). However, there are some special mathematical domains where proving tasks are more common: in geometry, in the context of verifications of solutions of differential equations and verification of formulas of trigonometric functions.

Example: *Show that* $2\sqrt{x}$ *is a solution for differential equation* $2xy' - y = 0$.

The pattern is the same as in the Ontarian textbooks, even if the percentages of the proof occurrences in Ontarian textbooks were substantially higher in geometry. Of all tasks in the geometry chapters in the Swedish textbooks, about 10% are proving tasks compared to the two Ontarian textbooks where 56% respectively 40% of the geometry tasks were proving tasks (Hanna and de Bruyn, 1999). Also in the French textbooks proof was concentrated in geometry (Grenier, 2000). In the two first courses (A-B), geometry is the only area where some proofs are explicitly given in the explanatory sections.

There are a few tasks dealing with counterexample. There are no indirect proofs presented in the ordinary course in either *Matematik 3000* or *Liber Pyramid*. In the Ontarian textbooks the concept is introduced in the geometry chapter and there are some exercises on it (Hanna and de Bruyn, 1999). Tasks encouraging pupils to make conjectures and own investigations are unusual in the Swedish textbooks. This was something Grenier (2000) also found out in the French textbooks.

There are some examples of potentially productive tasks for enhancing students' understanding of proof, but we argue that they are occasional and not systematically presented. Further, the range of the types of proofs given in the explanatory sections or as exercises are too narrow to be representative of mathematical practice. This is something that Hanna and de Bruyn (1999) also noticed in their study. The textbooks do not offer practice enough for the students to learn to construct different kinds of proofs.

4.2. How the different aspects of proof are visible in the textbooks

4.2.1. Inductive/ Deductive approaches

Students have often difficulties in dealing with algebraic symbols. That is why it is an important task for the teaching of mathematics to accustom the students to algebraic symbols and help them to realise the generality and the power of using such symbols. We found that a clear distinction between examples and general proofs is not always made visible. Below follows four different modes we found in *Liber Pyramid* and *Matematik 3000* of introducing some rules and formulas:

(1) The rule is given through some specific examples.

This mode is used when introducing the rule for multiplication of negative integers in *Liber Pyramid* for course A. In *Matematik 3000* an explanation of the role is given by a generic example. In *Liber Pyramid* this kind of explanation is given in a subsequent course, i.e., the C-course.

(2) The general rule is given with algebraic symbols followed by a special case as an example.

(3) The rule is introduced by generic examples.

This mode is quite usual in both of the textbook series. After the examples, the authors write: 'Generally (holds)…' , 'It is possible to show that generally…', 'One can strictly show…', 'In the same way one can prove'… , 'We can obviously formulate the following rule…', 'Without a proof we accept the following rules…'. Sometimes these comments are lacking and no clear distinction is made between the special and the general.

Example (*Liber Pyramid*):

$7^2 \cdot 7^4 = (7 \cdot 7) \cdot (7 \cdot 7 \cdot 7 \cdot 7) = 7^6$

So we get $7^2 \cdot 7^4 = 7^{2+4} = 7^6$

In the same manner we get that $3^3 \cdot 3^5 = 3^{3+5} = 3^8$

Generally holds $a^m \cdot a^n = a^{m+n}$

Also, in *Matematik 3000* exponent laws are introduced in the same way. In *Liber Pyramid*, a general proof is demanded as a middle-level exercise in the same section. However, the difference between the generic example and the proof is not visible. The proving task starts by 'Explain that...'. We think that the generic example can already be seen as an explanation. Then it would be more honest to say '*Prove* that it holds for all positive integers'. In the solutions a complete algebraic proof is given.

(4) The general rule with a deductive proof is given.

There are deductive proofs in the textbooks but they are not always called proofs, especially in the textbooks for the two first courses. Later, they are sometimes called proofs, sometimes derivations. Very often a specific or a generic example is introduced in parallel to the proof but often without an explanation of what the general and what the specific is. There is one example when the authors first test some properties with specific examples and then investigate if 'it is possible to show the general case'. There are some tasks encouraging pupils to make conjectures and own investigations but they seldom lead to a construction of a deductive proof.

4.2.2. Aspects of Conviction/Explanation

The proofs in the textbooks are obviously not given as obligatory rituals but in order to explain why the statement is true. In *Liber Pyramid* the proofs in the first textbooks are even presented as synonyms for explanations. However, some of the proving tasks start with 'Make believable that… (Troliggör att…)'.

4.2.3. Aspects of Formality, level of rigour and language

In contrast to the French textbooks (Grenier, 2000), the aspects of writing proofs are not dealt with at all. The style in proofs in the explanatory sections and in the solutions and hints is everyday language rather than formal mathematical language. Words like *proof, definition, assumption* are generally avoided in the textbooks, especially in the first two courses (A-B). This aspect can have some significance to the students' access and transparency because there are very few examples on how to write proofs in the explanatory sections.

When trying to avoid the word proof a confusion of different notions is easily created. In *Liber Pyramid*, proof is first defined as "logical reasoning without gaps". The word *explanation* is used later, instead of the word proof in the beginning of the book. Later the authors tell that the Greeks *discovered* that $\sqrt{2}$ was an irrational number, instead of the Greeks proved or showed that $\sqrt{2}$ was an irrational number. In *Liber Pyramid* for course A the derivations of the formulas for areas of polygons are called *justification (motivering)*. However, the proof of the area for trapezoids has not even a heading *justification* and the proof is hidden in the text.

The textbooks do not enhance the students' abilities to *differentiate between guesses and assumptions from given facts, as well as deductions and proof*, which is one of the curriculum goals for all the students. The words like assumption, guess/conjecture, given facts or deduction are not even mentioned. There are no exercises that shed light on the meaning of definitions. For example, in the geometry chapter in *Liber Pyramid*, the word definition is not used at all even if a definition of the concept 'definition' is given in the summary of the same chapter.

4.2.4. Aesthetics

We found two sentences referring to aesthetic aspects of proofs and theorems, both of them in *Liber Pyramid*: 'This beautiful and powerful formula can be used to…' (de Moivre's formula). 'Pretty and useful!' (the formula for a geometric sum). Aesthetics is an aspect that mathematicians often refer to when talking about proof (Nordström, 2004). So it would be natural to make also the aesthetic aspects of theorems and proofs visible in the textbooks.

4.3. Transparency and the students' access

Differentiating the tasks in the textbooks is made in a way, which indicates that the textbook authors consider that proofs are accessible only for already motivated and high achieving pupils. The proving tasks are most often placed in the second or third level of difficulty. This is in harmony with the criteria of judgement, which the Swedish curriculum states. However, an exception is the chapters of trigonometric formulas and differential equations, where examples on how to verify equalities are given in the explanatory section followed by exercises at all levels of difficulty.

None of the textbooks included in this study pursued to make proofs visible in a manner that focus on the logic and the structure of the proofs. However, there are some slight differences between the two sets. *Matematik 3000* offers some examples on how to prove some geometrical statements and contains more proving tasks in geometry than *Liber Pyramid*. There is also a rational order in *Matematik 3000*, in which the examples and the tasks follow each other and the structure gives the students a possibility of building deductive chains and start from clear theorems or axioms. That is not always the case in *Liber Pyramid* where you for example meet a middle level proving task that demands the knowledge of the similarity of triangles that is given before, but outside the ordinary course. Even if *Matematik 3000* offers some structure in presentation of proofs there are examples on how proof is made invisible as well. For instance the word 'proof' is omitted when presenting theorems and proofs.

Unlike *Matematik* 3000, where we could not find any discussions or explanations on proof or theorem, the authors of *Liber Pyramid* make some attempts to explain these notions. In the connection to the theorem *The sum of the angles in a triangle is 180°* the following explanations are given in *Liber Pyramid* (our translation):

'A statement that is true is often called a *theorem* in mathematics, e.g. Pythagorean theorem.'

'An explanation for why the theorem is true is called a *proof*.'

The following definitions are from the summary in the end of the geometry chapter in *Liber Pyramid* (our translation).

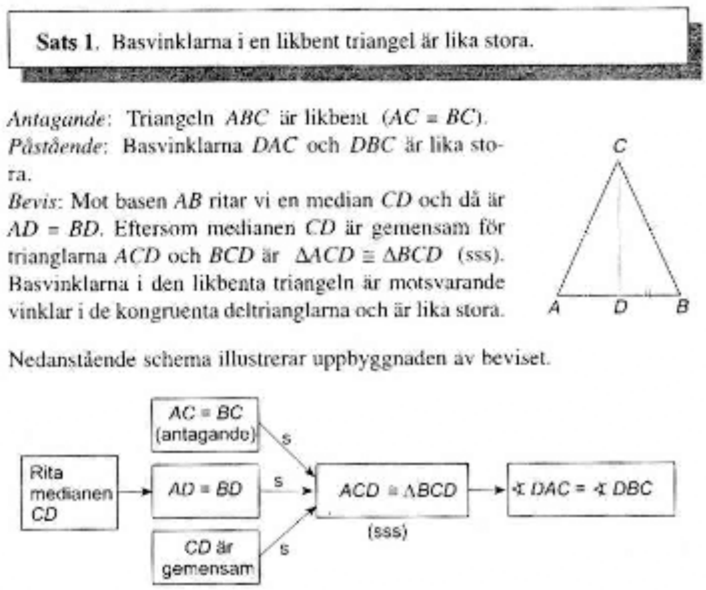
Theorem	A statement which has been proved. For example: The Pythagorean theorem.
Proof	A logical reasoning without gaps.
Axiom	A statement that is accepted without a proof. For example: The Parallel axiom in Euclidian geometry.
Definition	Determines (explains) what something is. For example: The definition of a right triangle.

We notice that there are different definitions for proof and theorem in the same textbook.

5. Discussion

An interesting theoretical question is to what extent and by what means it is possible to make the different aspects of proof visible at the upper-secondary level. The authors of *Liber Pyramid,* for example, attempt to explain the fundamental notions *statement, theorem, proof* and *truth* but as we have shown, it is rather confusing. This is not surprising because there is not a consensus on these notions among all mathematicians. The definitions are different in *classical* mathematics and *constructive* mathematics (e.g. Richman, 1997). In classical mathematics a *statement* is something which is true or false. A *proof* of a statement is a logical reasoning which makes it evident that the statement is true. The notion of *truth* is left undefined. In constructive mathematics, a *statement* is defined if one knows what a *proof* of the statement consists of. A statement is *true* if there is a proof of the statement. Hence truth is defined as provability and this is in fact implicitly also done in *Liber Pyramid*, since they have the following two definitions of *theorem*: 1) a true statement, 2) a statement which is proved. Students consider the act of proving theorems as more cumbersome than the act of computing. In the textbooks there are no rules for proving and no discussion of how proofs are created and also very few examples. In constructive mathematics there is no real difference between the activities of proving theorems and of computing (Martin-Löf, 1985). It is possible that knowledge of this fact could contribute to the way these notions are treated in school. This could be a subject for further investigations.

Even if it is difficult to give a correct definition of proof there are several ways of enhancing students' understanding of proof. The following example from a Finnish textbook shows how, for example, the structure in geometrical proofs can be made more visible. In the textbook above the figure, there is first a discussion on how to find out from the formulation of the theorem (the base angles in an isosceles triangle are equal) what is the assumption and what is the statement, which is not necessarily easy for the students to decide. The proof then begins with the *assumption* (*Antagande*): ‘The triangle ABC is isosceles.’ It follows by the *statement* (*Påstående*): ‘The base angles *DAC* and *DBC* are equal.’ After the proof the logical structure of the proof is illustrated with a figure. Such figures are employed also in the exercise sections and in all the following geometrical proofs given in the explanatory section. The figures illuminate the process of proving by showing its logical structure and how the necessary arguments needed for the conclusion are obtained from the assumptions, definitions, constructions, axioms or theorems.


Sats 1. Basvinklarna i en likbent triangel är lika stora.

Antagande: Triangeln *ABC* är likbent ($AC = BC$).
Påstående: Basvinklarna *DAC* och *DBC* är lika stora.
Bevis: Mot basen *AB* ritar vi en median *CD* och då är $AD = BD$. Eftersom medianen *CD* är gemensam för trianglarna *ACD* och *BCD* är $\Delta ACD \cong \Delta BCD$ (sss). Basvinklarna i den likbenta triangeln är motsvarande vinklar i de kongruenta deltrianglarna och är lika stora.

Nedanstående schema illustrerar uppbyggnaden av beviset.

Figur 1 Geometri, Gymnasiematematik, lång kurs, p. 88

6. Conclusion

The results of our study can be understood when considering them against the historical background. In the Swedish national curriculum 1994, the word proof was not mentioned. The style of making proof invisible in the textbooks can be a reaction against the earlier, very formal way of presenting theorems and proofs. It is not an easy task to decide how to treat proof at this level. However, the recent curriculum states some goals concerning proof and it is important to discuss how the textbooks meet the new requirements.

References

Adler, J.: 1999, ‘The dilemma of transparency: Seeing and Seeing Through Talk in the Mathematics Classroom’, *Journal for Research in Mathematics Education* 30, 1.

Bremler, N.: 2003, *Matteboken som redskap och aktör. En studie av hur derivata introduceras i svenska läroböcker 1967-2002,* Licentiate thesis. Lärarhögskolan i Stockholm.

Englund, B.: 1999, 'Lärobokskunskap, styrning och elevinflytande', *Pedagogisk Forskning i Sverige* 4(4), 327-348.

Grenier, D.: 2000, 'Learning proof and modelling. Inventory of teaching practice and new problems', A paper presented at topic study group on Proof and Proving in Mathematics Education, ICME 9, Tokyo.

Hanna, G.: 'Challenges to the importance of proof', *For the Learning of* Mathematics, 15(3), 42-49.

Hanna, G., de Bruyn, Y.: 1999, 'Opportunity to learn proof in Ontario grade twelve mathematics texts', *Ontario Mathematics Gazette,* 37(4), 23-29.

Lave, J. and Wenger, E.: 1991, *Situated Learning: Legitimate Peripheral Participation*, Cambridge University Press, Cambridge.

Martin-Löf, P.: Constructive mathematics and computer programming, In: .A.R. Hoare and J.C. Shepherson (eds.) Mathematical Logic and Computer Languages. Prentice Hall, 1985.

Nordström, K.: 2002, 'Swedish university entrants experiences about and attitudes towards proofs and proving', a paper presented at TG on argumentation and proof, CERME 3, Italy.

Nordström, K.: 2004, 'A pilot study on five mathematicians' pedagogical views on proof', a paper presented at TSG 14 on argumentation and proof, ICME 10, Copenhagen.

Renshaw, P.:2002, 'Community and learning', *Professional Lecture Series*, 1-21.

Richman.: 1997, Constructive mathematics, Internet: (http://www.math.fau.edu/Richman/html/construc.htm.

Skolverket. : 2002, National Science Programme, Internet: http://www3.skolverket.se/ki/eng/nv-eng.pdf.

Waring, S.: 2001, 'Proof is back. A proof-oriented approach to school mathematics', *Mathematics in School,* January.

Textbooks:

Björk, L-E., Brolin, H.: 2000. *Matematik 3000.* A-E. Natur och Kultur. Stockholm.

Wallin et al.: 2000. *Liber Pyramid.* A-E. Liber AB. Stockholm.

Silfverberg et al.: 2001. *Geometri*. Gymnasiematematik, lång kurs. Schildts, Vasa.

THE MEANING OF PROOF IN MATHEMATICS EDUCATION

David A Reid, *Acadia University, Canada*

Abstract: *The issue of what mathematics education researchers mean by "proof" and "proving" has been the topic of three recent papers. The discussion in those papers are analysed in terms of a common terminology for identifying characteristics of the meanings of proof current in research.*

> Die Bedeutung eines Wortes ist sein Gebrauch in der Sprache (The meaning of a word is its use in the language). (Wittgenstein, 1963, §43.).

Perhaps it is a sign of the maturity of research into the teaching and learning of proof and proving that we are beginning to reflect on what it is we are researching, and whether, as a community, we are successful in communicating our work to each other. Three papers have been presented in the past seven years exploring the different meaning for "proof" and "proving" used in different research communities within mathematics education. In this paper I will continue this exploration and attempt to create a common terminology through which these meanings can be interpreted.

At PME 21 in Finland, Godino and Recio (1997) described some of the meanings proof has in the domains of research in mathematical foundations, mathematics, the sciences, and in classrooms.

At PME 25 in Utrecht I identified four usages of "proof" in mathematics education research and one from everyday life (Reid 2001). I classified these under the headings "the concept of proof", "proofs", "proving" and "probing".

At the Taipei International conference on "Mathematics: Understanding Proving and Proving to Understand" Nicholas Balacheff (2002/2004) described seven different perspectives on proof from which research has been done, which he characterised as being based on different researcher epistemologies.

In this paper I will revisit these three papers, as well as referring to the work of some other authors to illustration the different meanings of proof and proving. I will discuss the differing meanings of proof and proving in the research literature in terms of several different dimensions along which their meanings can differ. These dimensions include the concept of proof, the purpose of teaching proof, the kinds of reasoning proving is seen to involve, the needs that proving is seen to address and the relationship seen between proof and language. These dimensions are not explicitly mentioned in the three papers, but rather I have identified them through comparison of the distinctions made in them. Other dimensions, or a different set of dimensions, might also be useful for characterising the different meanings of proof and proving in use in the mathematics education research literature, but my reading of these three papers did not suggest them. For example, the word "rigour" is often used in discussions of proof, but it is only used in one of these three papers and so it did not provide a useful dimension for characterising the distinctions being made in all three.

The concept of proof

Godino and Recio point out that what counts as proof is different in formal logic, mainstream mathematics, science and schools. This is not really terribly surprising, as each of these is a different "domain of explanation", a phrase Maturana (1988) coined to describe communities that have different criteria for accepting an argument as an explanation.

> … the observer accepts or rejects a reformulation … as an explanation according to whether or not it satisfies an implicit or explicit criterion of acceptability …. If the criterion of acceptability applies, the reformulation … is accepted and becomes an explanation, the emotion or mood of the observer shifts from doubt to contentment, and he or she stops asking over and over again the same question. As a result, each … criterion for accepting explanatory reformulations … defines a domain of explanations, …. (1988, p. 28)

The concept of proof that Godino and Recio claim operates in the domain of formal logic is believed by some researchers in mathematics education to operate in mainstream mathematics as well (as I pointed out in my 2001 paper). It is the belief that, in the words of Fischbein & Kedem (1982) "a formal proof of a mathematical statement confers on it the attribute of a priori universal validity." This concept of proof is very old, and here I will call it the "traditional" concept of proof.

Another concept was ascribed to the mathematical domain of explanation by Godino and Recio. It has also been proposed by mathematicians, sociologists of mathematics, and philosophers of mathematics (e.g., Davis 1972, Lakatos 1976, Tymoczko 1986 , Crowe 1988). They point out that proofs are produced by fallible mathematicians and so cannot establish absolute truth. Instead they see proof as part of a quasi-empirical process, in which proofs operate to clarify and make the detection of errors easier (but never complete).

Whether a researcher has a traditional or quasi-empirical concept of proof can have a significant impact on the research done and the results reported. For example, Fischbein and Kedem (1982) concluded that high school students did not understand proof, because the students did not have the traditional concept of proof. Researchers operating from a quasi-empirical concept may have reached a different conclusion, or more likely, would have done the study differently.

The purpose of teaching proof

Balacheff's first example of a researcher's epistemology is that of Fawcett (1938). Fawcett undertook a teaching experiment motivated by the belief that the study of proof in mathematics could develop his students' ability to think critically in other domains of explanation. In his introduction he makes it quite clear that the only justification he sees for teaching geometry in upper secondary school is the transfer of the method of proof learned in that context to other contexts. This can be contrasted with the purpose for teaching proof that can be inferred from the quotation Balacheff selects from the work of Healy and Hoyles (1998): "Proof … exemplifies the distinction between mathematics and the empirical sciences." This suggests a belief that the criterion of acceptability of explanation that defines mathematics as a domain of explanation cannot be transferred to other domains, as Fawcett claimed. Instead the motivation for teaching proof is a better understanding of the nature of mathematics itself, not better reasoning in other domains. This dimension of the meaning of proof can be labelled Critical Thinking versus Mathematics.

It is worth pointing out that this dimension is distinct from the dimension of the concept of proof. Fawcett is an example of the combination of the traditional concept of proof with the Critical Thinking purpose. Based on their emphasis on justification and the generality of a valid proof, I believe Healy and Hoyles also work from the traditional concept of proof, but as noted above their purpose for teaching proof is Mathematics. In some of my work (Reid 1995) I advocated a quasi-empirical concept of proof, but with a critical thinking purpose, suggesting that showing students the limits of mathematics would make them more critical of the use of numerical arguments in other domains.

Reasoning

Balacheff tries to contrast Fawcett's "epistemology" with that of Harel and Sowder (1998), an attempt that fails, I believe, because Fawcett is most definitely working from a traditional concept of proof, and Harel and Sowder explicitly state that the "proof schemes" they describe are not to be confused with mathematical proof at all. At least part of this confusion stems from their use of "prove" in the sense it has in everyday English, which is different from the mathematical use of the word, so perhaps Harel and Sowder could be said to be using the "everyday" concept of proof. One might expect that this confusion is lessened in Romance languages in which word like "démonstration" and "demostración" exist to refer specifically to "mathematical proof", but as the everyday words "preuve" and "*prueba*" are also used to refer "mathematical proof" confusion can still occur. (The possibility for confusion becomes almost absurd when comparing work written by native speakers of English and romace languages. For example, Sutherland, Olivero & Weeden (2004) describe the work of a teacher who stressed the difference between proof and demonstration in her teaching. Unfortunately, while one might think it reasonable to translate "proof" as "preuve/prueba" and "demonstration" as "démonstration/ demostración", the current usage of these words in the UK schools is such that "proof" should be translated as "démonstration/demostración" and "demonstration" as "preuve/prueba".

Harel and Sowder's work does, however, serve to illustrate another important dimension along which meanings of proof can differ: the kind of reasoning employed in proving. Their "proof schemes" include reference to authority and empirical reasoning as well as deductive reasoning, each proof scheme being, in fact, a specific kind of reasoning. Other researchers have suggested other kinds of reasoning (abductive, analogy) and sub-types (generic example) that will be familiar to most readers.

There is another meaning of proving in which it can involve several types of reasoning at the same time. Godino and Recio point out that in science proving involves a mixture of kinds of reasoning and the larger patterns of proving processes described in Knipping (2003) and Reid (2002) also involve other kinds of reasoning combined with deductive reasoning.

The majority of research on proof, however, uses "proving" in ways that assume it must involve deductive reasoning. Some, in fact, define proving in terms of the kind of reasoning involved and restrict it to deductive reasoning. For example, ten years ago I defined proving as "investigating using deductive reasoning" (Reid 1995 p. 7), and the NCTM Principles and Standards describes proofs as: "arguments consisting of logically rigorous deductions of conclusions from hypotheses"(NCTM 2000 p.56).

Thus, in the mathematics education research literature, one can observe three different points on the reasoning dimension of the meaning of proof: Several distinct kinds of reasoning are involved in proving; or several kinds of reasoning in combination are involved in proving; or deductive reasoning alone constitutes proving. This dimension is distinct from the dimensions of concept and purpose, but it seems reasonable to suppose that researcher with a traditional concept of proof would be more likely to see proof as purely deductive, while those with a quasi-empirical concept of proof might be more inclined to view proving as a combination of kinds of reasoning.

Needs

One of the most astonishing claims in the paper by Godino and Recio is their assertion that "mathematics students must [use proof to] convince themselves, and convince their teacher of the necessary and universal truth of theorems" (p. 2-318). It is not clear how a student can convince a teacher of something that the teacher either already believes (if the teacher has a traditional concept of proof) or cannot believe (if the teacher has a quasi-empirical concept of proof). And as to convincing themselves, Harel and Sowder's work suggests that students are unlikely to use mathematical proof to convince themselves of anything. This raises the question, when students engage in proving, what need are they trying to address?

Some researchers, notably de Villiers (1990) and Hanna (2000), have listed a range of functions for proof, or needs that proof fulfils for individuals, including verification, exploration, explanation, systematisation, communication and social acceptance. Most researchers in mathematics education take one of four approaches to considering what need motivates proving. Many, especially those who have the traditional concept of proof, assume implicitly or explicitly that the need of verification is the only one that motivates proving. Others (e.g., Thurston 1995, Mariotti 1997) describe the needs that motivate research mathematicians to prove and imply that the same needs *should* motivate students in schools to prove. Still others (e.g., Hanna 1989) describe the specific context of schools and suggest a need for proving appropriate to the school context, not necessarily a need that motivates research mathematicians. Finally, there are researchers (e.g., Reid 1995) who examine the mathematical activity of students and who identify the needs that in the existing school contexts do motivate students to prove.

The implications of differing beliefs about the needs that motivate proving for research and teaching are clear. Teachers who begin with an assumption that proving is about verification will create situations of doubt to motivate proving, while those who see explanation as motivating proving will create situations of wonder. Researchers who associate proof with verification (e.g., Harel & Sowder, 1998) will focus on the kinds of reasoning students use to verify, while those who see proving as motivated by many different needs will focus on other issues.

It is unlikely that all the possible combinations of the dimensions of the concept of proof and the need for proving can coexist. For example, a researcher who has a traditional concept of proof is likely to suggest that students' needs to verify or systematise could or should motivate them to prove. On the other hand, a researcher who has a quasi-empirical concept of proof might stress the needs to explore, to communicate, to gain social acceptance or to explain.

Proof and language

One of the distinctions Balacheff makes concerns the relationship between language and proof. He claims that for Duval (1991) proofs must rely only on syntactical elements, while others would claim that proofs rely on semantic or social elements for acceptance (Balacheff's examples are Pimm, 1987, and Burton and Morgan, 2000). Godino and Recio suggest a distinction that is useful for clarifying Balacheff's. They describe analytical and substantive arguments (from Krummheuer 1995, following Toulmin). Analytical arguments are based on specified, deductive rules of inference and explicit axioms; as such they are logical tautologies. Substantive arguments expand the meaning of what is argued. Godino and Recio use these categories to distinguish proofs in formal logic from other proofs, but Balacheff's distinction is between epistemologies of researchers in mathematics education.

It is evident that a researcher whose concept of proof is quasi-empirical is unlikely to mean by "proof" something analytical, however the substantive position seems compatible with both concepts.

The meaning of proof

Maturana (1988) emphasises that the criterion of acceptance used in a domain of explanation is often (necessarily) implicit. This may be part of the difficulty we have, as researchers in mathematics education, in being clear about what we are talking about when we refer to "proof" and "proving". That there are differences is apparent from the examples presented by Balacheff and in my 2001 paper, but it is only by considering many such examples that patterns of difference become evident. I speculate that it is possible to place each researcher in mathematics education somewhere in the space defined by the dimensions I have outlined above, but it is impossible to do this working only from existing publications, as we, as a community, do not make our positions clear. Perhaps the work of Balacheff, Godino and Recio, and my papers, will make it easier for us to communicate our positions, and thus provide a foundation for the kind of interaction needed to strengthen a community.

One reason we have difficulty in being clear is that the category "proof", like most categories, is not well defined, and probably cannot be. As Balacheff asserts, "almost all researchers will agree on a more or less formal definition of mathematical proof" (p. 1) but that definition, as Godino and Recio note, is useful only when working in the domains of logic and the foundations of mathematics. Most research mathematics and teacher of mathematics must contend with an informal and implicit definition, based on experiences they have had with prototypical examples of proofs.

The linguist Eleanor Rosch (Varela, Thomson & Rosch, 1991) has analysed the way categories are described in language, and has concluded that we do not have a definition in our heads for the words we use to classify our experiences. Instead we have what she calls "prototypical" examples. For example, if I ask you to think of a bird, you are likely to think of a common bird that lives where you live. You are unlikely to think of a penguin or an ostrich. For birds which resemble our prototype we recognise them as birds without making any recourse to a definition. For birds far from our prototype we rely on expert opinions to establish the borders of our categories. The experts tell us that penguins are birds, even though they do not fly and behave very much like seals, because they lay eggs. But on the other hand, they tell us that platypuses are not birds, even though they lay eggs and behave in many ways like ducks.

An experiment

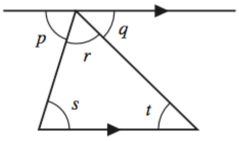
At CERME 4 the working group of proof brought together fifteen researchers with an interest in proof, from ten countries. I asked the fourteen others to form seven pairs, and to select the most exemplary proofs from a set of six I provided.

I have collected about fifty proofs, mostly from research papers on the topic (the full set is available on request). I distributed 40 of these (plus two duplicates) to the pairs. After choosing not more than three proofs as more exemplary than the others, they were to join another pair, and again select the most exemplary and discard the least exemplary. My plan was that either we would arrive at a consensus of a few very good examples of what a proof is (demonstrating that we share a common prototype for the category) or we would fail to reach a consensus (demonstrating that there are serious differences within the mathematics education community concerning what "proof" means). What actually occurred was a lot of interesting discussion within the pairs and groups of four, but the task of comparing and choosing exemplary proofs was so time consuming that the original plan had to be abandoned.

Nonetheless, some results can be reported. Three pairs (coded A, B, C) made two rounds of selections, and in the end they had selected three exemplary proofs from the 18 proofs given them (which included one duplicate). They selected #16, a visual "dot" proof that the sum of two even numbers is even, #19/44 a proof that the angle sum of a triangle is 180°, and #23, a "Behold!" proof of the Pythagorean theorem. #19/44 was the duplicate, a proof copied from one of the Hoyles-Healy-Kuchemann questionnaires (see Figure 1).

Proof that the sum of the interior angles of a triangle is 180 degrees

I drew a line parallel to the base of the triangle.



Statements	*Reasons*
$p = s$	Alternate angles between two parallel lines are equal
$q = t$	Alternate angles between two parallel lines are equal
$p + q + r = 180°$...	Angles on a straight line
$\therefore\ s + t + r = 180°$.	

Figure 1: A exemplary proof, #19/44, from Hoyles et al.

Prove: The sum of the first n positive integers is $n(n + 1)/2$.

Let $S(n) = 1 + 2 + 3 + \cdots + n$.
Then $S(n) = n + (n - 1) + (n - 2) + \cdots + 1$.

Taking the sum of these two rows,

$$2S(n) = (1 + n) + [2 + (n - 1)] + [3 + (n - 2)] + \cdots + (n + 1)$$
$$= (n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1)$$
$$= n(n + 1).$$

Therefore, $S(n) = n(n + 1)/2$.

Figure 2: An exemplary proof, #10, cfrom Hanna (1989)

This proof was judged to be a better example of a proof than 13 others, via direct or indirect comparisons (the indirect comparisons occurred when it was judged to be better than a proof that had previously been preferred to others). Of course, as it wasn't compared to most of the proofs, or seen by most of the participants, it may not really be the best example. All three of the proofs chosen by this group use diagrams. In the second round they eliminated two other proofs, also using diagrams, of the Pythagorean theorem (#43) and the commutativity of multiplication (#30). In the first round the three groups A, B, C eliminated 12 proofs, 2 with diagrams and 10 without.

Pair G selected three proofs that have an algebraic element (#10, see Figure 2, #13 & #24, of the sum of the first *n* integers, the sum of two even numbers, and the Pythagorean theorem). The other formats available were an action proof (from Tall 1995), a narrative proof from Euclid (Book IX, Prop 20, the infinitude of primes) and an empirical argument. Their selection seems to have been influenced by the use of algebra, at least more than by the reputation of the author (Euclid) or the accessibility (action) of the proofs.

Pair E selected three narrative proofs in the domain of number theory (#8, #27, & #28, of the sum of the first *n* integers, the irrationality of $\sqrt{5}$ and the infinitude of primes). #8 also includes a visual generic example. They rejected a textbook geometry proof of an insignificant theorem, a proof from Euclid (Book I, Prop 1) and a non-standard proof of the angle sum of a triangle (based on tessellations, from Hoyles et al.). It may be that the narrative element in the proofs they chose is coincidence, but their preference for number theory over geometry seems clear.

When a similar experiment was done at Universität Oldenburg with only ten proofs, #8 and #10 (which are included in the choices of groups G and E) were selected as highly exemplary.

Note that the nine proofs chosen by these five pairs are proofs of only six theorems, all of them well known. This is partly a result of the set of proofs they had to choose from, as half the proofs given were proofs of these six theorems, but the fact that these and only these were selected suggests that the mathematical significance or familiarity of the theorem proven may make a proof more exemplary.

Pair D combined with either Pair E or Pair G in the second round (I have no method of determining which pair they joined), at which point the two proofs they had selected in the first round were eliminated. One of their selections, #11, is similar in structure to #8 and #10, but uses a numerical generic example. The other, #31, is a diagram based proof of an algebraic identity. Neither of their choices include algebra or a narrative element, and this might have been a factor in their elimination when Pair D joined another pair.

Pair F spent a great deal of time discussing the proofs they were given, and by the end of the allotted time they had eliminated one, a formal proof from a logic text.

What does all this mean? There seems to be some common ground about what a proof is, at least in the context of this working group. But there seems also to be some differences. There is more work to be done before we have a clear idea of what those differences are and what significance they have for the research we do. But if we can acknowledge that there is an issue here, and discuss the characteristics of proof, we may be able to come to, if not agreement, then at least agreement on how we differ.

References

Balacheff N. (2002/2004) The researcher epistemology: a deadlock from educational research on proof. Fou Lai Lin (ed.) 2002 *International Conference on Mathematics - "Understanding proving and proving to understand"*. Taipei: NSC and NTNU (pp. 23-44). Reprinted in *Les cahiers du laboratoire Leibniz*, no 109, August 2004, Online at http://www-leibniz.imag.fr/NEWLEIBNIZ/LesCahiers/Cahier109/ResumCahier109.html

Burton L., Morgan C. (2000) Mathematicians writing. *Journal for Research in Mathematics Education* 31(4) 429-452.

Crowe, M. (1988). Ten misconceptions about mathematics and its history. In W. Aspray & P. Kitcher (Eds.) *History and philosophy of modern mathematics, Minnesota Studies in the philosophy of science, Volume XI,* (pp. 260-277). Minneapolis: University of Minnesota Press.

Davis, P. (1972). Fidelity in mathematical discourse: Is one and one really two? *American Mathematical Monthly, 79*, 252-263. (Reprinted in Tymoczko, 1986).

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras,* n. 24, pp. 17-24.

Duval R. (1991) Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics 22*(3) 233-263.

Fawcett, H. P. (1938). *The nature of proof: a description and evaluation of certain procedures used in a senior high school to develop an understanding of the nature of proof.* (NCTM yearbook 1938). New York: Teachers' College, Columbia University.

Fischbein, E. & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), *Proceedings of the Sixth Annual Conference of the International Group for the Psychology of Mathematics Education,* (pp. 128-131). Antwerp.

Godino J. D., Recio A. M. (1997). Meaning of proofs in mathematics education In Pekhonen, Erkki (Ed.), *Proceedings of the Twenty-first Conference of the International Group for the Psychology of Mathematics Education.* (Vol. 2 pp. 313-320). Lahti, Finland.

Hanna, G. (1989). Proofs that prove and proofs that explain. In *Proceedings of the Thirteenth International Conference on the Psychology of Mathematics Education,* (pp. 45-51). Paris.as

Hanna G. (2000) : Proof, Explanation and Exploration: An Overview. *Educational Studies in Mathematics*, *44*(1&2), Keith Jones, Ángel Gutiérrez Maria Alessandra Mariotti (eds.) 5-23.

Harel G., Sowder L (1998) Students' proof schemes: Results from exploratory studies. In: Schonfeld A., Kaput J., and E. Dubinsky E. (eds.) *Research in collegiate mathematics education III*. (Issues in Mathematics Education, Volume 7, pp. 234-282). American Mathematical Society.

Healy L. & Hoyles C. (1998) *Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students* *in the UK*. Research Report. Mathematical Sciences, Institute of Education, University of London.

Hoyles C., Healy L. & Kuchemann, D. (1995-2003) Justifying and Proving in School Mathematics Project and Longitudinal Proof Project. Description and list of publications online at: http://www.ioe.ac.uk/proof/index.html.

Knipping, C. (2003). *Beweisprozesse in der Unterrichtspraxis – Vergleichende Analysen von Mathematikunterricht in Deutschland und Frankreich*. Hildesheim: Franzbecker.

Krummheuer G. (1995) The ethnography of argumentation. In: Cobb. P., Bauersfeld H. (eds.) *The emergence of mathematical meaning: interaction in the classroom culture*. Hillsdale, NJ: Erlbaum.

Lakatos, I. (1976). *Proofs and Refutations*. Princeton: Princeton University Press.

Mariotti, M.A., (1997) Justifying and Proving in Geometry: the mediation of a microworld, Revised and extended version of the version published in: Hejny M., Novotna J. (eds.) *Proceedings of the European Conference on Mathematical Education* (pp.21-26). Prague: Prometheus Publishing House. Online at http://www.lettredelapreuve.it/resumes/mariotti/mariotti97a/mariotti97a.html.

Maturana, H. (1988). Reality: The search for objectivity or the quest for a compelling argument. *The Irish Journal of Psychology, 9*(1) 25-82.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston VA: Author.

Pimm D. (1987) *Speaking mathematically. Communication in the mathematics classroom*. London: Routledge and Kegan Paul.

Reid, D. (2002) Conjectures and Refutations in Grade 5 Mathematics. *Journal for Research in Mathematics Education, 33*(1) 5-29

Reid, D. (2001) Proof, Proofs, Proving and Probing: Research Related to Proof. Paper based on a Short Oral Presentation at the Twentieth-Fifth Annual Conference of the International Group for the Psychology of Mathematics Education, Utrecht, Netherlands. Online at http://ace.acadiau.ca/~dreid/publications/proof/proof.htm

Reid, D. (1995). *The need to prove*. Unpublished doctoral dissertation, University of Alberta, Department of Secondary Education.

Sutherland, R., Olivero, F. & Weeden, M. (2004) Orchestrating mathematical proof through the use of digital tools. In Marit Johnsen Høines &Anne Berit Fuglestad (Ed.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4 pp. 265–272). Bergen, Norway.

Tall D. (1995) Cognitive Development, Representations & Proof. *Justifying and Proving in School Mathematics,* Institute of Education, London, (pp. 27-38).

Thurston, W. (1995). On proof and progress in mathematics. For the Learning of Mathematics, 15(1), 29-37.

Tymoczko, T. (Ed.), (1986). *New directions in the philosophy of mathematics: An anthology*. Boston: Birkhäuser.

Varela, F., Thompson, E., & Rosch, E. (1991), *The Embodied Mind*. Cambridge, MA: MIT Press.

Wittgenstein, L. (1963). *Philosophical Investigations*. New York: The Macmillan Company.

ABOUT A CONSTRUCTIVIST APPROACH FOR STIMULATING STUDENTS' THINKING TO PRODUCE CONJECTURES AND THEIR PROVING IN ACTIVE LEARNING OF GEOMETRY

Oleksiy Yevdokimov, *Kharkov State Pedagogical University, Ukraine*

Abstract: *The paper describes processes that might lead secondary school students to produce conjectures in a plane geometry. It highlights relationship between conjecturing and proving. The author attempts to construct a teaching-learning environment proposing activities of observation and exploration of key concepts in geometry favouring the production of conjectures and providing motivation for the successive phase of validation, through refutations and proofs. Supporting didactic materials are built up in a way to introduce production of conjectures as a meaningful activity to students.*

Background and theoretical framework

Conjecturing and proving activities have traditionally been given significant attention throughout the Grades in the schools. In the most of the countries current mathematics education standards contain a special section on reasoning and proof, investigations, conjectures, evaluation of arguments and the use of various methods of proofs (e.g. NCTM Standards, 2000; MIUR, 2004; UMES, 2003). Therefore, different questions concerning argumentation and proof are constantly in the focus of researchers and mathematics educators' interest.

During the last years a considerable amount of research, concerning the processes of conjectures production and construction of proofs was connected with interactive learning environment and different software (Harel and Papert, 1990; Davis, 1991), in particular, dynamic geometry software (Arzarello et al, 1998; Furinghetti and Paola, 2003; Mariotti, 2000). Neither considering a technology aspect within the theme of argumentation and proof in full nor restricting the use of technology in that direction, we would like to put into consideration a constructivist didactical approach, which, on the one hand, can be successfully used in stimulation students' active research interest while learning mathematics, in particular, geometry, on the other hand, provides possibility for investigation students' abilities in their development to produce conjectures.

Within the long-term research project on active learning of mathematics in a constructivist framework for secondary school students the author gave a course on plane geometry for 45 students (8^{th} Grade, 14-15 years old) during the year. All students didn't take any courses before, which were aimed on formation of skills to produce conjectures specially. Teaching programme of the course consisted of 16 theoretical modules with support of the didactical materials for each of them.

The aim of the paper is to describe a didactical approach to organise students' thinking leading to conjectures production on the base of materials of this long-term research project, which were worked out and used as support for active learning of geometry in a constructivist framework. These materials were built up in a way to introduce conjectures production as a meaningful activity to students.

Our theoretical position is grounded in the theory of active learning processes in mathematics (Hiebert, 1992; Wang, Haertel and Walberg, 1993). We took into account that current learning perspectives incorporate three important assumptions (Anthony, 1996):

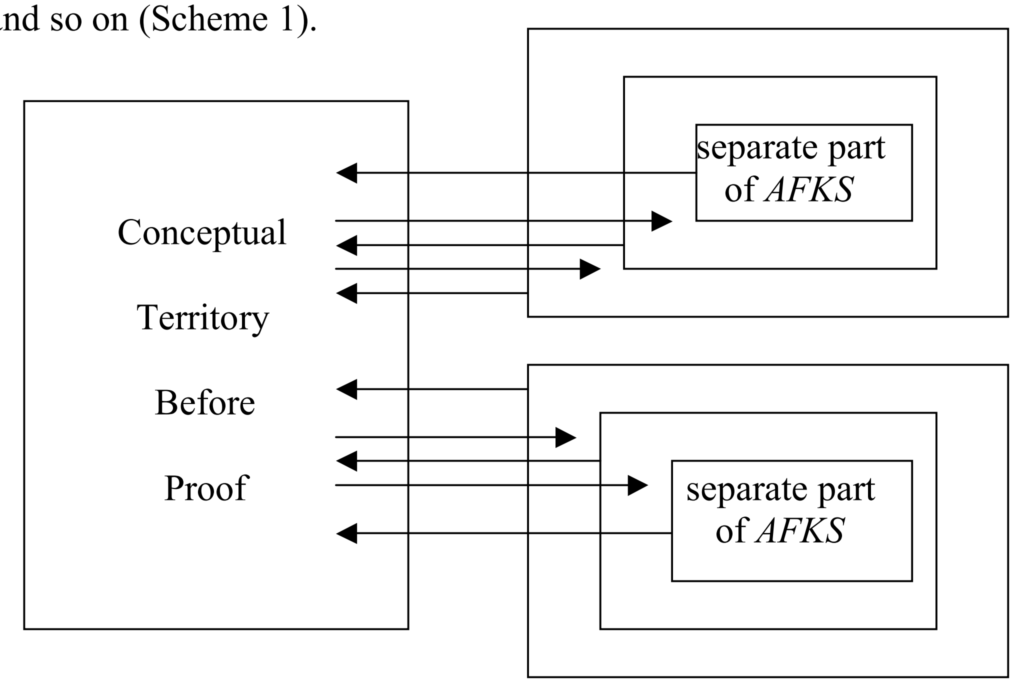
- learning is a process of knowledge construction, not of knowledge recording or absorption;
- learning is knowledge-dependent; people use current knowledge to construct new knowledge;
- the learner is aware of the processes of cognition and can control and regulate them.

From a constructivist perspective it is easier for a student, under appropriate arrangement of teaching, to act as an architect, to reveal the truth and construct new knowledge, than to learn ready-made knowledge without understanding its origin, meaning and interrelations (Davis, 1991). In other words, *"learning is a process of construction in which the students themselves have to be the primary actors"* (von Glasersfeld, 1995). Thus, the view of the learner has changed from that of a passive recipient of knowledge to that of an active constructor of knowledge.

We would like to consider conjecturing process from the didactical point of view, i.e. to show how it can be constructed similarly to the process of mathematical research and how this kind of teaching contributes to the development of students' mathematical thinking. de Villiers (1996) and Hanna (2000) noted that in actual mathematical research mathematicians have to first convince themselves that a mathematical statement is true and then move to a formal proof. At the same time conjecturing with verification, exploration and explanation constitute the necessary elements that precede formal proof (*"conceptual territory before proof"*, Edwards, 1997). Yevdokimov (2003) proposed to consider an *Active Fund of Knowledge of a Student* (*AFKS*) in the given area of mathematics. As *AFKS* we called student's understanding of definitions and properties for some mathematical objects of that domain and skills to use that knowledge. The key question in the context of conjecturing and proving was the following: where, when and for what of mathematical objects a student would apply a certain mathematical property for proving and whether it would be necessary to apply that property generally in that case.

Modifying this idea, we distinguished the separate (smaller) parts of *AFKS*, which deal with a certain mathematical object, for consideration and, after that, motivated student's using that separate parts of *AFKS* altogether as a whole *AFKS* for production new conjectures and constructing their proofs, even for other mathematical objects. Therefore, on the first stage of our research we attracted students' attention and focused exploratory work on using and developing the separate parts of their *AFKS* as much as possible. On the second stage, we started to increase *AFKS* properly, stimulating students for active mental work with using all separate parts of *AFKS* simultaneously.

We would like to emphasize a dual correspondence between separate parts of *AFKS* (personal attribute) and "conceptual territory before proof" (impersonal attribute). More precisely, parts of *AFKS* act in "conceptual territory before proof" and vice versa, Edwards' term contributes to increasing every part of *AFKS*, which acts in it again and so on (Scheme 1).



Scheme 1.

Thus, on the first stage of the research within the every module the main task for developing and increasing the separate parts of *AFKS*, was the following:

"Find out as many properties of a certain mathematical object as you can".

On the second stage of the research within the every module we proposed for students the following task:

"Find out properties of something using your previous findings for certain mathematical objects".

Now we need to clarify what it means: to find out or reveal a property. On the one hand, any property gives relationship between conjecturing and proving, since for obtaining a property we have to produce conjecture and then prove it. On the other hand, any property gives a clear structure from premise to result (Scheme 2). And such form of any property is a final product of students' findings for the task.

Premise ⇒ Result

Scheme 2.

At the same time, in generalising context a property can be considered in three more forms, which highlight importance of all-sided investigation in relationship between conjecturing and proving (Scheme 3).

Premise	⇒	?
?	⇒	Result
?	⇒	?

Scheme 3.

However, a mathematical conjecture does appear from nowhere, without links to its *"historical-mathematical neighbourhood"* (Yevdokimov, 2004). Conjecture is a result of applying *AFKS* to different mathematical objects within *"conceptual territory before proof"* taking into account possible generalisation, systematisation and analogue. Like Brown and Walter (1990) we propose to consider *"situation"*, an issue, which is a localised area of inquiry activities with features that can be taken as given or challenged and modified.

Turning to the forms, it is necessary to note that for finding any property students have to move through its forms of Scheme 3 for obtaining that property in the form of Scheme 2. Moreover, the first two forms of Scheme 3 refer to the first stage of the research and the last form refers to the second stage. Of course, it doesn't relate to all properties, which were known for students before. Here, we speak of the development of conjecturing process in the student's mind only.

For illustration the schemes above we would like to demonstrate one well-known property of bisectors of a triangle in all forms (property B_1 from Appendix). It is shown in the following table.

<table>
<tr><td colspan="2">Property</td><td colspan="3">All bisectors of a triangle intersect in one and the same point</td></tr>
<tr><td colspan="2">Scheme 2</td><td>We have a triangle and three its bisectors</td><td>⇒</td><td>Bisectors intersect in one and the same point</td></tr>
<tr><td rowspan="3">Scheme 3</td><td rowspan="2">Property in the generalising context</td><td>We have a triangle and three its bisectors</td><td>⇒</td><td>?</td></tr>
<tr><td>?</td><td>⇒</td><td>We have a point inside a triangle</td></tr>
<tr><td>"Situation": a triangle and its components</td><td>?</td><td>⇒</td><td>?</td></tr>
</table>

Table 1.

Description of students' activities and data collection

Taking to attention the scope of the paper we would like to describe and characterise students' conjecturing and proving activities for a part of the module concerning a triangle and some of its properties and present a sample of the didactical material (see Appendix) for this part.

We proposed three tasks in succession for students' work on their own. Sufficient period of time was given for every task. Our first task was the following:

Task 1.

"Find out as many properties of a bisector of a triangle as you can".

We would like to note that some bisector's properties were connected with height and median of a triangle. Therefore, in the first part students had been asked to explore the similar tasks for height and median too:

Task 2.

"Find out as many properties of a height of a triangle as you can".

Task 3.

"Find out as many properties of a median of a triangle as you can".

As we mentioned above such tasks were aimed on developing and increasing separate parts of AFKS ($AFKS_{bisector,}$, $AFKS_{height,}$, $AFKS_{median}$ correspondingly).

It is necessary to note that some properties were known for students before, therefore, at first, we asked students to point out all known properties. After that, students could begin their inquiry work to produce conjectures and prove them. At the same time students had been asked to prove every property without any dependence whether it was indicated as known before or proposed as a conjecture. If a student proposed at least 5 conjectures independently, at least 3 of them with proofs, and was not able to produce other conjectures, then he/she had access to didactical material with extended list of properties for bisector (see Appendix), height and median. After that students had been asked to prove those properties, which were unrevealed or unproved for them before. At these stages help of a teacher was an acceptable, but not necessary condition for students. Instead of teacher's help students could take advantage of extended didactical material (it is omitted in the paper), which contained not only properties themselves, but short remarks and instructions for proving every property. After getting acquaintance with proofs students took for consideration the next task.

This procedure was repeated in succession three times for every task. Results became better every time, it witnessed that most of the students increased their AFKS and developed their abilities to produce conjecture. Full data collection of students' progress in increasing their $AFKS_{bisector}$ in the first stage of the research is given in Table 2. The same tables were formed for $AFKS_{height,}$ and $AFKS_{median}$ correspondingly (they are omitted in the paper).

Property	B_1			B_2			B_3		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	37	6	41	39	6	43	24	20	42
Property	B_4			B_5			B_6		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	39	2	40	5	38	43	32	10	34
Property	B_7			B_8			B_9		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	2	7	32	3	29	30	2	-	30
Property	B_{10}			B_{11}			B_{12}		
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given
Number of students	3	-	34	-	28	19	2	4	19
Property	B_{13}			B_{14}					
Status of the property	was known before	proposed as a conjecture	proof was given	was known before	proposed as a conjecture	proof was given			
Number of students	4	2	18	1	29	15			

Table 2.

Summary data collection of students' progress in increasing of their $AFKS_{bisector}$, $AFKS_{height,}$ and $AFKS_{median}$ in the first stage of the research is given in Table 3.

We would like to remark that didactical materials contained 14 properties for bisector, 16 ones for height and 12 properties for median correspondingly.

Properties	B	H	M	B, H, M altogether
Number of students, who proposed at least 12 conjectures	2	5	1	1
Number of students, who proposed at least 11 conjectures	3	6	1	1
Number of students, who proposed at least 10 conjectures	5	8	2	2
Number of students, who proposed at least 9 conjectures	7	9	7	6
Number of students, who proposed at least 8 conjectures	13	14	14	9
Number of students, who proposed at least 7 conjectures	14	16	17	13
Number of students, who proposed at least 6 conjectures	17	20	21	17
Number of students, who proposed at least 5 conjectures	29	30	31	28
Number of students, who proposed at least 4 conjectures	35	39	38	35
Number of students, who proposed at least 3 conjectures	38	41	41	36
Number of students, who proposed at least 2 conjectures	41	42	43	40

Table 3.

On the second stage we proposed the following task for students' work on their own:

"Find out properties of something using your previous findings and exercises for bisector, height and median".

As we mentioned above such task was aimed on developing and increasing of AFKS properly, using $AFKS_{bisector,}$, $AFKS_{height,}$ and $AFKS_{median}$ simultaneously.

Shortly, we would like to attract attention for students' strategy to produce conjectures, which was prepared in our teaching-learning environment and used by students on their own with full understanding and not by chance.

On the base of B_2 (see Appendix) 21 students proposed to construct different geometrical objects in a triangle. Thus, in that way 15 students came to the concept of symmedian of a triangle. After having a geometrical object, where content of their AFKS could apply to, 14 students produced a conjecture of existing a point of intersection for three symmedians of a triangle. As a result Lemoine point of a triangle was revealed. Property B_1 significantly contributed for students' inquiry work in that direction. At the same time property B_4 led to the students' conjecturing work (5 students did it successfully) for finding the following property:

"Symmedians of a triangle divide the opposite sides of it into the parts, which are proportional to the squares of the corresponding adjoining sides of this triangle".

It is important to note that other properties of Lemoine point were conjectured and proved by students in the similar way in our teaching-learning environment.

Final remarks

At the end we would like to emphasize peculiarities of students' conjecturing and proving activities in our research. At first, they do know direction of their work, i.e. a mathematical object for which they have been asked to produce conjectures, but the ways for achieving this aim are not indicated for students. At the same time initial information and further ideas can be taken from that properties, which were pointed out by students right away as known for them before. At last, directions of their further work are not indicated for students, but most of them are able to imagine how possible properties would look like.

Our results, although local, support hypothesis that most of the students can be successfully involved in the conjecturing and proving activities in different levels, if learning has active and constructive nature. In this work we have shown effectiveness of a constructivist approach to organise students' thinking leading to production of conjectures. We found out that a specific learning-teaching environment can significantly contribute to students' progress in learning geometry.

References

Anthony, G. (1996). Active learning in a constructivist framework. *Educational Studies in Mathematics*, 31, 349-369.

Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O. (1998). Dragging in Cabri and modalities of transition from conjectures to proofs in geometry. In: A.Olivier & K.Newstead (Eds) *Proceedings of PME22*, Stellenbosch, South Africa, Vol.2, 32-39.

Brown, S. I. and Walter, M. I. (1990). *Problem Posing*. Lawrence Erlbaum: Hillsdale NJ.

Davis, R. (1991). Constructivist views on the teaching and learning of mathematics. *Journal for Research in Mathematics Education*, Monograph 4, National Council of Teachers of Mathematics.

de Villiers, M. (1996). *Some adventures in Euclidean geometry.* Durban: University of Durban-Westville.

Edwards, L. (1997). Exploring the territory before proof: Students' generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning*, 2, 187-215.

Furinghetti, F. and Paola, D. (2003). To produce conjectures and to prove them within a dynamic geometry environment: a case study. In N.A.Pateman, B.J.Dougherty and J.T.Zilliox (Eds) *Proceedings of the 2003 Joint Meeting of PME and PMENA,* USA, Vol.2, 397-404.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44 (1), 5-23.

Harel, I. and Papert, S. (1990). Software Design as a Learning Environment. *Interactive Learning Environments*, 1(1), 1-33.

Hiebert, J. (1992). Reflection and communication: Cognitive considerations in school mathematics reform. *International Journal of Educational Research*, *17*, 439-456.

Italian Ministry of Education. (2004). *New mathematical standards for the school from 5 through 18 years.* MIUR.

Mariotti, M. A. (2000). Introduction to proof: the mediation of a dynamic software environment. *Educational Studies in Mathematics*, 44 (1), 25-53.

National Council of Teachers of Mathematics. (2000). *Principle and standards for school mathematics.* Reston, VA: NCTM.

Ukrainian Ministry of Education and Science. (2003). *Programmes for schools. Mathematics*. Kiev, UMES.

Wang, M. C., Haertel, G. D. and Walberg, H. J. (1993). Toward a knowledge base for school learning. *Review of Educational Research, 63*, 249-294.

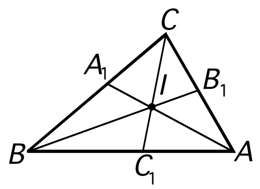
Yevdokimov, O. (2003). Intuitive proofs as a tool for development of student's creative abilities while solving and proving, In N.A.Pateman, B.J.Dougherty and J.T.Zilliox (Eds) *Proceedings of the 2003 Joint Meeting of PME and PMENA,* USA, Vol.1, 264.

Yevdokimov, O. (2004). Using materials from the history of mathematics in discovery-based learning, *Proceedings of the History and Pedagogy of Mathematics ICME-10 Satellite Meeting*, Sweden, 457-465.

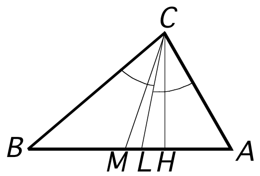
Appendix

Bisector's properties

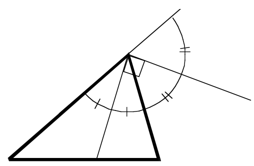
B$_1$. All bisectors of a triangle intersect in one and the same point, it is a center of the circumference, which is inscribed in that triangle.



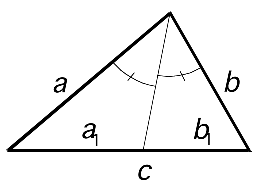
B$_2$. A bisector is between height and median from the same vertex of a triangle. In isosceles triangle bisector, height and median coincide.



B$_3$. Bisectors of interior and exterior angles of the same vertex of a triangle are perpendicular.



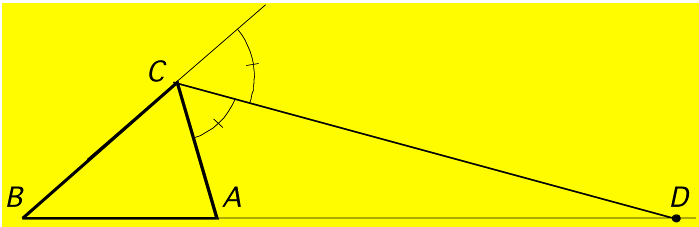
B$_4$. The bisectors of a triangle divide the opposite sides of it into the parts, which are proportional to the corresponding adjoining sides of this triangle. $\frac{a_1}{b_1} = \frac{a}{b}$.



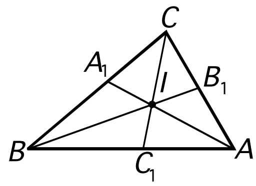
B$_5$. The bisector of the triangle with sides *a*, *b*, *c* divides the opposite side *c* on the segments $a_1 = \frac{ac}{a+b}$, $b_1 = \frac{bc}{a+b}$.

B$_6$. If a segment, which connects a vertex of a triangle with a point on the opposite side of that triangle, divides the opposite side into the parts, which are proportional to the corresponding adjoining sides of this triangle, then it is a bisector.

B$_7$. If *BD* is a bisector of exterior angle *C,* then $\frac{BD}{AD} = \frac{BC}{AC}$.



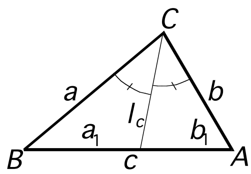
B₈. If bisectors of a triangle intersect in the point I, then it divides bisector CC_1 in the following relation $\frac{CI}{IC_1} = \frac{a+b}{c}$.



B₉. Length of bisector

1) $l_c = \dfrac{2ab\cos\frac{C}{2}}{a+b}$;

2) $l_c^2 = ab - a_1 b_1$.

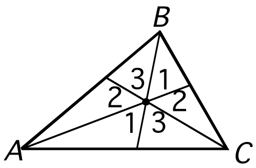


B₁₀. Angles between bisectors:

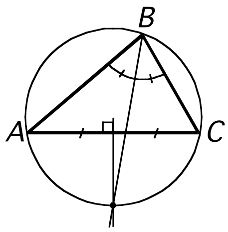
$\angle 1 = \frac{A+B}{2}$; $\angle 2 = \frac{A+C}{2}$; $\angle 3 = \frac{B+C}{2}$;

$\angle 1 + \angle 2 = 90° + A/2$;

$\angle 1 + \angle 3 = 90° + B/2$; $\angle 2 + \angle 3 = 90° + C/2$.

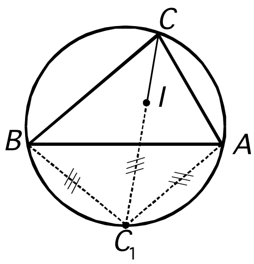


B₁₁. A bisector of a triangle and a mid-perpendicular to the opposite side of a triangle intersect in a point, which belongs to the circle described around this triangle

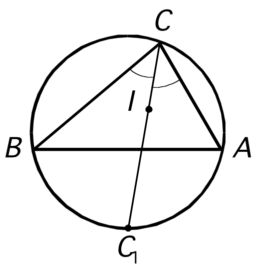


B₁₂. If point I is a centre of inscribed circumference into triangle ABC and point C_1 belongs to described circumference around this triangle and line CI simultaneously, then

$$C_1A = C_1B = C_1I = 2R\sin\frac{C}{2}.$$



B₁₃ If point I is a centre of inscribed circumference into triangle ABC and point C_1 belongs to described circumference around this triangle and line CI simultaneously, then $CI \cdot IC_1 = 2Rr$.



B₁₄. A bisector divides an angle between radius of described circumference around that triangle and its height from the same vertex of the triangle on two equal parts.