WORKING GROUP 6
Algebraic thinking

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WORKING GROUP 6. ALGEBRAIC THINKING

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Keywords: algebraic thinking, history, epistemology, epistemography, semiotics, linguistics, language, symbols.

INTRODUCTION

How to write a pretty boring introduction to a chapter made with various authors' contributions? That is easy: by pasting summaries of one contribution after another, shortened to the point where they lose their meaning. Therefore we, the Algebraic Thinking Working Group leaders, decided to avoid this tedious and pointless rewriting exercise, and instead wanted to present here the main outcomes which had sprung up during the group discussions[1].

Our first question was: on what can we *work* in a *Working* Group? After having read the contribution proposals, our first idea was to split the whole "algebraic thinking" theme into various domains related to the students' levels: Linear Algebra, Pre-Algebra, Elementary Algebra etc. However we felt that by doing it this way, we would miss the point: working together is not just about communicating (scientific) facts between sub-domain specialists, and much less about trying to convert others to one's own faith. Rather, it should be an exchange about the pros and the cons of the different frameworks used in order to interpret the problems participants face, and a way to promote in-depth scientific cooperation.

Moreover, many problems of miscomprehension tend to arise when discussing from different frameworks: sometimes different words are used to describe the same phenomenons, sometimes (more frequently) the same word (like "language" or "obstacle") is used with quite different meanings (which is worse). Therefore we decided that the working group sessions would be devoted to uncovering the possible misunderstandings about the various words or concepts the presenters were using. We then decided to organise presentations according to the predominance of two perspectives: the historical perspective and the semiotic perspective.

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[1] Everyone in the group discussion could *understand* French, most *spoke* and *understood* better French than English, but some could not *speak* French. The situation was very similar for Spanish, except for one member who did not understand Spanish. Therefore, we unanimously decided that, given the circumstances, everybody would speak in the idiom, French or English, with which he or she felt most comfortable. This pragmatic decision improved the fluidity of the discussion dramatically as well as the possibility to discuss subtle and deep ideas.

It had been clear from the beginning that our aim was not to study either the history or the semiotics of mathematics for themselves. Actually, no group member is a specialist in these research domains. Instead, we conceived of historic and epistemologic studies on the one hand and semiotic and linguistic studies on the other as tools for research in mathematics education.

It is interesting to note that the use of the words “semiotics,” “linguistics,” “history” or “epistemology” already raised thorny definition problems.

Regarding "semiotics" and "linguistics" and the world "language": there is an immense variety of ways to represent mathematical objects or facts, and these representation systems (including the way representations are interpreted and transmitted) are described by a science called “semiotics” (see Drouhard & Teppo, 2004). Charles S. Peirce can be considered the founder of semiotics, and Umberto Eco one of the prominent contemporary scientists in this field. Within this framework, the well-defined phrases “semiotic representations” and “semiotic representation systems” are used rather than the ambiguous ones “representations” and “ways to represent”.

However, some particular semiotic representation systems show specific additional characteristics: this is the case for “natural languages” like English or Spanish ("*the positive number which square is two*" is written in the natural language English) and for “symbolic languages” ("$\sqrt{2}$ " is written in the symbolic language of algebra). These special semiotic representation systems are therefore described both by semiotics (being semiotic systems) and by linguistics (being languages) (Drouhard & Panizza, to appear a, b). Ferdinand de Saussure can be considered the founder of linguistics and Noam Chomsky one of the prominent scientists in this field.

The problem with the word “language” is that its meaning differs very much according to the theoretical framework in question. Some linguists define a language as the set of sentences produced by a generative grammar[2], (this is tipically the case with Chomsky and computer scientists). Other ("functionalist" linguists like Jakobson or semioticians like Eco) define language by its functions, in other words, by what it allows one to do (to transmit information or orders, to ask questions, to describe facts, to express ideas or feelings etc.). Thus, when speaking of "language" it is important never to forget to add "in the sense of” Chomsky or Eco, for instance.

There is also a problem with the word "epistemology" which is used by scientists in different domains; firstly by those who study the philosophy of mathematics (for instance the nature of mathematical objects); but also, by some mathematics historians. Even Piagetian psychologists call their domain "genetic epistemology"! It is not a mere question of vocabulary, since there are passionate arguments surrounding the relationship between history and the philosophy of mathematics. All positions can be found in the scientific literature, including the extreme ones (i.e. that

[2] A generative grammar is basically a set of rewrite rules

philosophy is a branch of history, or the opposite). Mathematics education is not immune from the controversy (if, for instance, we consider the notion of "epistemological obstacle"), never really being sure whether it is history, philosophy, both or neither. The aim of our discussion group was not to take a position on this debate, but rather to shed light on the risk of misunderstanding and the necessity on being clear about what we are referring to when speaking of "epistemology."

EPISTEMOLOGY AND HISTORY

Often, the group discussions turned to epistemological considerations of algebra. Some members objected to the notion of the validity of the recapitulation principle (i.e. that ontogenesis recapitulates phylogenesis). Although the question had been discussed in the past (see e.g. Furinghetti and Radford, 2002; Radford, 1997) the members felt that it was important to deal with this question in order to better grasp the role of the history of algebra as a means to explain the difficulties that students encounter when they learn algebra. In this context, some members mentioned the idea of epistemological obstacles. By definition, epistemological obstacles (in the sense of Brousseau, 1983) are those which are intrinsic to knowledge (as opposed to ontogenetic, didactic and cultural obstacles). In the course of the discussion, doubts were raised concerning this concept of knowledge. The opposition between epistemological and cultural obstacles was related to the problematic idea that mathematical knowledge could have a kind of intrinsic kernel, independent of the cultural context from which such knowledge arises and evolves (Radford, ibid.). The discussion stressed the importance of being careful with the notion of epistemological obstacle and taking into account the cultural context in which a notion appears. For instance, it was argued, using Araya Chacón's example of negative numbers, that, for ancient Chinese mathematicians, negative numbers were very 'natural' and that the question is rather to see the cultural conditions that made those numbers thinkable. The Chinese episteme rested on the cultural idea of opposites (yin-yang) while the Greek episteme was based on a non-symmetrical opposition between being and non-being.

Moreover, when the cultural context changes, problems and difficulties change too: this is one of the reasons that led us to decide that the recapitulation principle had little relevance.

The discussion then focused on the value of the history of mathematics for mathematics education. Some members argued that the fact that the recapitulation principle is not valid does not mean that the history of mathematics loses its relevance in the educational realm. The problem is to determine which kinds of historical studies are suitable for mathematics education.

G.T. Bagni's paper explicitly tackles this problem, stressing that different uses of history imply different epistemological assumptions, and arguing that a social and cultural account of the history of mathematics better fits the needs of research on the didactics of mathematics.

The history of mathematics, it was contended, can shed some light on the conceptual development of mathematics, but in order to do so, it has to be conceived in non-essentialist ways (i.e. in ways that do not assume that mathematics evolve according to a supposedly internal teleology). History has to attend to the cultural settings in which mathematics evolve, and to see those settings not as mere picturesque and charming backgrounds but as integral parts of conceptual developments. This perspective is not without its own difficulties. For one thing, it requires us to revisit the ampler problem of knowledge formation and cognition.

The historical-cultural theoretical framework presented by L. Radford was discussed in the group. This framework is an attempt to go beyond the classical way of conceiving the role of history and culture in mathematics education –a way that can be summarized as follows: (1) it sees history as a sequence of events disconnected from their cultural settings without paying attention to the cognitive-epistemic dimension (i.e. what makes mathematical ideas possible at certain periods) and (2) it sees culture as a descriptive background with no organic ties to the cognitive domain. Thus Radford's position is to consider the cognitive dimension of historical developments and to consider simmetrically the cultural dimension of cognitive developments. This theoretical framework leads to a relationship between ontogenesis and phylogenesis different from the recapitulationist one.

The depths of this problem were illustrated through the emergence of new non-rationalist epistemologies, such as cultural, feminist and post-modernist epistemologies, each one opening different routes through which to conceive knowledge and knowledge production. One concrete example was the following. In the research conducted by Radford and his students on algebraic generalizations (see the work presented by Radford, Bardini and Sabena at this Conference) or on equations (e.g. Radford, 2002), an important role is given not only to symbols, but also to social interaction, gestures, language and artifacts in the emergence of algebraic thinking. The role of gestures or the rhythm of actions, for instance, does not have a significant role to play in rationalist epistemologies (even in Piaget's genetic epistemology where kinesthetic actions fade away as soon as the sensori-motor stage is supposedly completed). However, in a different epistemology –one that conceives cognition not only as involving the mental dimension of the mind but also as including gestures, rhythm, perception, etc.– new forms of knowledge production are considered and cognition is cast in different terms.

The cultural-historical theoretical framework is based on the premise that each act of knowing is imbricated in the history of the object of knowledge and the cultural sense of knowing in which the act of knowing occurs. The theoretical framework acknoweldges the following fundamental *limitation* in the use of history for didactic purposes. In *empirical* studies, it is possible (even if not always easy) to have access to the complex learning processes of contemporary students (e.g. their culturally situated sources such as textbooks, classroom discourse, written material). In the study of past mathematicians' discoveries, there is no other available source than written texts.

HISTORICAL DATA

This serious problem was analysed firstly from the point of view of the selection of historical data when dealing with a new account of the history of algebraic notation, as raised by Bagni. We considered the difficulties that we have to deal with in our research due to the fact that our main source of historical data is ancient mathematical texts. If we do not conceive the history of mathematics as the discovery of eternal mathematical objects and truths, and if we think that the system of signs used to write a mathematical text is not a means to expressing those eternal mathematical objects and truths—but rather an essential component of the construction of mathematical objects—we cannot rely on the translation of ancient mathematical texts to modern concepts and systems of signs. For instance, we cannot use Witmer's translation of Viète to study the history of algebraic notation, because Witmer translates the relevant parts of Viète's text into modern algebraic language.

Dealing with the original texts is not an easy task for those of us who are researchers on the didactics of mathematics and not professional historians of mathematics, but we have to take care to at least be aware of the transformations made to the original texts in the editions we use.

Next we had a second look at the use of written texts, taking into account the risk of anachronism. It is especially important to keep in mind that, when looking at ancient mathematical texts, we cannot project our modern concepts on them. Besides taking into account the cultural and social dimensions that differ through time and place, if we use Freudenthal's historical phenomenology, we know that what is relevant to didactics is to analyse which phenomena where organised by concepts that we can see as historical precursors to modern concepts. In this sense, we considered that if we track the history of "integer numbers" back to ancient times, for instance back to Diophantos' or al- Khwārizmī's texts (See Puig, 2004), what we can find is algebraic expressions in which there are quantities that are being subtracted from other quantities. There are not positive and negative quantities, but quantities that are being added to others (additive quantities) and quantities that are being subtracted from others, and the latter cannot be conceived on their own but only as being subtracted from others. Thus, al-Khwārizmī may even go so far as to speak of "minus thing" when he is explaining the sign rules, but he is always referring to a situation in which that thing is being subtracted from something

> When you say ten minus thing by ten and thing, you say ten by ten, a hundred, and minus thing by ten, ten "subtractive" things, and thing by ten, ten "additive" things, and minus thing by thing, "subtractive" treasure; therefore, the product is a hundred dirhams minus one treasure. (Rosen, 1881, p. 17 of the text in Arabic)

However, as the subtractive quantities are conceived as something that has been subtracted from something, an expression in which there is a subtractive quantity represents a quantity with a defect, a quantity in which something is lacking. Diophantos' sign system expresses this way of conceiving the subtractive in an especially explicit way, as in his sign system all the additive quantities are written together, juxtaposed in a sequence one after another, and all the subtractive quantities are written afterwards, also juxtaposed, preceded by the word *leipsis* (what is lacking). Thus, the algebraic expression

$$x^3 - 3x^2 + 3x - 1$$

is written as

$$\mathrm{K}^{\Upsilon}\overline{\alpha}\,\varsigma\overline{\gamma}\ \Lambda\ \Delta^{\Upsilon}\overline{\gamma}\,\overset{o}{\mathrm{M}}\overline{\alpha}$$ (Tannery, 1893, vol. I, p. 434, l. 10),

an abbreviation of "cubos 1 arithmos 3 what is lacking dynamis 3 monas (units) 1," in which the expressions corresponding to x^3 and $3x$ are juxtaposed on one side, and x^2 and 1 on the other, separated by the abbreviation for "what is lacking" (Greek letters lambda and iota).

Thus, the main phenomena that are organized in al- Khwārizmī and Diophantos are the phenomena of "the subtractive," "what is subtracted (from a quantity)" or "what is lacking (to a quantity)".

Examples from Diophantos, al-Khwārizmī, Viète, Chuquet and Bombelli's algebraic expressions give us the opportunity to further discuss Nesselman's frequently quoted three stages in the evolution of algebraic language: rhetorical, syncopated, and symbolic (See Section 3 of Puig and Rojano, 2004). It was pointed out that in this case it is also worth looking at the literality of Nesselman's text. Nesselman's characterization of syncopated algebra stresses that syncopated algebra is algebra in which the exposition is also of a rhetorical nature "but for certain frequently recurring concepts and operations it uses consistent abbreviations instead of complete words" (Nesselmann, 1842, p. 302). What is really important from Nesselman's point of view is the rhetorical nature of the exposition, and not the use of signs that are mere abbreviations of words. That is the reason why in this stage Nesselmann places not only Diophantos, but even Viète "although in his writings Viète had already shown the seed of modern algebra, which nevertheless only germinated some time after him" (Nesselmann, 1842, p. 302). For example, Viète writes "*A* quad – *B* in *A* 2, æquetur *Z* plano" for the equation that we write: $x^2 - 2bx = c$, using abbreviations like "quad," the abbreviation of "quadratum (square)," instead of using numbers (2, in this case). Viète is using letters to represent quantities, but this is not enough to characterize his sign system as symbolic from Nesselman's point of view. For him, the fundamental feature of symbolic language is not the mere fact of the existence of letters to represent quantities or of signs foreign to ordinary language to represent operations but the fact that one can operate with this sign system without having to resort to translating it into ordinary language. In Nesselmann's own words: "We can perform an algebraic calculation from start to finish in a wholly understandable way without using a single written word […] (Nesselmann, 1842, p. 302).

"Symbolic," in Nesselmann's sense, means then the possibility of calculating on the level of the expressions without resorting to the level of content. We also discussed what "calculating on the level of the expressions" really means. It was pointed out that Chuquet's and Bombelli's idea of using numbers instead of abbreviations of words to stand for "thing" or "root," "square" (or "census" in Latin Medieval texts, and "treasures" in Arab Medieval texts), "cube," etc. was crucial because rules for transformations like "a thing by a square is a cube," that are grounded on the level of content, could be replaced by arithmetical equalities like $1 + 2 = 3$, that are meaningful even if we do not resort to the level of content (the relations among the type of quantities involved). In this sense, one has the possibility of calculating on the level of the expressions with the current symbolic language of algebra.

SEMIOTIC AND LINGUISTIC ASPECTS

During the conference, some contributors presented studies which could be clearly seen through semiotical lenses (see for instance Novotná & Sarrazy and the question of the spontaneous non-linguistic schemas done by the children) or through linguistical lenses (see for instance the tipically linguistical concept of "deictic" in Radford et al.). We found many different points of interest according to the linguistic/semiotics axis.

On the one hand, the different works presented within this theoretical frame allowed us to approach it from different perspectives and with increasing levels of generality. On the other hand, this framework offered a complementary perspective for the analyses of other works presented within other theoretical frames.

The discussions that sustained our analyses can be included in three main lines:

1. Algebraic thinking is not always associated with the use of the present algebraic symbolism.
2. Very different contexts may favour the development of algebraic symbolism.
3. It is important to make a clear distinction between one- and two-dimensional symbolic writings.

1) Algebraic thinking is not always associated with the use of modern algebraic symbolism.

This sentence was interpreted in two ways, in accordance with mathematical education literature: neither does the use of modern algebraic symbolism always involve algebraic thinking; nor does algebraic thinking always involve the use of the modern algebraic symbolism. Various contributions were discussed in the WG according to these interpretations of the use of modern algebraic symbolism.

The difficulties in understanding the complex relationship between algebraic writings and algebraic thinking was also enlightened with the analysis offered by Puig from the historical perspective (see previously in this text the points concerning the negative quantities and the "minus sign" in algebraic expressions, and Nesselman's own criteria for clasifying the evolution of algebraic language; see also Radford (1995)).

2) Very different contexts may support the development of algebraic symbolism

This dimension was especially centred on the analysis of different situations supporting the construction of algebraic symbolism. Lins and Kaput (2004) stressed an emphasis on what students *can* do as opposed to the perspectives centred on highlighting difficulties or characterising errors. Several contributions involving the first perspective were discussed. Various authors showed different contexts which could gradually lead the students towards algebraic symbolism.

The paper of Radford and al. led us to a discussion of the role of *students' gestures and language*: *what they actually do, do not do, say or do not say* as objects of analysis in their construction of algebraic representations. In this context, the use of "deictics" in relation to students' interactions was especially taken into account. In particular, the role of perception in the use of deictics was suggested as an instrument for analyzing the "point of view" in the subject-object and student-students relationship.

All this discussion led us to consider the enunciative theories as frames that can contribute to interpreting students' mathematical speech. Different works by Radford can be mentioned in this direction. In Radford (2000, 2002) he demonstrates–through several analyses of the students' language in class—several functions language plays in the construction of algebraic generality (for example the *deictics function* and the *generative* function). A theoretical line is offered by Duval, who established different components of the sense of a proposition, founded on the fact that a proposition is posed in an enunciation context (Duval, 1995).

3) Distinction between one- and two-dimensional symbolic writings

A word alphabetically written is read from left to right (or from right to left in Arabic or Hebraic writings); in any case, in just one direction (from beginning to end). On the contrary, algebraic writing is bi-dimensional: not just the succession of letters and symbols is relevant but their relative vertical placement too (Drouhard, 1992).

It is the case for fractions: $\frac{\textit{(higher than the current line)}}{\textit{(lower than the current line)}}$ but also for powers:

$\textit{(current position\&size)}^{\textit{(higher\&smaller)}}$. The vertical reading order may be top-down (as in fractions) or down-top, as in: $\int_a^b f(x)\,dx$. Moreover, Kirshner (1989) showed that for algebraic writings, horizontal spacing is relevant too (although redundant), as in "2×4+3" (wider horizontal space around the "+" mark than around the "×" mark).

During the session discussions, Puig noted that the origin of bi-dimensional writings (which already appear in Greek notation, see above) could be found in the use of schemes, tables and drawings (obviously bidimensional) from the earlier times of mathematics. Bidimensional writing, he added, is technically difficult when passing from paper-and-pencil to print characters or keyboards: this is the case for calculators or spreadsheets (one must type 2^3 to obtain 2^3).

As time has gone on, there has been pressure to "linearise" writings, essentially for printing reasons. A lot of examples can be found in Cajori (1928, 1993). For instance the "vinculum" upper straight line was used to express the aggregation of terms, equivalent to our modern parentheses; nowadays this vinculum is used only with fractions and roots. However, with typewriters, even fractions and roots became difficult to write and there was a tendency to replace "$\sqrt{ax+b}$" by "√($ax+b$)". It is possible to think, Puig pointed out, that the failure of the Gottlob Frege's *Begriffschrift* (Ideography, 1879) may rely on the intensive use of two dimensions in the symbolism he proposed; actually only few of Frege's unidimensional notations remain, like "¬", "⊨" or "⊢". The situation reverted dramatically with word processors; and maybe we would use *Begriffschrift* notations on an everyday basis if Frege could have used LaTeX to write his articles!

During the session discussions, Drouhard stressed the analogy between mathematical writings and Chinese ideograms. Firstly, some ideograms use two dimensions, like mathematical writings. For instance: the ideogram for *shuāng*:

隹×2 + 又 = 雙 which is "two birds in the right hand": "pair".

Then again, ideograms just note "ideas" and not sounds; therefore they are pronounced totally differently according to the idiom. For instance:

> "Thus, although the number one is "yi" in Mandarin, "yat" in Cantonese and "tsit" in Hokkien, they derive from a common ancient Chinese word and still share an identical character ("一")"[3]

It is the same for mathematical writings: you will pronounce the same writing "$2x$+3": "dos equis más tres," "deux *x* plus trois" or "two *x* plus three" etc. according to your mother tongue. From this point of view, it is possible to consider that mathematical writing is, by far, the most widespread written language in the world.

Mathematical writings, however, are not ideograms (even if they are close relatives); in particular they are characterized by a virtually infinite possiblity to combine, like: $10^{10^{10^{10^{\cdots}}}}$, which is not the case of ideograms.

CONCLUSION

A last theoretical point was presented in order to sum up some fragmented remarks. It is a model of knowledge called "paradigmatic perspective," which was briefly presented at the previous CERME conference in Bellaria (Drouhard & Panizza, 2003b). This is not the place to describe it in detail (see Drouhard & Panizza, 2003a, 2005, Panizza & Drouhard, 2003, Sackur et al., 2005, Bagni, to appear). We will just

[3] Source: http://www.answers.com/topic/chinese-language

recall that the organisation of the knowledge to be taught, called "epistemographical model," consists in: Conceptual knowledge, Semiotic and Linguistic knowledge, Instrumental knowledge, knowledge of the rules of the mathematical game and knowledge that allows for identification of domains. This general framework allowed us:

- To focus on the importance of semiotic analysis (Peirce) in some presentations (e.g. Novotná & Sarrazy)
- To better analyse the knowledge involved in other presentations (e.g. about the pre-requisite knowledge needed at the beginning of university by De Vleeschouwer, or on difficulties with matrixes by Viola)
- To avoid looking at history from just one aspect of knowledge: in this case there is a risk of remaining at an either superficial or biased level. On the contrary, it is really fascinating to observe and analyse how semiotic progress (like the invention of notations for variables or parameters) is related to instrumental progress (more convenient notations permit one to better solve more problems) and to conceptual progress (see also Duval, 1988).

This last point provided yet another occasion to fruitfully intertweave discussions on history, as well as semiotic and epistemological issues.

REFERENCES

Bagni, G. T. (2006). Quali saperi sono acquisiti da chi fa matematica? Una conversazione con Jean-Philippe Drouhard. To appear in *La matematica e la sua didattica*.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en Didactique des Mathématiques*, **4**/2, pp. 165-198.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*, Edited and translated by N. Balacheff, M. Cooper, R. Sutherland, V. Warfield. Dordrecht: Kluwer.

Cajori, F. (1993). *A History of Mathematical Notations*. New York: Dover. First Edition: The Open Court Publishing, La Salle (Ill.), 1928.

Drouhard, J-Ph. (1992). *Les Écritures Symboliques de l'Algèbre élémentaire*, Thèse de Doctorat, Université Paris 7.

Drouhard, J-Ph., Panizza M.(To appear): "De Saussure ou Pierce? Chomsky ou Eco? Bosch, Chevallard ou Duval? Tous à la fois? Complémentarité et incompatibilités des différents paradigmes linguistiques et sémiotiques en didactique de l'algèbre". To appear in L. Bazzini (dir.) *Proceedings of SFIDA 21-24*. University of Turin.

Drouhard, J-Ph., Panizza M. (To appear): "Pierce, Frege, Saussure: trois grands courants sémiolinguistiques pour la didactique (2ème partie)". To appear in L. Bazzini (dir.) *Proceedings of SFIDA 21-24*. University of Turin.

Drouhard, J-Ph., Panizza, M. (2003a), Les trois ordres de connaissances: un essai de présentation synthétique. Bazzini, L. (Ed.), *Atti del Seminario Franco Italiano di Didattica dell'Algebra, V, (SFIDA 17, 18, 19, 20)*, Dipartimento di Matematica, Università degli Studi, Torino.

Drouhard, J-Ph., Panizza, M. (2003b), What do the Students Need to Know, in Order to be Able to Actually do Algebra? The Three Orders of knowledge. *Atti della 3rd European Conference on Research on Mathematics Education*, Bellaria, Italy, CD, Università di Pisa.

Drouhard, J-Ph., Panizza M. (2005), Perspective Paradigmatique et Ordres de Connaissances. A. Mercier, Cl. Margolinas (Dir.). *Actes de la 12ème école d'été de Didactiques des Mathématiques*. 1 ouvrage + 1CD, le texte se trouve dans le CD. Grenoble: La Pensée Sauvage.

Drouhard, J-Ph., Teppo, A. (2004). Symbols and Language. In K. Stacey, H. Chick,, M. Kendal (Eds.), *The teaching and learning of algebra: The 12th ICMI study* (pp. 227-264). Norwood, MA: Kluwer Academic Publishers.

Duval R. (1988), Signe et objet I et II *Annales de Didactique et de Sciences Cognitives* **6** (pp. 139-163 ;165-196) IREM de Strasbourg

Duval, R. (1995). *Sémiosis et Pensée Humaine, Registres sémiotiques et apprentissages intellectuels*. Berne: Peter Lang.

Duval, R. (1999). Écriture, raisonnement, et découverte de la démonstration en mathématiques. *Actes de la Xème Ecole d'Eté de Didactique des Mathématiques*, Houlgate. 29-50.

Frege, G. (1879). *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle. In French: *L'idéographie, un langage formulaire de la pensée pure construit d'après celui de l'arithmétique*. Introduction et traduction par C. Besson. Vrin. ISBN: 2-7116-1388-7.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Reidel.

Furinghetti, F., L. Radford (2002). Historical conceptual developments and the teaching of mathematics: from phylogenesis and ontogenesis theory to classroom practice. In L. English (Ed.), *Handbook of International Research in Mathematics Education (631-654)*. New Jersey, Lawrence Erlbaum.

al-Khwārizmī, Muhammad ibn Mūsa (1939). *Kitāb al-mukhtasar fī hisāb al-jabr wa'l-muqābala* (Edited by Alī Mustafā Masharrafa and Muhammad Mursī Ahmad, Reprinted 1968). Cairo: al-Qahirah.

Kirshner, D. (1989). "The Visual Syntax of Algebra", *Journal for Research in Mathematics Education*, 20-3, 276-287.

Lins, R., Kaput, J. (2004). The Early Development of Algebraic Reasoning: The Current State of the Field. In K. Stacey, H. Chick,, M. Kendal (Eds.), *The teaching and learning of algebra: The 12th ICMI study* (pp. 47-70). Norwood, MA: Kluwer Academic Publishers.

Nesselman, G. H. F. (1842). *Versuch einer kritischen geschichte der algebra, 1. Teil. Die Algebra der Griechen*. Berlin: G. Reimer.

Panizza, M., Drouhard, J-Ph. (2003). Los órdenes de conocimiento como marco para significar las prácticas evaluativas. Palou de Maté, C. (Ed.), *La Enseñanza y la* *evaluación: una propuesta para matemática y lengua.* pág. 51-73. Buenos Aires: GEEMA (Grupo Editor Multimedial). Colección: Estudios Universitarios.

Puig, L., Rojano, T. (2004). The history of algebra in mathematics education. In K. Stacey, H. Chick, and M. Kendal (Eds.), *The teaching and learning of algebra: The 12th ICMI study* (pp. 189-224). Boston / Dordrecht / New York / London: Kluwer Academic Publishers.

Puig, L. (2004). *History of algebraic ideas and research on educational algebra,* Regular Lecture presented at the Tenth International Congress of Mathematical Education, Copenhagen, Denmark. Text available at http://www.uv.es/puigl/

Radford, L. (1995). "Before the Other Unknowns Were Invented: Didactic Inquiries on the Methods and Problems of Mediaeval Italian Algebra." *For the Learning of Mathematics* **15**(3). 28-38.)

Radford, L. (1997). On Psychology, Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics. *For the Learning of Mathematics,* **17**(1). 26-33.

Radford, L. (2000). "Signs and meanings in students' emergent algebraic thinking: A semiotic analysis." *Educational Studies in Mathematics* **42**(3). 237-268.

Radford, L. (2002). "Generalizing Geometric-Numeric Patterns: Metaphors, Indexes and Other Students' Semiotic Devices." *Mediterranean Journal for Research in Mathematics Education* **1**(2). 63-72.

Radford, L. (2002). Algebra as tekhne. Artefacts, Symbols and Equations in the Classroom. *Mediterranean Journal for Research in Mathematics Education*, **1**(1). 31-56.

Rosen, F. (1831). *The algebra of Mohammed Ben Musa.* London: Oriental Translation Fund.

Sackur, C., Assude, T., Maurel, M., Drouhard, J-Ph., Paquelier, Y. (2005). L'expérience de la nécessité épistémique. *RDM*, **25**/1, pp. 57-90.

Tannery, P. (Ed.) (1893). *Diophanti Alexandrini Opera Omnia cum graecis commentariis*. Edidit et latine interpretatus est Paulus Tannery. 2 vols. Stuttgart: B. G. Teubner. [Reprinted in 1974.]

Van Schooten, F. (1646). *Francisci Vietæ Opera Mathematica*. Lugduni Batavorum: Ex Officinā Bonaventuræ & Abrahami Elzeviriorum.

Witmer, T. R. (Ed.) (1983). *François Viète. The analytic art.* Kent, OH: The Kent State University Press.

DIFFICULTIES FOUND BY THE STUDENTS DURING THE STUDY OF SUBSTRACTION OF INTEGER NUMBERS

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Abstract: *The objective of the investigation presented is to determine the possible causes of the students' difficulties during the study of subtraction of integers. Considering as principal reference certain elements of the Theory of the Didactic Transposition, the results are formulated in terms of the evolution of the "scholarly knowledge"(historical note) and its transformations to become a school content according to the programs of study and some textbooks of upper secondary school in France (knowledge to be taught). This previous work is useful to contextualize and finally analyze the knowledge taught in two classes of fifth grade, explaining the difficulties found by the students.*

Keywords: Algebraic thinking, students' difficulties, subtraction of integers.

1. INTRODUCTION

The purpose of this document is to share the analysis made in order to determine the possible causes of the difficulties that the students have during the study of the subtraction of integer numbers. It begins with a brief description of some previous works that justify the elaboration in conjunction with the experience considered. Then the conceptual reference in which the analysis is placed will be exposed. After this, the methodology used is presented coherently with the transformations suffered by the "scholarly knowledge" to the "knowledge to be taught" and to the "taught knowledge".

Following this exposition, the results show some indicators that help identifying the categories of errors that the students may more commonly commit, besides the four difficulties and their possible causes.

Finally, the conclusions and one annotation regarding the procedure used by certain university students that show another point of view are exposed.

2. PRELIMINARIES AND JUSTIFICATION

The topic of the integer numbers has been of interest for the specialists in the area of Mathematics and Mathematics Education, due to its particular delay in being accepted as a mathematical object (about 1500 years) and for the difficulties of the professors for building situations of teaching, and of the students for being competent in them.

G. Schubring (1986) has worked about the ruptures of the mathematical status of the negative numbers, doing an analysis of certain textbooks between 1750 and 1850 in three European countries: France, Germany and England. In the first one, the calculations of the negative quantities is adequate, even though, the negative numbers preserve an ambiguous status.

George Glaeser (1981) proposes a study of the epistemology of the integer numbers. He details 10 obstacles to his mathematical acceptance, from the contributions of authors ranking from Diofanto to Hankel.

In a more recent study in Spain, Bruno and Martiñón (1995-1996) discuss the dimensions in which the comprehension of the integer numbers can be established. Among the results, it is pointed out that a student can be able to solve correctly a problem applying the subtraction, without understanding why this mathematical operation might be used.

This last result, the suggestion of Glaeser to study the actual consequences of the obstacles that he proposes, and the ambiguity of the treatment in textbooks reported by Schubring may justify performing the research, with the goal of defining a tentative inventory of the difficulties found by 49 students of fifth class, and their possible causes.

3. CONCEPTUAL REFERENCE

3.1 Didactic Transposition. Didactic Contract. Ostension Contract

Chevallard (1991), has shown that the “knowledge to be taught” cannot be considered a reduction of a more complex knowledge, resulting from a “scholarly knowledge”. It is necessary that a learning content, after being designated as “knowledge to be taught”, suffers some adaptive transformations that should turn it apt as an objective of teaching; that is, it has to pass through a process of Didactic Transposition (Chevallard 1991).

As part of the transposition process, the ancient/new dialectic adds two faces to the object. In one side, its character of novelty is necessary to satisfy and justify the rising of the new content in one situation. On the other side, its ancient character guarantees the recognition of certain elements previously learned by the students; that is, it authorizes an identification that restores it in the panorama of the ancient understandings.

The process of didactic transposition also determines the paper of the teacher and the student. This distinction is recognized at least in two forms: first because the educator possesses “more knowledge” than students and secondly because he/she is able to anticipate what the students may know. Thus two registers of epistemological acts or two “ways of knowledge” are defined: one is what the educator may teach and the way of doing it, the other one is what students may know and how it may be learned. So, in this way, each character can be identified with a role to follow and establish a relationship (generally implicit) between what each one is responsible of doing in front of the other one. This relationship is called the *didactic contract* (Brousseau, 1986).

Brousseau brings one definition of the contract of “ostention” as a part of the didactic contract. His definition affirms that the professor “shows” an object or a property and the student accepts seeing it as a representative of a class of objects. He thus might recognize the elements of this class of objects in other circumstances. This implies that the “ostention” of the solution of a particular problem is supposed to give the necessary tools for solving the exercises thatwill follow it, grading the level of application.

3.2 Mathematical Organizations. Didactic Organizations

Chevallard (1999) places the mathematical activity and its study in the group of human activities and of social institutions, as a way to describe it with a unique model briefed under the name of praxeology.

Among its fundamental elements, we find the idea of “type of tasks”, synonym of “action”, that supposes the definition of a precise object to which apply that action. The idea of “technique”, as the way of performing a task; the “technology”, referring principally to the rational justification that ensures the validity of the technique, and the notion of “theory” as the discourse that justifies and explains the technology. We call *specific* praxeology or specific mathematical organization the block formed by a type of tasks, the correspondent technique, the technology and the respective theory.

The didactic organizations are understood as a group of types of tasks, techniques, technologies and theories, used for the concrete study in a specific institution. These organizations can be constructed by what Chevallard has called the moments of study or didactic moments and that can be considered as dimensions or situations that succeed regularly in a didactic process.

The first moment is named the moment of the first encounter and corresponds to the first near drawing to the object of study. It is followed by the moment of exploration of the type of tasks and the elaboration of a technique. Then comes the constitution of the theoretical-technological environment, that supports the forth moment, the work of the technique, necessary to improve the dominion of it and explore its achievements. The institutionalization moment appears when it has to be cleared out what the students may know about the constructed mathematic organization. There is, finally, the moment of the evaluation.

It is important to notate that the presence of these moments in a didactic process is not certain nor chronologic. In certain occasions, they are not present or they succeed in a simultaneous way.

4. METHODOLOGY

In order to reach the general objective of determining the origin or the possible causes of the difficulties that the students have during the study of the subtraction of integer numbers, the research has been divided in three parts. In the first part, a bibliographical research has been done, where elements were selected in order to create a description of the evolution of the mathematical object, the subtraction of integers. Next, from the criteria extracted from the Theory of the Didactic Transposition, the notions of praxeologies, were analyzed the programs of instruction and the French textbooks, of the most significant improvements, of the years 1950, 60, 71, 78, 85 and the current.

In the second part, two classes of fifth grade of one school in Toulouse were selected. The first one, G1, with 28 pupils, in charge of the teacher P1, with more than 35 years of teaching experience. The second one, G2, with 28 pupils also, guided by a teacher (P2) with about 15 years of experience. In each one of them four observations were audio-recorded and transcipted in their whole. Using the same criteria that in the previous stage, the interpreted data in the observations were described and analyzed, in this way analyzing, the current transposition.

In the last part, work was done with four students from each class, a couple (woman-man) with high academic performance and another one with medium-low academic performance, according to the criteria of the teachers.

Starting from the revision of the notebooks, written evaluations and considering the applied didactic and mathematical organizations, a guide was elaborated with the semi-directed interviews to the couple of students. Each one was audio-recorded. From the interpretation of the collected data in these interviews, a list of the most common errors was created, forming categories with them with the respective indicator for their identification, living also, the possible causes. From these indicators a test was built and was applied in an anonym way to the students that attended both classes that day (49 in total).

The Figure 1, summerized the methodolody used

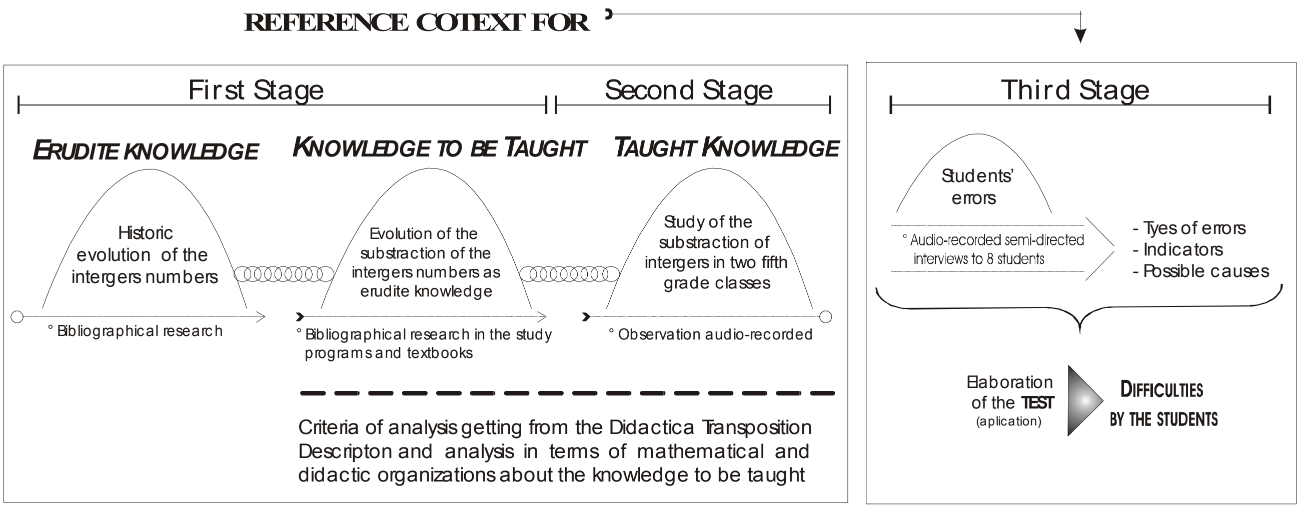


Figure 1: Outline of methodology reference

"Difficulty for the students", was defined as a recurrent error, defining it thus: if it was detected in at least 25% of the students that answered the question of the first part and at least a 40% for the items of the second part. This is due to the open or closed character of the question. In this paper the results and conclusions corresponding to the third part are presented.

5. RESULTS

The results corresponding to the third part of the investigation are the indicators built as analysis criteria of the test, the difficulties of the students related to the subtraction of integer numbers, and the possible causes that are read from the analysis.

The indicators can be classified in two types: in a first kind the relative to the mathematical and didactic organizations where the students participate, and that we consider them as possible sources of errors; in a second kind the description of such errors.

The rule of subtraction studied in both classes was: *in order to subtract two integers numbers, the opposite of the second is added to the first one*; that is, $a - b = a + (-b)$. This involves during the beginning of the study, a stage of "re-write" of the expression (that allow us to appeal to a "known" equivalent expression) and another of "simplification". For example, in the expression

Stage of simplification

$$(+5) - (+8) = (+5) + (-8) = (-3)$$

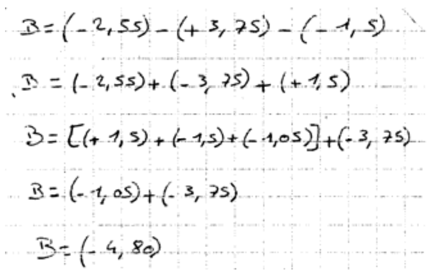
Stage of re-write

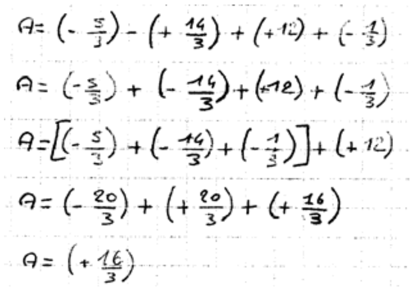
We will take this difference into account when we present the indicators of the second type. In square brackets is indicated the percentage of students in which their developments of the test, the described indicator is found.

<u>Indicators of the first kind</u>

1. The equivalence $c + b = a \leftrightarrow a - b = c$ is not immediate (evident) for the students [44]
2. Such the teachers as the students use the numeric straight line in order to "better explain" themselves
3. Given that the integer numbers are seen as natural numbers with a sign before them, the students have difficulty distinguishing between negative numbers and the ones that are subtracted [59]
4. In order to find the distance between two numbers a subtraction of the bigger one minus the smaller one (always positive) is done, but in order to find the difference between two numbers, the first given number is subtracted from the second one. [8]
5. The use of other writing of a number, without justification of its validity, can provoke errors during the calculation. [9]

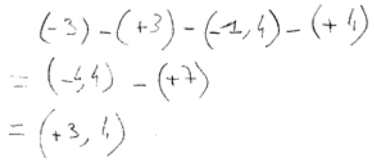
As mentioned during the previous results, P_1 promotes the re-writing of the numbers in order to simplify the calculations. For example, the procedure of E11 as an answer to the question of the quiz:





But some students do not develop as well, methods of cross-check and as clause of the contract, seek to apply also, the re-writing of numbers to simplify expressions that involve the subtraction For example the development of E13 during an exercise for the interview.

(–2,7) – (+3,2) – (–1,5) + (–4)

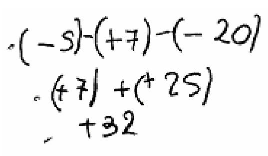


<u>Indicators of the second kind</u>

Stage of re-writing

A. The student do not applies correctly the properties of subtraction of integer numbers (not commutativity, not associativity) [43]

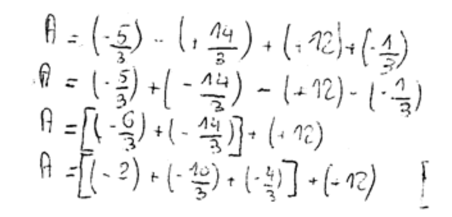
   For example, the answer given by E22 for the first exercise of calculation of the test:



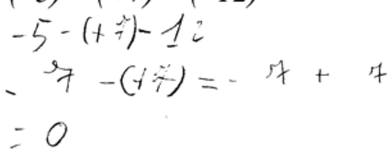
The subtraction is not commutative nor associative. E22 writes *(+7)–((–20)+(–5)) = (+7)–(–25) = (+7) + (+25)*, disappearing the "–" before (+7).

B. Changing all the signs that indicate an operation, even if these indicate an addition [15]

   It is the case in which a subtraction is changed by an addition. For example the procedure proposed by a student of G1 during the quiz,



(–5) – (+7) – (–12)

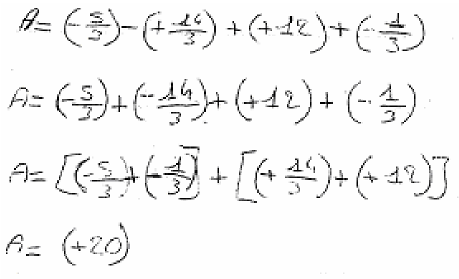


C. When subtracting two negative numbers, find "two minus signs" following each other, so just one is left [15]

We interpret this error as a result of the interviews to two students of G2. For example, for the first calculation a student writes:

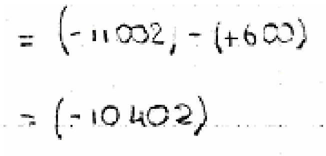
D. Add a negative number, equals to add the opposite [13]

It is the case of the procedure of a student of G1,



Indicators of simplification

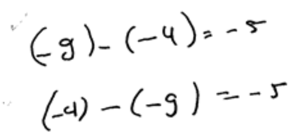
E. In order to subtract two integers of different signs, subtract the absolute value of the bigger one, minus the absolute value of the smaller one and left the sign of the one farther from zero [28]



For example, the student E12, in a quiz applied by P_1, wrote : *Subtract two integer numbers of different sign, subtract and leave the sign of the number farther from zero.*

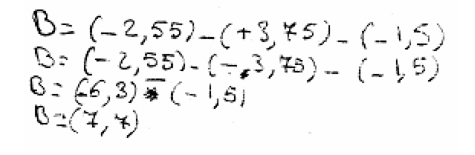
F. In order to subtract two integers of equal sign, subtract the absolute value of the bigger one, minus the value of the smaller one and leave the common sign. [2]

Though we also can consider that the sign of the number farther from zero is kept. For example, in the procedure of E14 that is presented.



G. In order to subtract two integer numbers of the same sign , add the numbers[28]

For example in the last two lines of the procedure of a student in a quiz applied by P_1



The test applied to the two observed classes was analyzed according to these indicators. According to the operationalization of "difficulty", we obtain that the errors of the kind A, E and G are difficulties for the students, because they are presented at least in a 25%.

The possible cause of the difficulty A, relapse in that during the study of the mathematical organizations, there was not a work that took into account the properties of the addition of two integers and the reasons why those are not extensive to the subtraction. Falls in the students the responsibility of making the integration of the properties for each operation about the integers and apply them to simplify expressions.

The difficulties E and G are product of an inadequate domain of the rules used to add integers. Moreover, these are destabilized by the rule of subtraction.

From the results of the second part, we interpret as valid hypothesis number 1 and number 3, of the indicators of the first kind. When comparing the incorrect answers by class, we obtain a separation of 28% that we explain thus: the scarce errors in G1 are explained by the confusion when interpreting the minus sign of the subtraction as the minus sign of the negative number. While in G2, we can foresee that they increase and has more variety as a result of the equivalences ($- +x = -x$, $- -x = +x$, …; element not present in G1), because the students tend to answer, using the procedure of calculation.

6. CONCLUSIONS

A more wide analysis realized in the investigation, indicates that certain difficulties found, do not depend upon the didactic choices of the teachers, they look like common to the students of both classes. The hypothesis 1 and 3 can be re-formulated in terms of what Brousseau calls obstacles of didactic origin, because only depend upon one selection of one project of the educational system, that consists in studying the addition before the subtraction.

The errors that are present in a primordial manner in a class, can be consequence of the lack of a protagonist role of the students in the moment of exploration, development and work of the technique; because finally, are the teachers who enunciate the rule of calculation.

In a theoretical way, and knowing that it does not exist a “correct way to teach”, since it is a complex process subject to the participants, we suggest a work that considers the treatment of the errors during the lessons. That is, not only correcting them in an oral or written way, o even worst, ignore them; but taking them into account when these rise and proposing exercises or questions that in an intentional way make them to appear. For example, propose to the students an incorrect procedure of an exercise and that they find the errors or explain the possible reasonings that take to it.

When proposing to five university students the simplification of an algebraic addition, we note that, referred to what we know of the effects of the advance of the didactic time, the new knowledge generally replaces the old one. This is the case, when applying the law of signs in order to simplify the subtraction or addition of integer numbers. For example in the expression $4 - (-6)$, the reasoning “minus multiplied by minus is plus, then $4 + 6$”, shows a mix of the notions relative to the addition, subtraction and multiplication, where the more general “absorbs” the more specific and weak ones, more if the original sense is altered and apparently not understood.

The probability that such phenomenon of absorption of more particular knowledge in the future, appears to be high, so we consider necessary that the study of them (first meeting) has to be significant or at least enough so it will not loose the original sense in the future (or know which is the change). From such restlessness arise the questions: ¿what are some of the possible knowledge to teach that have the risk of being replaced for other more general knowledge?, ¿what will be situations that can be built to give sense and mathematical justifications to such knowledge?

7. BIBLIOGRAPHY

Briand J. and Chevalier M.C.: 1995, *Les enjeux didactiques dans l'enseignement des mathématiques*, Hatier, Paris.

Brousseau G.: 1998, *Théorie des situations didactique*, La pensée sauvage, Paris

Bruno A.; Martiñón, A.: 1995-1996, 'Les nombres négatifs dans l'abstrait, dans le contexte et sur la droite', *Petit X*, 42, 59-78.

Chevallard Y.: 1991, *La transposition Didactique*, 2nd ed., La pensée sauvage, Paris.

Chevallard, Y.: 1999, 'L'analyse des pratiques enseignantes en théorie anthropologique du didactique', *Recherches en didactique des mathématiques*, 19 (2), 221-266.

Glaeser, G.: 1981, 'Epistémologie des nombres relatifs', *Recherches en didactique des mathématiques*, 2 (3), 303-346.

Schubring, G.: 1986, 'Ruptures dans le statut mathématique des nombres négatifs', *Petit X*, 12, 5-32.

INEQUALITIES AND EQUATIONS: HISTORY AND DIDACTICS

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Abstract: *The historical development of equations and inequalities is examined, in order to underline their very different roles in various socio-cultural contexts. From the educational point of view, historical differences must be adequately taken into account: as a matter of fact, a forced analogy between equations and inequalities, in procedural sense, would cause some dangerous phenomena.*

Keywords: Algebraic language, Equations, Historico-cultural epistemology, History of mathematics, Inequalities

1. INTRODUCTION: ALGEBRAIC EQUATIONS AND INEQUALITIES

Frequently, from the educational point of view, algebraic inequalities are introduced to pupils after algebraic equations, and the solving techniques are strictly compared; nevertheless, in classroom practice, techniques for equation solving, when applied to inequalities, lead sometimes to wrong results: so didactic connections between equations and inequalities are not simple to be stated (a number of papers can be found; for instance: Linchevski & Sfard, 1991 and 1992; Fischbein & Barash, 1993; Tsamir, Tirosh & Almog, 1998). Some experimental studies by L. Bazzini and P. Tsamir (2002) clearly pointed out several meaningful situations.

Let us note that the word *equation*, in English, denotes the mathematical statement of an *equality*. For instance, by writing "$x+2 = 5$" (*equation*) we state that the $x+2$ is *equal* to 5: and this is true if and only if $x = 3$ (solution of the considered equation). Of course we can consider an equality also without a proper equation, e.g. without an unknown: when we write, for instance, "2+7 = 9" we state that the sum of the numbers 2 and 7 is equal to 9 (frequently a statement of an equality that is true for all values of a variable, e.g. "$2x+7x = 9x$", is indicated by the word *identity*) and this is true. From the logical point of view, "2+7 = 9" is a sentence that expresses a proposition with the truth value "true"; "$x+2 = 5$" is not a sentence: it does not express a proposition, but a condition regarding the values which may be assigned to the variable involved (Bell & Machover, 1977, p. 12) and it will assume a truth value, either "true" or "false", depending on which number is assigned to x as a value.

Let us now consider the *inequality* "$x+2 < 5$": by that we state that $x+2$ is *less than* 5 and this is true if and only if $x < 3$. In several languages the word *inequality* can assume two different versions, so it is translated by two different words: for instance, in French, these words are *inégalité* (in Italian: *disuguaglianza*) and *inéquation* (*disequazione*).[1] With reference to these words, the mentioned difference would be summarised as follows: an *inéquation* is the mathematical statement of an *inégalité*.

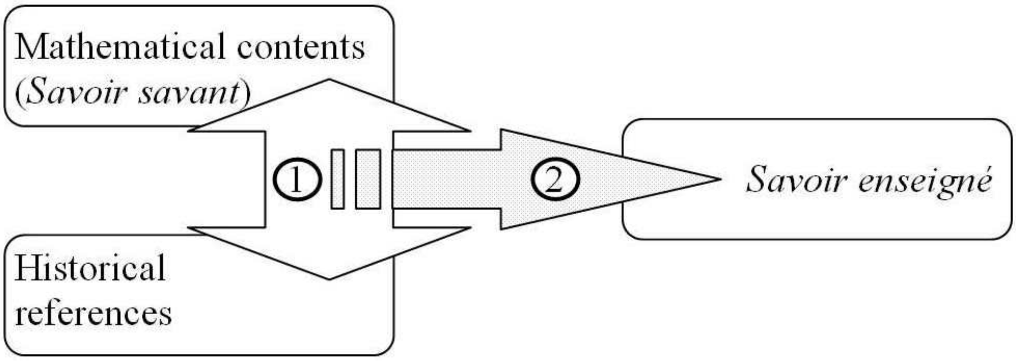
Both from a logical point of view and from an educational point of view, there is a great difference between an inequality like "$x+2 < 3$" and an inequality like $1+2 < 5$: their epistemological status is clearly different.

We shall denote the first inequality by the term *inéquation*, the second by *inégalité*.

2. HISTORY AND DIDACTICS: DIFFERENT THEORETICAL PERSPECTIVES

Our work will take into account some references from the history of Algebra. As several studies have pointed out, the historical approach can play a valuable role in mathematics teaching and learning and it is a major issue of the research in mathematics education, with reference to all school levels (Heiede, 1996).

The use of the history into education links psychological learning processes with historical-epistemological issues (Radford, Boero & Vasco, 2000, p. 162) and this link is ensured by epistemology (Moreno & Waldegg, 1993). Concerning the features of interactions between history and educational practice, a wide range of views can be examined. Different levels can be considered with reference to teaching-learning processes: a first is related to anecdotes presentation (and it can be useful in order to strengthen pupils' conviction: Radford, 1997); higher levels bring out metacognitive and multidisciplinary possibilities. Let us consider the following representation:



(where some well known terms by Y. Chevallard are employed). Of course this is just a schematic outline: for instance, the passage from the *savoir savant* to the *savoir enseigné* is not simple. However two sets of connections must be analysed:

- connections (1) between mathematical contents and historical references;
- connections (2) between mathematical contents linked to historical references and knowledge presented to pupils in classroom (after the *transposition didactique*).

Different uses of the history into didactics do not reflect just practical educational issues: they imply different epistemological assumptions (Radford, 1997 and 2003). For instance, the selection of historical data to be presented in classroom practice is

[1] Sometimes, in English, an *inequation* denotes a statement that two quantities or expressions are *not the same*, or *do not represent the same value* (written by a crossed-out equal sign: $x \neq y$).

epistemologically relevant: this selection reflects some epistemological choices by the teacher, too. Important problems are related to the interpretation of historical data: this is frequently based upon our cultural institutions and beliefs (Gadamer, 1975).

Frequently the role of the history into didactics is considered from an introductory point of view[2]: sometimes a parallelism between the historical development and the cognitive growth is assumed (since E. Haeckel's "law of recapitulation", 1874; see: Piaget & Garcia, 1989). As a matter of fact, a new concept is often encountered by mathematicians in operative stages, for instance in problem solving activities, and it will be theoretically framed many years or several centuries later (Furinghetti & Radford, 2002); a parallel evolution can be pointed out in the educational field: often the first contact with a new notion takes place in operative stages (Sfard, 1991; see the discussion in: Radford, 1997): in fact, pupils' reactions are sometimes rather similar to reactions noted in mathematicians in history (Tall & Vinner, 1981) and such correspondence would be an important tool for mathematics teachers.

The mentioned parallelism would require a theoretical framework: as a matter of fact, it leads to epistemological issues. A major issue is related to the interpretation of history: for instance, is it correct to present the history as a path that, by unavoidable mistakes, obstacles overcoming and critical reprises, finally leads to our modern theories? What is the role played by social and cultural factors that influenced historical periods? Mathematical contents deal with non-mathematical context, too, and knowledge must be understood in terms of cultural institutions (Bagni, 2004).

According to the "epistemological obstacles" perspective by G. Brousseau, one of the most important goals of historical studies is finding problems and systems of constraints (*situations fondamentales*) that must be analysed in order to understand existing knowledge, whose discovery is connected to the solution of such problems (Brousseau, 1983; Radford, Boero & Vasco 2000, p. 163). Obstacles are subdivided into epistemological, ontogenetic, didactic and cultural ones (Brousseau, 1989) and this subdivision points out that the sphere of the knowledge is considered isolate from other spheres. This perspective is characterised by other important assumptions (Radford, 1997): the reappearance in teaching-learning processes, nowadays, of the same obstacles encountered by mathematicians in the history; and the exclusive, isolated approach of the pupil to the knowledge, without taking into account social interactions with other pupils and teachers.

With reference to the above-presented schematic picture, we can summarise epistemological assumptions as follows:

(1) knowledge exists and represents the best solution of relevant problems;

epistemological obstacles recur either in history or in educational practice;

[2] Teachers can be induced to apply historical knowledge to classroom practice according to a naïve approach (as noted in Radford, 1997): for instance the educational introduction of a topic would take place just by the ordered presentation of all the historical references related with it.

(2) the sphere of knowledge is separated from educational and cultural spheres; pupils approach knowledge individually.

The crucial point is the following (Gadamer, 1975): is it possible, nowadays, to see historical events without the influence of our modern conceptions? As a matter of fact, we can explicitly accept the presence of our modern point of view: in other words, we can take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero & Vasco, 2000, p. 165). Concerning the nature of mathematics, "the historical approach encourages and enables us to regard mathematics not as a static product, with *a priori* existence, but as an intellectual process; not as a complete structure dissociated from the world, but as an on-going activity of individuals" (Grugnetti & Rogers, 2000, p. 45; see also the "voices and echoes" perspective: Boero & Al. 1997).

According to the socio-cultural perspective by L. Radford, knowledge is linked to activities of individuals and, as we above noted, this is strictly related to cultural institutions; knowledge is not built individually, but into a wider social context (Radford, Boero & Vasco, 2000, p. 164). The role played by the history must be interpreted with reference to different socio-cultural situations (Radford, 2003) and it gives us the opportunity for a deep critical study of considered historical periods.

With reference to the above-presented picture, we can summarize two different epistemological assumptions from the previous ones as follows:

(1) knowledge is related to actions required in order to solve problems; problems are solved within the socio-cultural contexts of the considered periods;

(2) knowledge is socially constructed; cultural institutions and beliefs of their own culture influence pupils.

3. THE SELECTION OF HISTORICAL DATA: THE HISTORY OF ALGEBRAIC NOTATION

We previously stated that the selection of historical data is epistemologically relevant to the historical introduction of a concept. A classical example (Radford, 1996 and 1997) is relevant to our research.

In 1842, G.H.F. Nesselmann characterised three main stages in the historical development of algebraic notation (see: Serfati, 1997):

Rhetorical Algebra
(Egyptians, Babylonians etc.)
Syncopated Algebra
(Pacioli, Cardan etc.)
Symbolic Algebra
(Descartes etc.)

(from) Words

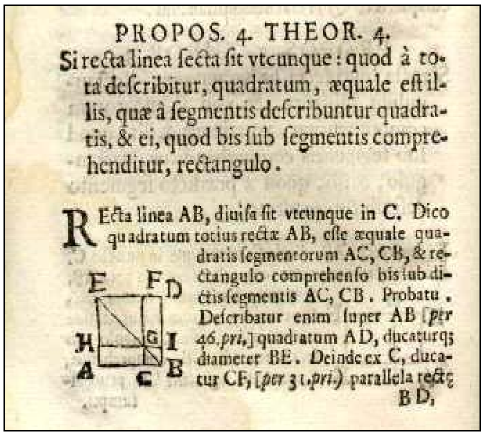
⇩

(to) Symbols

(concerning rhetorical algebra, original Nesselman's approach would be referred to Arabs: the interpretation of Babylonian mathematical texts is not so ancient).
This sequence can suggest a progressive elimination of non-mathematical verbal expressions: mathematical objects would be "purified by taking away all their insane physical substance" (Radford, 1997, p. 28); it suggests the existence of a definitive algebraic language, so that the historical development is the progressive approaching to our modern, pure expression. But this traditional summary can be considered as a full expression of the history of algebraic language? Important steps are still missing: for instance, we must remember the Greek "Geometric Algebra" (this denomination was given by H.G. Zeuthen, with reference to the 2nd Book of Eulclid's *Elements*) and the symbolism introduced by Diophantus of Alexandria (3rd-4th centuries).

Roots of the "Geometric Algebra" are related to Eudoxus of Cnidus (408-355 B.C.) who introduced the notion of a magnitude standing for entities such as line segments, areas, volumes (Kline, 1972, p. 48). No quantitative values were assigned to such magnitudes (so Eudoxian ideas avoid irrational numbers as numbers) and this allowed Greeks to give general results: the figure is referred to the 4th Proposition of the 2nd Books of *Elements*.

Nowadays this proposition is expressed by: $(a+b)^2 = a^2+b^2+2ab$, but in *Elements* only the picture gives the proof of this statement.



PROPOS. 4. THEOR. 4.
Si recta linea ſecta ſit vtcunque: quod à tota deſcribitur, quadratum, æquale eſt illis, quæ à ſegmentis deſcribuntur quadratis, & ei, quod bis ſub ſegmentis comprehenditur, rectangulo.

REcta linea AB, diuiſa ſit vtcunque in C. Dico quadratum totius rectæ AB, eſſe æquale quadratis ſegmentorum AC, CB, & rectangulo comprehenſo bis ſub dictis ſegmentis AC, CB. Probatu. Deſcribatur enim ſuper AB [*per 46. pri.*] quadratum AD, ducaturq; diameter BE. Deinde ex C, ducatur CF, [*per 31. pri.*] parallela rectæ BD,

(Rondelli, G.: 1693, *Euclidis Elementa*, Longo, Bologna, p. 80)

Six centuries later, Diophantus of Alexandria introduced an algebraic symbolism, and this is "one of Diophantus' major steps" (Kline, 1972, p. 139).[3] This symbolism is complicated and it is not complete (the main difference between Diophantine symbolism and our modern algebraic notation is the lack of symbols for operations and relations: Boyer, 1985, p. 202); Diophantine Algebra has been called *syncopated* (see: Boyer, 1985, p. 201; Kline, 1972, p. 140), but if we compare Diophantus' syncopation and, for instance, Cardan's one we realise that they are very different: Diophantus obtained fundamental achievements (Greek Algebra "no longer was restricted to the first three powers or dimensions": Boyer, 1985, p. 202), while European syncopated Algebra (15th-16th centuries) seems to be "a mere technical strategy that the limitations of writing and the lacks of printing in past times imposed on the diligent scribes that had to copy manuscripts by hand" (Radford, 1997, p. 29).

[3] Some Diophantine symbols appear in a collection of problems probably antedating Diophantus' *Arithmetica* (as noted in: Boyer, 1985, p. 204; Robbins, 1929).

If we rewrite our summary taking into account those new elements, we have:

Rhetorical Algebra (Egyptians, Babylonians etc.)	***Words***
Greek "Geometric Algebra"	***Pictures***
Diophantus of Alexandria	*Incomplete symbolism (?)*
Syncopated Algebra (Pacioli, Cardan etc.)	*Abbreviated words (?)*
Symbolic Algebra (Descartes etc.)	***Symbols***

So how can we describe the history of Algebra only in the sense of a progressive "purification", if we consider Geometric Algebra and Diophantine symbols?

4. FROM HISTORY TO DIDACTICS: EQUATIONS AND INEQUALITIES

Previous discussion underlines that algebraic processes have not been expressed by symbols for a long time, but the evolution of algebraic notation does not reflect just the progressive elimination of "insane physical substance" (Radford, 1997, p. 28). Several elements must be taken into account: for instance, it is important to point out that mathematical expression was initially oral. More generally, relevant non-mathematical elements must be considered: the development of western mathematical symbolism is to be framed into the correct cultural context, towards a systematization of human expression.

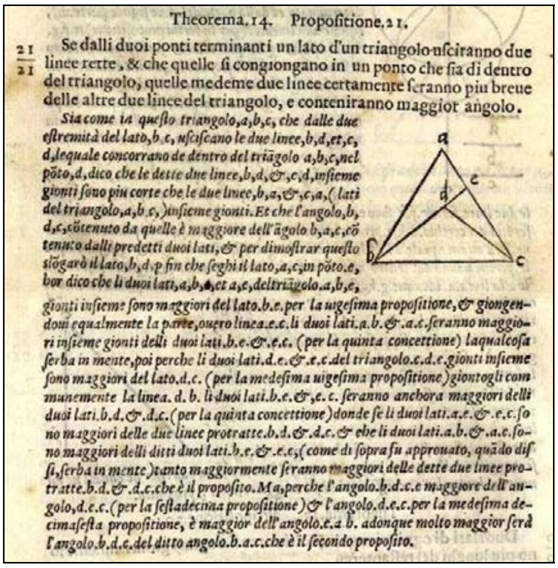
Historical evolution is complex: for instance, G. Lakoff and R. Núñez note: "It may be hard to believe, but for two millennia, up to the 16^{th} century, mathematicians got by without a symbol for equality" (Lakoff & Núñez, 2000, p. 376). Of course the role of "=" cannot be considered too simple: "Even an idea as apparently simple as equality involves considerable cognitive complexity. […] An understanding of what "=" means requires a cognitive analysis of the mathematical ideas involved" (Lakoff & Núñez, 2000, p. 377; Arzarello, 2000). In the first paragraph we noted several differences between equalities and equations, and other important differences can be mentioned (see: Lakoff & Núñez, 2000, p. 376).

Let us now sketch some historical references regarding equation and inequalities.

The history of equations is rich and different mathematical cultures in many part of the world dealt with processes that can be related to equations; in the Renaissance, the so-called *Regola d'Algebra* (algebraic rule) was the process for arithmetic problem solving based upon the resolution of an algebraic equation (Franci & Toti Rigatelli, 1979, p. 7).

As we shall see, the history of inequalities is not so rich. Ancient inequalities, too, were expressed by verbal registers; it is important to underline that an inequality (see the picture, referred to a geometric inequality dealing with 21st Proposition of the 1st Book of *Elements*) is often only the expression of an *inégalité*.

Some inequalities in the proper sense of *inéquation* can be related to the development of the Calculus, e.g. to majorizing/minorizing (see: Hairer & Wanner, 1996).[4] Let us now consider some texts published in 19th century; two treatises by P. Ruffini (1765-1822) were included in the 3rd-5th parts of *Corso di Matematiche* (Modena, Italy, 1806 and 1808).



Theorema.14. Propositione.21.

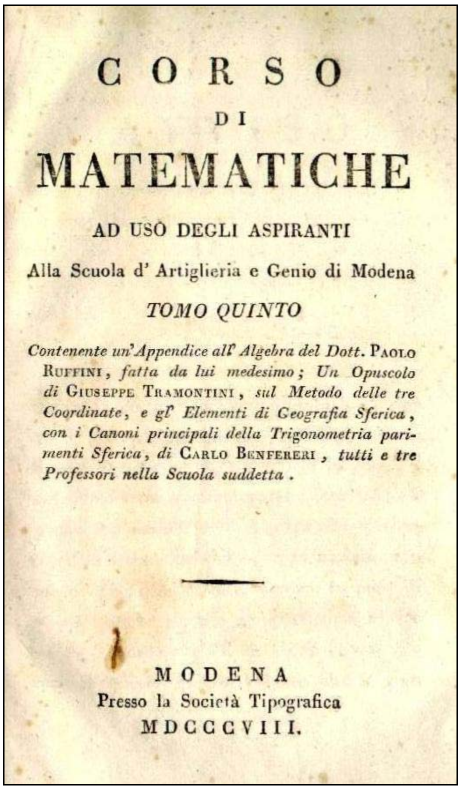
21/21 Se dalli duoi ponti terminanti un lato d'un triangolo usciranno due linee rette, & che quelle si congiongano in un ponto che sia di dentro del triangolo, quelle medeme due linee certamente seranno piu breue delle altre due linee del triangolo, e conteniranno maggior angolo.

Sia come in questo triangolo,a,b,c, che dalle due estremità del lato,b,c, uscischano le due linee,b,d,et,c,d,lequale concorrano de dentro del triãgolo a,b,c,nel põto,d,dico che le dette due linee,b,d,&,c,d,insieme gionti sono piu corte che le due linee,b,a,&,c,a,(lati del triangolo,a,b,c,)insieme gionti. Et che l'angolo,b,d,c,cõtenuto da quelle è maggiore dell'ãgolo b,a,c,cõtenuto dalli predetti duoi lati,& per dimostrar questo slõgarò il lato,b,d,p fin che seghi il lato,a,c,in põto.e, hor dico che li duoi lati,a,b,et a,c,del triãgolo.a,b,e, gionti insieme sono maggiori del lato.b.e.per la uigesima propositione,& giongendoui equalmente la parte,ouero linea.e.c.li duoi lati.a.b.&.a.c.seranno maggiori insieme gionti delli duoi lati.b.e.&.e.c. (per la quinta concettione) laqualcosa serba in mente,poi perche li duoi lati.d.e.&.e.c.del triangolo.c.d.e.gionti insieme sono maggiori del lato.d.c. (per la medesima uigesima propositione)giontogli communemente la linea. d. b. li duoi lati.b.e.&,e.c. seranno anchora maggiori delli duoi lati.b.d.&.d.c.(per la quinta concettione)donde se li duoi lati.a.e.&.e.c.sono maggiori delle due linee protratte.b.d.&.d.c.& che li duoi lati.a.b.&.a.c.sono maggiori delli ditti duoi lati.b.e.&.e.c,(come di sopra fu approuato, quãdo dissi,serba in mente)tanto maggiormente seranno maggiori delle dette due linee protratte.b.d.&.d.c.che è il proposito. Ma,perche l'angolo.b.d.c.e maggiore dell'angolo,d.e.c.(per la sestadecima propositione)& l'angolo.d.e.c.per la medesima decimasesta propositione, è maggior dell'angolo.e.a.b. adonque molto maggior serà l'angolo.b.d.c.del ditto angolo.b.a.c.che è il secondo proposito.

(Tartaglia, N., 1569: *Euclide Megarense*, Bariletto, Venezia, p. 27)

Let us propose some quotations:

- in the 3rd vol. (*Algebra*), p. 24, a property of equivalence for equations is explicitly stated: "Given the equation A–B–C = –D+E, we can carry the terms from the first to the second member and from the second to the first member, and we shall have: D–E = –A+B+C" (the translation is ours); it is important to underline that in the considered treatise no similar properties are stated with reference to inequalities;
- in the 3rd vol., p. 146, inequalities are proposed and solved in order to express some particular conditions for the solutions of some given equations. Frequently examples deal with similar conditions (in the 5th vol., *Appendice all'Algebra*, too): so inequalities are often combined to equations and to simultaneous equations in order to express some conditions.



CORSO

DI

MATEMATICHE

AD USO DEGLI ASPIRANTI

Alla Scuola d' Artiglieria e Genio di Modena

TOMO QUINTO

Contenente un' Appendice all' Algebra del Dott. Paolo Ruffini, *fatta da lui medesimo; Un Opuscolo di* Giuseppe Tramontini, *sul Metodo delle tre Coordinate, e gl' Elementi di Geografia Sferica, con i Canoni principali della Trigonometria parimenti Sferica, di* Carlo Benfereri, *tutti e tre Professori nella Scuola suddetta.*

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[4] We cannot forget the well known statement by J. Dieudonné in the *Préface* of his *Calcul infinitesimal* (Hermann, Paris 1980): "En d'autres termes, le Calcul infinitésimal, tel qu'il se présente dans ce livre est l'apprentissage de maniement des *inégalités* bien plus que des égalités, et on pourrait le résumer en trois mots: *majorer, minorer, approcher*".

Moreover, an interesting quotation can be considered with reference to the 20th century. P. Odifreddi writes: "A contribution by von Neumann was the solution, in 1937, of a problem posed by L. Walras in 1874. […] He noted that a model must be expressed by inequalities (as we usually do nowadays) and must not be expressed just by equations (as mathematicians were accustomed to do in that period), then he found a solution by Brouwer's theorem".[5]

So we can point out an interesting historical asymmetry: mathematicians usually expressed the problem to be solved by equations (Franci & Toti Rigatelli, 1979, p. 7); then, by inequalities (in the proper sense of *inéquation*), they expressed some conditions for the solutions of the considered equations. Moreover, in the history, the resolution of an inequality (*inéquation*) has been often obtained by solving an equation that practically replaced the assigned inequality. Social and cultural contexts must be taken into account: frequently the "practical solution" has been considered the main result to be obtained, much more important than the "field of possibilities". So a meaningful social importance has been attributed to the process by which the solution can be obtained (see the use of practical methods in order to improve the precision of the solutions: Hairer & Wanner, 1996).

5. FINAL REFLECTIONS

Although recently the autonomous role of inequalities (in the sense of *inéquation*, too) has been educationally recognised, in classroom practice there is still an operative dependence, a relevant "subordination". For instance, an inequality characterises a subset of the set of real numbers, frequently an infinite subset, a segment or a half-line. Main features of these subset are sometimes their "boundary points" (for instance, the ends of the segment): and they can be obtained by solving the equation obtained by replacing "<" with "=" in the given inequality.

Important metaphors (related to Arithmetics) are based upon "physical segments": for instance, we can propose the correspondence of a number with a distance that "can be measured by placing physical segments of unit length end-to-end and counting them" (Lakoff & Núñez, 2000, p. 68). Moreover: "When we move in a straight line from one place to another, the path of our motion forms a physical segment […]. There is a simple relationship between a path of motion and a physical segment. The origin of the motion corresponds to one end of a physical segment, the endpoint of the motion corresponds to the other end of the physical segment; and the path of motion corresponds to the rest of the physical segment" (Lakoff & Núñez, 2000, pp. 71-72). So in the framework of the *embodied cognition* the physical description of a segment (and of an half-line) has its origin in an end of the segment and its endpoint in the other end (in the case of an half-line, it goes on indefinitely): this underlines once again the importance of mentioned "boundary points".

Frequently the first step (and, in many cases, it is the main step) of the resolution of an algebraic inequality (*inéquation*) is the resolution of an equation: as a matter of

[5] Quotation from the website www.matematicamente.it/articoli; the translation is ours.

fact, in order to solve $a(x) < b(x)$ we must solve the equation $a(x) = b(x)$. So a forced but sometimes improper educational analogy can be considered, besides the historical asymmetry. This can cause some dangerous phenomena, in procedural sense: in order to avoid breaks between sense and denotation of algebraic expressions, L. Bazzini and P. Tsamir suggest a functional approach, an integrated introduction of equations and inequalities based upon the concept of function (Bazzini & Tsamir, 2002). The use of historical references, correctly considered in their own contexts, can help us to present the different roles and to underline the procedural differences between equations and inequalities.

Let us finally note that the 2nd order knowledge (we make reference to: Drouhard & Panizza, 2003) is relevant to the correct educational presentation of equations and inequalities. For instance, concerning the semiotic representation, the forced analogy between algebraic equations and inequalities, implicitly considered in their sequential presentation, can be referred also to employed representation registers: as a matter of fact, symbolic registers can suggest similar operative approaches to $f(x) = g(x)$ and to $f(x) < g(x)$. So the use of non-symbolic registers (for instance the visual register, directly involved in the functional approach) can be useful; of course the co-ordination of employed register is a very important point (Duval, 1995).

ACKNOWLEDGMENTS

The Author would like to thank Luis Puig and Luis Radford for the valuable help provided for improving this paper. A draft of this paper was presented at the "Séminaire Franco-Italien de Didactique de l'Algèbre 22" in Genova, Italy, 28 may, 2004: the Author wishes to thank Ferdinando Arzarello, Paolo Boero, Jean-Philippe Drouhard and Catherine Sackur who gave him, during the Séminaire, important suggestions and references.

REFERENCES

Arzarello, F.: 2000, Inside and outside: space, time and language in proof production, *PME-24*, 1, 23-38.

Bagni, G.T.: 2004, Historical roots of limit notion. Development of its representative registers and cognitive development, *Canadian Journal of Science, Mathematics and Technology Education*, forthcoming.

Bazzini, L. & Tsamir, P.: 2002, Algorithmic models: Italian and Israeli students' solutions to algebraic inequalities, *PME-26*, 4, 289-296.

Bell, J. & Machover, M.: 1977, *A Course in Mathematical Logic*, North-Holland, Amsterdam-New York-Oxford.

Boero, P.; Pedemonte, B. & Robotti, E.: 1997, Approaching theoretical knowledge through voices and echoes: a Vygotskian perspective, *PME-21*, 2, 81-88.

Boyer, C.B.: 1985, *A History of Mathematics*, Princeton University Press, Princeton, New Jersey (1st ed.: John Wiley & Sons, 1968).

Brousseau, G.: 1983, Les obstacles épistémologiques et les problèmes in mathématiques, *Recherches en Didactique des Mathématiques*, 4, 2, 165-198.

Brousseau, G.: 1989, Les obstacles épistémologiques et la didactique des mathématiques, Bednarz, N. & Garnier, C. (Eds.), *Constructions des savoirs, obstacles et conflits*, Agence d'Arc, Montreal, 41-64.

Drouhard J.-Ph. & Panizza, M.: 2003, What do the Students Need to Know, in Order to be Able to Actually do Algebra? The Three Orders of Knowledge, *CERME-3 Proceedings*, forthcoming.

Duval, R.: 1995, *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*, Peter Lang, Paris.

Fischbein, E. & Barash, A.: 1993, Algorithmic models and their misuse in solving algebraic problems, *PME-17*, 1, 162-172.

Franci, R. & Toti Rigatelli, L.: 1979, *Teoria e storia delle equazioni algebriche*, Mursia, Milano.

Furinghetti, F. & Radford, L.: 2002, Historical conceptual developments and the teaching of mathematics: from philogenesis and ontogenesis theory to classroom practice, English, L. (Ed.), *Handbook of International Research in Mathematics Education*, Erlbaum, Hillsdale, 631-654.

Gadamer, H.-G.: 1975, *Truth and Method*, Crossroad, New York (2nd edition: 1989).

Grugnetti, L. & Rogers, L.: 2000, Philosophical, multicultural and interdisciplinary issues, Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education*, Kluwer, Dodrecht, 39-62.

Hairer, E. & Wanner, G.: 1996, *Analysis by Its History*, Springer-Verlag, New York.

Heiede, T.: 1996, History of Mathematics and the Teacher, Calinger, R. (Ed.), *Vita Mathematica*, The Mathematical Association of America, 231-243.

Kline, M.: 1972, *Mathematical thought from ancient to modern times*, Oxford University Press, New York.

Lakoff, G. & Nuñez, R.: 2000, *Where Mathematics come from? How the Embodied Mind Brings Mathematics into Being*, Basic Books, New York.

Linchevski, L. & Sfard, A.: 1991, Rules without reasons as processes without objects, the case of equations and inequalities, *PME-15*, 2, 317-324.

Linchevski, L. & Sfard, A.: 1992, Equations and inequalities. Processes without objects?, *PME-16*, 3, 136.

Moreno, L. & Waldegg, G.: 1993, Constructivism and mathematical education, *International Journal of Mathematics Education in Science and Technology*, 24, 5, 653-661.

Nesselmann, G.H.F.: 1842, *Versuch einer kritischen Geschichte den Algebra, Nach den Quellen bearbeitet*, Reimer, Berlin.

Piaget, J. & Garcia, R.: 1989, *Psychogenesis and the History of Science*, Columbia University Press, New York.

Radford, L.: 1996, An Historical Incursion into the Hidden Side of the Early Development of Equations, Giménez, J., Campos Lins, R. & Gómez, B. (Eds.), *Arithmetic and Algebra Education*, Univ. Rovira I Virgili, Tarragona, 120-131.

Radford, L.: 1997, On Psychology, Historical Epistemology and the Teaching os Mathematics: Towards a Socio-Cultural History of Mathematics, *For the Learning of Mathematics*, 17(1), 26-33.

Radford, L.: 2003, On Culture and Mind. A post-Vygotskian Semiotic Perspective, with an Example from Greek Mathematical Thought, Anderson, M. & Al. (Eds.), *Educational Perspectives on Mathematics as Semiosis: From Thinking to Interpreting to Knowing*, Legas, Ottawa, 49-79.

Radford, L., Boero, P. & Vasco, C.: 2000, Epistemological assumptions framing interpretations of students understanding of mathematics, Fauvel, J. & van Maanen, J. (Eds.), *History in Mathematics Education*, Kluwer, Dordrecht, 162-67.

Robbins, F.E.: 1929, P. Mich. 620: A Series of Arithmetical Problems, *Classical Philology*, 24, 321-329.

Serfati, M.: 1997, *La constitution de l'écriture symbolique mathématique*, Thèse de doctorat de l'Université Paris 1.

Sfard, A.: 1991, On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coins, *Educational Studies in Mathematics*, 22, 1-36.

Tall, D. & Vinner, S.: 1981, Concept image and concept definition in Mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-69.

Tsamir, P., Tirosh, D. & Almog, N.: 1998, Students' solutions of inequalities, *PME-22*, 129-136.

PUPIL'S AUTONOMOUS STUDYING: FROM AN EPISTEMOLOGICAL ANALYSIS TOWARDS THE CONSTRUCTION OF A DIAGNOSIS

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Abstract: *Seeking viable didactic projects to help the learning process, we focus on the difficulties encountered by pupils when studying mathematics in an autonomous mode. Even establishing a diagnosis for such difficulties is a problem. We use here an epistemological concept, named "mathematical site", to describe the field of objects and relations which are relevant during the studying process. By integrating the "site" in our research at the theoretical and experimental level, we want to develop a didactic model for a diagnosis.*

Keywords: studying process, diagnostic approach, mathematical site, algebraic-functional framework, curriculum of the secondary cycle, epistemological and didactic analysis

1. INTRODUCTION

The personal work or *autonomous studying* in mathematics that the pupil must do during "physical" absence of a teacher has an important place in learning this discipline. From Bachelard (1949) until the first formulations of the concept of a student-learner by Chevallard (1988, 1995) for the didactics, the studying activity is always supposed to be inevitable for the learning and a particular position is assigned to the pupil in this activity. When Bachelard, more than one half-century ago, treated "the rational pupil", he defined this activity as "the studying of the composition of the knowledge." According to him, the pupil who studies "by repeating" this composition, makes "his own being". As for Chevallard (Chevallard, 1988; Chevallard, Bosch & Gascón, 1997), he considers the *student* as an acting subject, who takes part in teaching process and defines the *studying* as the missing link between teaching and learning.

Nowadays, we face an expansion of the systems concerning the studying process. Many new teaching systems were set up at the secondary school level in France ("modular" teaching, personal framed work, individualized aid), and, additionally, numerous organizations propose help outside the school (particular lesson, helping forums to the exercises on Internet etc.).

These systems can be interpreted only as the result of a need. And their disputed effects on the learning show that pupil's autonomous studying poses serious problems (Erdogan, 2001; Mathéron & Noirfalise, 2002).

The causes could be several and might be the objects of several researches. For instance, one can consider that pupils nowadays do not work hard or that they have lost the will of work a long time ago and that the school continues to deny this reality because it is not able to do anything.

Our hypothesis is that the success of the studying in autonomous mode is strongly related to the nature of the work expected and to the pupil's real possibilities with respect to this work. We think in particular that it is necessary to identify the actions we expect from a pupil, to understand as for as possible the difficulties he/she encounters and to analyze the cognitive and didactic conditions of their appearance. In other words, the construction of a *diagnosis* is the most important task for us in the problem of the autonomous studying.

Our aim in this paper consists in developing a didactic model adapted for the diagnosis. We use here an epistemological concept, the *"mathematical site"* to describe the field of objects and relations which are relevant during the studying process. Our model integrates the *site* both for theoretical and experimental purposes.

The concept of a "mathematical site" as organizer of a diagnosis

From epistemological and historical point of view, it is not possible to consider the mathematical objects as isolated and developed independently (see for instance Giusti, 1999). Each object has to be studied in narrow relations with other and any situation is to be related to other situations. Actually, to any specific question can be assigned a paradigmatic location in some "site" of relevant objects and relations. This "mathematical site" seen as an organized unit of knowledge, may be more precisely defined as follows.

The field of the mathematical items whose studying appears relevant – or is supposed such - for the understanding of a given scientific object O can be considered as a network of objects and relations, *the mathematical site of O.* Some of these objects and relations are visible, others are hidden. For any subject in position of a student, the mathematical site arises in the shape of a field of significance, investigation and experiment, a sufficiently stable field which confers on his studying a reliable reference.

On purpose to use the site for establishing a diagnosis, we conceive an epistemological and didactic analysis on two complementary levels:

At the first level, we analyse the ecologic conditions of objects and relations with respect to a given institution (Artaud, 1988). At the second level, we are concerned with the part of the site which is relevant, *a priori,* for didactic situations.

We claim that, on the basis of a comparison between what is revealed by the preceding analyses, a diagnosis can be so elaborated to explain the difficulties encountered by the pupils in actual situations of autonomous studying.

2. METHODOLOGY OF ANALYSIS AND DATA

We are interested by pupil's autonomous studying of the 10^{th} grade class (15-16 old pupils, entering high schools in France) within the "algebraic-functional" framework. We observed a class of a school located in a suburb of Paris, considered as "good" in the educational circles, especially for its scientific teaching. According to the teacher, the class is quite motivated in mathematics and shows constant progress. Besides, two thirds of the class wishes to choose the scientific class for the year to come.

Following the guide-lines described above, we begin by carrying out an analysis of the curriculum, to which most textbooks refer, throughout the secondary level teaching what concerns the main objects that are relevant for the studying of a particular object. We are focusing both on the introduction and the developments of these objects as mathematical *notions*, and on the semantic relations between these. Then we seek to delimitate and to organize the mathematical site corresponding to our questioned framework. Finally, we analyse an actual situation of autonomous studying (study of the variations of a function of the second degree) in the moment of an examination. This moment appears important to us. We regard it as the indirect witness of conditions of studying in the institution. The pupils are supposed to work individually for being ready to examination and the professor is supposed to satisfy mathematical and didactic conditions for this preparation.

3. ANALYSIS OF TEACHING CONTENTS AND ITS CORRELATION WITH THE CORRESPONDING MATHEMATICAL SITE

The "algebraic-functional" framework of the 10^{th} grade class

Equations and "*inequations*" *(= problematic inequalities*) are principal objects of this framework and take a large part in mathematical contents of the secondary cycle. At this level, the teaching of mathematics tries to develop pupils' capacities concerning the translation of data in equation and the solution of equations or inequations as basic mathematical capacities but also as transverse capacities. At the 10^{th} grade level, the studying of equations and inequations relies on two pillars: On one hand, algebraic calculations, studied since the secondary school, should become reliable before entry in precalculus courses (French basic "Analyse") of the 11^{th} grade; on the other hand, with the studying of non linear functions, the framework of equations and inequations is widened and reveals new dimensions.

The *"algebraic-functional"* framework at the 10^{th} grade should then be based, in theory, on a great number of objects and relations: its studying starts from the earliest secondary school and goes on after the 10^{th} grade.

As our goal here is limited to show how we build our model, we will only consider five concepts as object: *equation, order, function, algebraic expression* and *variation*. By the way, we will not examine here the concept of a number and avoid the questions relative to numerical calculations whose analysis reveals a high complexity.

At the secondary school

The curriculum of mathematics at the secondary school in France is composed of three parts: *geometrical studies, numerical studies, organization and management of data-functions*. The acquisition of algebraic language and its employment appears as one of the three objectives of *numerical studies*. In fact, we can formulate the aims of these numerical studies as (a) acquisition of numbers (operations and representations) on one hand (b) acquisition and use of algebraic language on the other hand.

The concept of an algebraic expression

Initiation in algebraic expressions starts as of the 6th class under the name of *"literal expressions"*. This introduction continues in 7th class with the distributive law of the multiplication with respect to the addition. Indeed this knowledge is to be considered as basic and essential for the algebraic solving of equations. The algebraic calculation with the meaning of writing transformation starts to develop in the 8th class (development/reduction of expressions, order properties relative to the addition and multiplication). Suggested work is articulated on two axes:

- *Use of literal expressions for numerical calculations*
- *Use of literal calculation in the setting in equation and the solution of various problems.*

Factorizations with the help of a common factor and remarkable identities occur in the curriculum of the 9th grade.

The concept of an equation

Introduction to equation solving also starts in the beginning of the 6th class, with the guess of the value of a missing data. In the 7th class, this introduction continues with an important goal: to understand an equation as a problematic equality concerning a number (pupils check the veracity of the equation by testing several values of the variables) This goal vanishes in the 8th class: at this level, the objective is the ability of formulating a problem with an equation, and of resolving so a class of problems leading to linear equations. Solving an equation of the form AB=0, where A and B are polynomials of degree 1, is in the curriculum of the 9th class, while the easiest cases of binomial equations will appear later, in the 10th class.

We thus note a simultaneous and "naturalized" introduction of algebraic expressions, and of equations in the 6th class, and an increasing complexity in next grades. In the later classes of the secondary school, techniques on algebraic expressions and equation algebraic solving develop at the same time.

The concept of an order

Ordering appears in the first years of the secondary school in the form of comparison of numbers and evaluation of magnitudes. The algebraic study starts in the 8th class with the effect on the order of additive operators and multiplicative ones when positive. In the 9th class, negative multipliers appear and, at the same time the solving of simple inequations by the use of the representation of the solutions on a graduated line. No introduction to the very concept of an inequation appears in the curriculum: the knowledge concerning the comparison of numbers involved in equations is not presented as an answer to a problem.

It should be noticed that the objects to be compared are more and more general from the 6th grade until the 9th: integers, non negative decimals, relative decimals, ... Then, the question of the comparison of the *letters* which refer to numbers evolves and becomes comparison of *algebraic expressions* and of *functional expressions* in the 10th class.

The concepts “function" and "variation"

In the first years of the secondary school, the curriculum envisages an implicit use of the concept of a function. In the 6th class for instance, the calculation of areas and perimeters involve expressions such as *"according to", "is related to»*. It is in the 9th class that the pupils discover the concept of a function, with the sense of a "mapping", a systematic process that associate to any element of a given unit another element of another unit. The study at this level is limited to *linear* functions and is mainly based on the idea of proportionality met in the preceding classes when *"organizing and managing data-functions"*.

The curriculum stresses that with the introduction of functions, the letters take a new statute: used in earlier classes in reference to sizes, then used to indicate unknown and/or indefinite values, they become "*variables"*.

However, no specific knowledge is associated to this denomination and, before the 10th grade the word "*variation*" only occurs in case of linear functions. Thus, the interplay between "function" and "variation of a variable" does not appear like a relation of conceptual nature: the concept of a function, as invested throughout the secondary school is highly numerical and limited to calculations. The concept works as an intermediary between "function" and "formula" and the sense corresponds mostly to the idea of an "algorithm".

In other side, the resolution of equations and inequations at the secondary school tends to be purely algebraic, namely with the factorization method which mainly consists in recognizing an algebraic pattern. No connection appears between algebraic and functional frameworks, a connection that would give a relevant field to the pupils of the 10th class.

Consequently, "objects/concepts" we consider as relevant for the autonomous studying in the 10th class within the "algebraic-functional" framework never seem really met by pupils. For some "good pupils", some connections are made between these objects but only under the form of a “meta-knowledge”.

In the 10th class

The curriculum of the 10th class is divided into three big chapters: *"Statistics", "Calculations and Functions", "Geometry*". The part "calculation and functions" deals with the study of the sets of numbers, that of the functions and the algebraic calculations. According to the writers of the curriculum, numerical and algebraic calculations should not constitute a matter of systematic revision but are to be treated through various sections. *"In particular, they will be treated in close connection with the study of the functions"*

In this way, by joining together under only one title the functions and the algebraic calculations, the curriculum of the 10th class seems to put an end to the previous distinction of the secondary school between *"numerical works"* and *"functions"*. It is probably with this objective that the curriculum lists capacities concerning the algebraic calculation under the title of *"function and algebraic expression";*

- *To recognize the form of an algebraic expression (addition, product, square, difference in two squares)*
- *To identify the sequence of the functions leading of x to f(x) when f is given by a formula.*
- *To recognize various writings of the same expression and to choose the form most adapted to the requested work (reduced form, factorized form ...)*
- *To modify an expression, to develop, to reduce, according to the aim.*

We see that the word "algebraic expression" starts to be used and the curriculum defines new objectives concerning its use. The algebraic study of the functions and the work on the algebraic expressions seem at the same time to be developed and to be introduced. Let us remind that the same phenomenon appeared at the secondary school for "literal calculation" and "equation solving". In other terms, it seems that no *situation* is needed to introduce the concepts themselves, but that the point is to develop tools presumed to be necessary to install precise *techniques*.

The part "function" of the curriculum is composed of the following sections: the qualitative study of the functions, increasing and decreasing functions, maximum and minimum of a function on an interval, functions of reference *(x^2, $1/x$)* and linear functions. The emphasis is particularly laid on the various aspects under which a function appears: graphs, numerical, qualitative. The curriculum takes also into account possible difficulties concerning the understanding of the concept of a variable and the notation *f*.

What concerns equations and inequations, the curriculum asks for the use of an array of signs to solve an inequation or to determine the sign of a function on an interval. It also asks for the combination of two different modes of resolution;

> *"For the same problem, we will combine the contributions of the graphic mode and of the algebraic resolution. We will specify the advantages and the limits of these various modes of resolution"*

For this combination, more explanations are found in the text accompanying the curriculum:

"A pupil having to solve an equation as $(x - 2)^2 = 9$ perceives rather easily that the equality is well checked for $x = 5$ and he is then satisfied to give this unique solution; he has even often some reserves to apply any technique making it possible to lead to the whole of the solutions. The chart of function $x \mapsto (x-2)^2$ which highlights well the existence of two solutions will encourage to exceed the first reasoning...

Another example is the use of the chart of function $x \mapsto x^2 + 3x - 10$ to conjecture that 2 is a solution of the equation $x^2 + 3x - 10 = 0$; calculation enables to check easily that it is well the case; it remains to pre-empt factorization a little and to check that $(x - 2)(x + 5)$ is a possible writing for the expression $x^2 + 3x - 10$ as well, leading so to the solution of the equation $x^2 + 3x - 10 = 0$

These examples show how the point of view of the functions can enrich the reflection on equation solving. These remarks apply even more to inequation solving, since the set of solutions is almost never reduced to a single value.

So, to solve the equation $f(x) = k$ on a given interval I, the pupil has to localize first the x-coordinates of the intersection points of the chart of the function with the line $y=k$, in order to conjecture that they are possible solutions of the equation. Then, he rewrites the equation as $f(x)-k=0$, factorizes this new expression using the conjectured values results and solves it by applying the cancellation law. Similarly, for the inequation $f(x) \leq g(x)$, one has to solve the equation $f(x) - g(x) = 0$, according to the above - described technique, to determine the sign of each factor, then to draw an array and, observing the sign rules, to determine finally the sign of $f(x)-g(x)$.

We see that this technique, originated at the secondary school, is rather powerful. Mainly based on an algebraic justification, the technique requires no specific work on the algebraic expressions in general, and no study of the functions. In this way, the variations of functions and their algebraic properties are not to be studied whereas they constitute one of the objectives of the curriculum!

Actually, the technique proposed by the curriculum poses several problems. We should mention for instance that in case of inequations whose set of solution is empty or is the whole set of real numbers. In those cases, the technique, based upon factorization and sign arrays, becomes impracticable: it is advisable to study the *variations* of the functions or to make some appropriate algebraic reasoning. In other words, a work of algebraic or functional type would be necessary, whereas it is dodged in favour of a single technique.

The site corresponding to the "algebraic-functional" framework of the 10th class

We propose here a diagram of the *site* relative to this field. In this diagram, when related by an arrow, x→y (in this order) the objects x and y are meant to occur together in some praxeologic system, with a higher degree of generality for y than for x: the arrow x→y marks a *relevance relation* and should be read as *"y is relevant to support the understanding of x"*.

Typically, this is a transitive relation: consequently, we drew only those arrows, which cannot easily be refined.

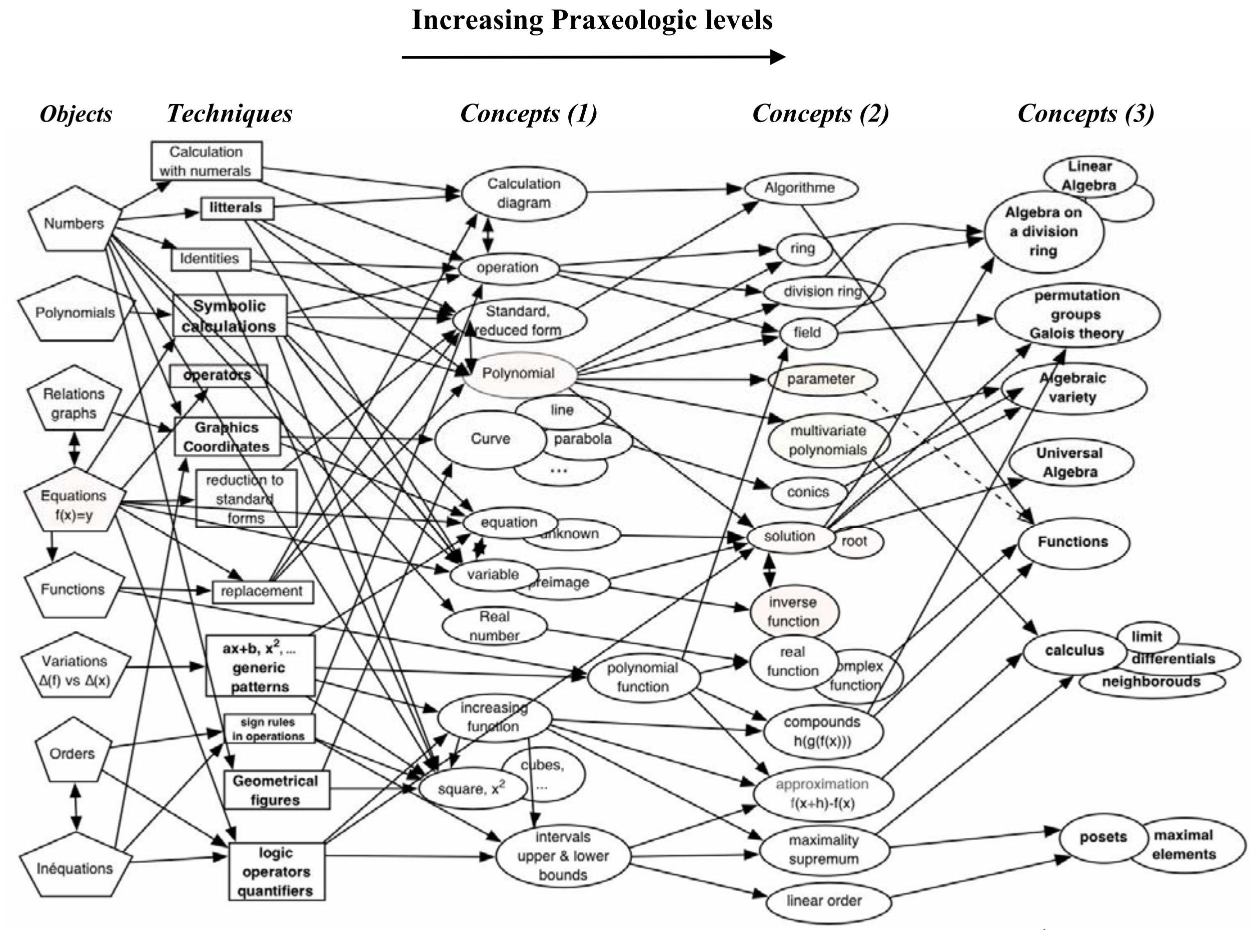


Fig-1: Diagram of the "site" relative to the "algebraic - functional" field of the 10th class

Beside the problematic objects themselves, the above diagram shows some of the techniques usually attached to them and presents also the main *concepts*, i.e. the relative theoretical tools. For the sake of clarity, what concerns concepts, we distinguish three main degrees of depth, although, from a rigorous epistemological point of view, no strict delimitation could be made.

Accordingly to the analysis we presented above, we can identify among the "*relevance relations*" (marked by arrows on the diagram) the ones which are sustained by the *didactic contract*, and many others, which are *de facto* excluded from the didactic process.

4. ANALYSIS OF AN ACTUAL SITUATION OF AUTONOMOUS STUDYING

The situation chosen is the study of variations of a second degree function on a given interval. The moment of pupils' autonomous studying is that of an examination. The precise mathematical situation corresponds to the first question of the examination and consists in studying the variations of the function $f(x)=4-(2x-8)^2$ on the interval $[4;+\infty[$.

Let us study in first how the pupils encounter the object “variation of a function”, starting with the curricular aspect.

The curriculum envisages initially a "qualitative" general study of the functions (definition, variations, chart). Further is presented the question of studying the variation of a function, i.e. deciding whether the function is increasing or decreasing on a given interval. Capacities defined by the curriculum give place to two quite distinct types of activities:

- *to describe the behaviour of a function defined by a curve, with an appropriate terminology or with the use of a variation-array,.*
- *to draw a chart compatible with a variation-array.*

The effect of the behaviour of a function on the order of the image values, mentioned in the comments, seems in the curriculum the only property which can be used as a tool for solving problems:

> *"The fact will be underlined that an increasing function preserves the order, while a decreasing function reverses the order; a formal definition is expected here".*

Then, what concerns the functions of reference (x^2 and $1/x$), the curriculum requires *"To determine the variations and to represent the functions graphically"* without specifying how to obtain the variations.

In order to show that a function is monotone increasing/decreasing on an interval, we know that the first technique consists in applying the definition, at the same time on a descriptive level (to specify the goal) and on an operational level (to check the answer): *a function is increasing (respectively decreasing) on an interval I, if for all real numbers a, b in I such as a<b, then f(a)<f(b) (respectively f(a)> f(b)).* According to this technique, it would initially be necessary to choose, if not given, the intervals where the function is increasing (or decreasing) and then to determine the order of *f(a)* and *f(b), a* being lower than *b*, what can be done according to different algebraic treatments.

Another way of determining the variation of a function is to see the function as a combination of other functions whose variations are known.

Now let us examine our situation of autonomous studying when according to the technique requested in our class. After the examination, the teacher was presenting the technique as follows:

> *"...That is to say a and b are two real numbers such as $4 \leq a < b$. Then $8 \leq 2a < 2b$ and $0 \leq 2a-8 < 2b-8$ thus 2a-8 and 2b-8 are two positive numbers, arranged in the ascending order. As the square function is increasing on* $[0;+\infty[$*, we can write;*
> $(2a-8)^2<(2b-8)^2 \Leftrightarrow -(2a-8)^2 > -(2b-8)^2 \Leftrightarrow 4-(2a-8)^2 > 4-(2b-8)^2$ *thus f(a)>f(b)*
> *Conclusion: for all a and b of* $[4;+\infty[$ *such as $4 \leq a < b$, f(a)>f(b) thus the function $f(x)=4-(2x-8)^2$ is decreasing on the interval* $[4;+\infty[$

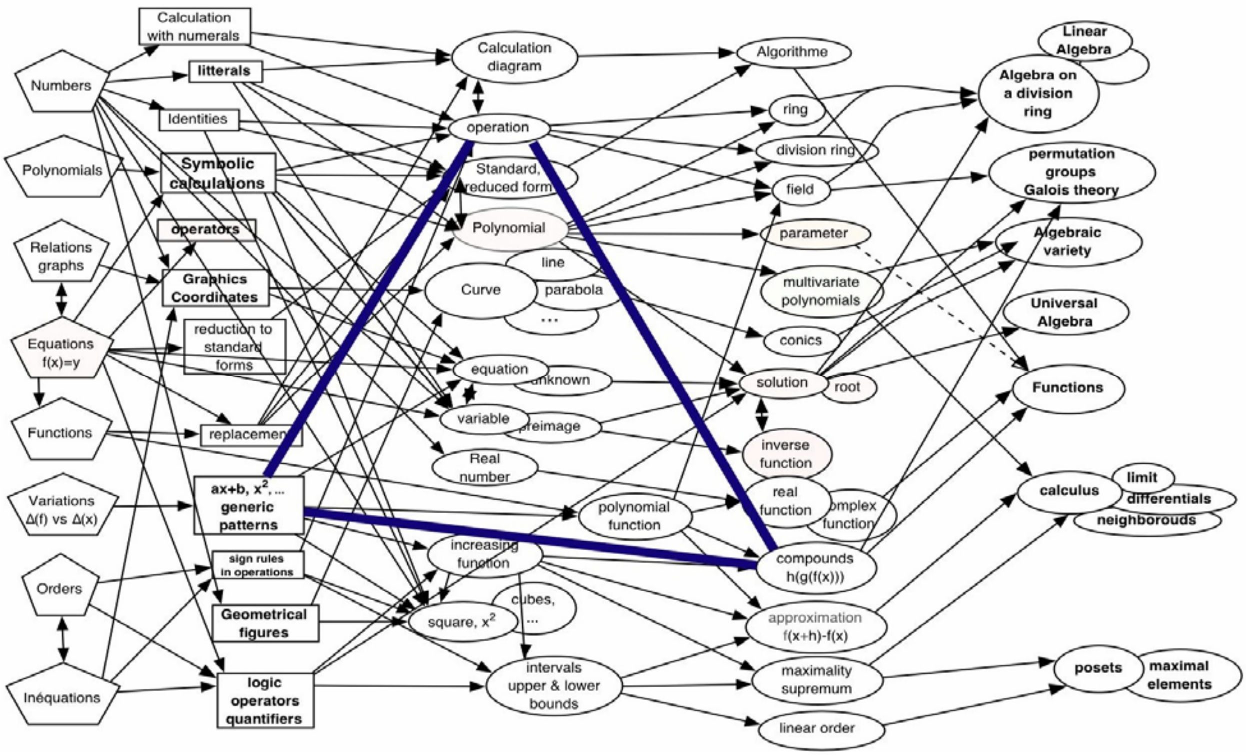
It is clearly seen that the core of this technique is the reference to the square function, of which the growing becomes the decisive criterion when comparing the squares of two numbers. But we must note that such a reference is not self evident:

the comparison rule appears initially in a "natural" way in the context of numerical calculations, but the use of this rule as a *tool* for the situation would require to take into account *the growth of the square function*, hence a passage to the "functional register" which, here, has been skipped.

Actually, the problem is that the requested technique is a mixture of two different techniques: algebraic and functional. The pupils who are working in the algebraic register, are asked to pass to the functional register in order to obtain a justification which cannot exist with no knowledge on the *composition of functions* or, for the least, with no knowledge on the *change of variable*.

While the algebraic calculations warrant the building scheme of the function x--> *4-(2x-8)²*, the variation of the square function intervenes in a non dispensable way for the technique realization and constitutes the very object of the didactic contract: it is precisely this type of use of the square function which constitutes the didactic project of the teacher and the aim of the curriculum.

Now let us try to locate on the diagram of the site the relations which the didactic contract establishes in connection with the object of knowing "variation of a function".



The relations offered to pupils' work in class appear in the site map as three new links which are far from being the most relevant. With other words, these relations appear only like "artificial" links, of solely didactic nature.

5. CONCLUSION

Our analysis around the concept "of a mathematical site" shows that the matter studied is presented at the pupils in a split up form, with fragmented objects and relations. The difference between the relations identified on the site and those created by work in class, attests that the relevant objects are evacuated from the scene, they do not form part of the objects to be taught on this level. In particular, the means for the validation of the techniques taught in the class are excluded from the process of teaching on this level; such a lack would explain impossibility for the pupils of controlling the procedures they follow during their autonomous study.

The analysis of the pupils’ exams’ papers confirms this impossibility. Only one pupil (out of 33) succeeded in making this exercise. Most pupils seemed to know that they must use anyway the growth of the square function but did not guess where and how. Apparently, they obeyed only the institutional request. Moreover, the teacher, reconfirming the request, allotted 0.5 point (out of 1) to any pupil who mentioned, in one way or another, the growth of the square function, even when no coherence with the rest of the answer could be detected.

Thus the institutional constraints on the didactic contract are such that the study in autonomous mode becomes inaccessible to the pupils and highly difficult to integrate into the didactic project of the teacher.

This conclusion leads us to raise further questions concerning the autonomous part of the process of studying. In particular, it seems essential to us to reconsider the role of the studying activity and its organization, to develop in class a genuine "*culture de l'étude*".

BIBLIOGRAPHY

ARTAUD, M. (1988), Introduction à l’approche écologique du didactique. L’écologie des organisations mathématiques et didactiques. *In actes de la IXème école d’été de didactique des mathématiques,* pp 101-139. Houlgate : ARDM et IUFM de Caen.

BACHELARD, G. (1949), *Le rationalisme appliqué,* Collection PUF, Paris, 1986.

CHEVALLARD Y. (1988), The student-Learner Gap, *Proceedings of the Third Conference on the Theory of Mathematics Education*, Anvers, 1-6 (unpublished).

CHEVALLARD, Y. (1995), ‘Familière et Problématique, la figure du professeur’, *VIII^{e} Ecole d’été de didactique des mathématiques*, Saint - Sauves.

CHEVALLARD, Y., BOSCH M. & GASCÒN J. (1997), *Estudiar matemáticas, el eslabón perdido entre enseñanza y aprendizaje*, Coll. Cuadernos de Educación, Horsori.

ERDOGAN, A. (2001), *La dimension didactique des forums de questions mathématiques*. Mémoire de DEA, Lirdhist, Université Lyon 1, France.

GIUSTI, E. (1999), *La naissance des objets mathématiques*. Translated from Italian, Les éditions ellipses, 2000.

MATHERON, Y., NOIRFALISE, R. (2002), Point de vue : L'aide Individualisée, entre système didactique auxiliaire inutile et déficit d'analyse didactique entravant son efficacité et son développement'. *Petit x* n°60, Grenoble.

AN EXPERIENCE WITH PARABOLAS

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Abstract: *In this paper we present an experience with mathematics teachers, using the computer as a tool and a dynamic software to study the parabola under two different points of view: canonical and developed, the first one unknown to the teachers and the second one, part of their previous knowledge. Having as theoretical framework Hoyles´ ideas and Fischbein´s aspects of mathematical knowledge, our intention was to analyse the changes the new approach would cause in the old one and whether the teachers would use it in their classroom. We have concluded that the teachers remain making interactions between intuitive and algorithmic aspects of the new content and we claim that they will not use it in their classroom.*

Introduction

Working on a known subject, under a new point of view, using the computer as a tool and a dynamic software, we had the intention to observe if the study of a new approach would contribute to the teachers' knowledge and would cause a change in their practice.

We have started from the following hypothesis: the teacher will not develop a certain content in the classroom unless he believes his own knowledge is enough to teach this content and if he has given mathematical meaning to it.

According to Hoyles (1999), mathematical meaning comes from connections between old and new facts. The planned use of software must propitiate these connections and not simply add up new contents.

According to Fischbein (1993), mathematical knowledge comes from the interaction between its intuitive, algorithmic and formal aspects.

Methodology

We have used for this experience two open activities whose aim is the study of the parabola, which were developed by CREEM (1992) to be studied using the software FONCTION. They have been adapted and published by a group of teachers from PUC/SP (SILVA, 2001). In the first of these activities, one studies the role of the parameters on the canonical form ($f(x) = a(x - u)^2 + v$). In the second one, the role of the coefficients on the developed form ($f(x) = ax^2 + bx + c$). In both, the subject is stimulated to act, observe, realise, reflect, make conjectures and validate them with the aid of the dynamic software.

The computer tool has been chosen to facilitate visualisation and allow the subject to make conjectures validate them by means of a lot more examples than in a paper and pencil environment. The FONCTION graphic interface allows dynamic access to function parameters and/or coefficients change on one hand and visualisation of the corresponding graph modification on the other.

This investigation described here was made with mathematics teachers from São Paulo´s public schools who participate in the research project "Mathematical thinking: research and teaching core", whose co-ordinators are Dr. Saddo Ag Almouloud and Dr. Tânia Maria Mendonça Campos.

The activities

In the activity with the study of canonical form (activity 1), the quick visualisation made easy by the software intend to show how parameter a variation alters the parabola concavity, while parameters u and v variation provoke translation. Subjects may see yet u and v as vertex co-ordinates.

In the activity with the study of the developed form (activity 2) we can see how coefficients a, b and c modify the graph. Computer helps subjects to realise that: coefficient c variation provokes vertical translations, while b leads to a rigid movement with vertices describing a new parabola. The a variation provokes a non-rigid movement.

By observing the graph it is possible to relate both forms, canonical and developed. By means of algebraic development one may obtain generic expressions for u and v as functions of a, b and c.

Workshops

We had observed three teachers from the group. Two of them (C. and O.) have worked in pairs and the other (A.), individually. Neither one knew parabola canonical algebraic form although all of them were familiarised with the developed one.

Activity 1

Parameter a

With the help of a dynamic software subjects realised that a>0 implies concavity up and a<0, concavity down and parabola "opens" and "closes". C. says: "I thought this movement (opening and closing) was linked to x value and it is not." As C. has not felt necessity to formally validate this statement (*formal aspect*) (or at least has not expressed it), we may say that for him observation and experimentation are enough to accepting a conjecture (*intuitive aspect*).

2

Parameter u

In questions involving parameter a variation, u and v remained equal zero and subjects have not connected them with the vertex co-ordinates, although they have obtained them by looking at the graph.

The variation of u parameter made easy by the software dynamics has allowed the perception (*intuitive aspect*), by subjects, of u as the vertex abscise, but they have only given vertex co-ordinates by means of u and v after the researcher intervention. Apparently they have not perceived that that parabola have not deformed because a value was constant and this leads us to say that their attention is just on u parameter, without making connections with a.

Parameter v

Observing the parabola movement when parameter v varies has finally allowed subjects to perceive that u and v are the vertex co-ordinates (*intuitive aspect*). C. starts to press u key instead of v one and when he sees the “wrong movement” (*intuitive aspect*)on the screen, quickly corrects himself. This leads us to say that C. has learned the differences between movements provoked by u and v, although he has not used the word translation to describe any of them. O. does not do algebraic calculus (*algorithmic aspect*) to determine vertex co-ordinates because he has learned they are in evidence in the algebraic expression of canonical form (*algorithmic aspect*).

Activity 2

To complete the table at the beginning of activity 2 (see annexe 2) all three subjects have used the graph to determine vertex co-ordinates and expressions given by the software to get each canonical form. O. mumbles something about “… how is it possible?” (*intuitive aspect*) meaning the algebraic manipulation for the passage from developed form to canonical one but do not goes on in his idea (*algorithmic aspect*). He seems pretty satisfied with the results made easy by the computer.

Parameter a

They do not have difficulties in understanding this parameter as been the same one in canonical form.

Parameter c

Subjects did observe the movement c parameter variation provoke: O. claims it is a translation in the vertical axe; A., just as a movement.

Parameter b

In answering the question “When a and c values are fixed and b varies which movement describes parabola’s graph?”, A. says: “The parabola walks, changes the

3

quadrant, changes the vertex, but has the same aperture". O. says: "Look, it is a rotation!". C.: "It describes a parabola with concavity in the opposite direction and vertex in (0, c)".

To complete the table at the end of activity 2, all three of them take the canonical form and u and v values from the software.

In a general way, none of them has demonstrated any preoccupation with formal validity for statements they have formulated (*formal aspect*), looking pretty satisfied with *intuitive and algorithmic aspects* of the new facts presented to them.

Theoretical framework

According to Hoyles (1999),

> "Mathematical meanings derive from connections – intramathematical connections which link new mathematical knowledge with old, shaping it into a part of the mathematical system; and extra-mathematical meaning derived from contexts and settings which may include the experiential world." (Hoyles, 1999.)

Looking for mathematical meaning on the study of parabolas, we have proposed two activities aiming to work on it under two different points of view, a known one (developed form) and a new one (canonical form), which subjects have declared not to know yet. The software dynamics was a chosen factor with the intention of promoting connections between old and new mathematical knowledge.

According to Fischbein (1993), mathematical knowledge has three aspects: formal, algorithmic and intuitive, and a subject must be apt to do constant and dynamic interaction between them.

The formal aspect is due to axioms, definitions and theorems and it is essential because

> "Axioms, definitions, theorems and proofs have to penetrate as active components in the reasoning process. They have to be invented or learned, organized, checked, and used actively by the students." (Fischbein, 1993.)

This aspect by itself is not enough since it does not give necessary background for individuals to dominate resolution procedures, which constitute the unavoidable algorithmic aspect inasmuch as

> "We need *skills* and not only *understanding*, and skills can be acquired only by practical, systematic training. ... Mathematical reasoning cannot be reduced to a system of solving procedures. ... Solving procedures that are not supported by a formal, explicit justification are forgotten sooner or later." (Fischbein, 1993.)

Nevertheless, we must not ignore the intuitive aspect that is related to intuitive cognition, understanding and solution, because intuitive interpretations are profoundly rooted in individual experience and they can cause a coercive action.

"... Intuitions may play a facilitating role in the instructional process, but, very often contradictions may appear: Intuition may become obstacles - epistemological obstacles (Bachelard) - in the learning, solving, or invention processes." (Fischbein, 1993.)

Related to the use of computer

"We have shown that technology can change pupils' experience of mathematics but with several provisos: the users of the technology, (teachers and students), must appreciate what they wish to accomplish and how the technology might help them; the technology itself must be carefully integrated into the curriculum and not simply added on to it, and most crucial of all, the focus of all the activity is kept unswervingly on mathematical knowledge and *not* on the hardware or software." (Hoyles, 1999.)

Conclusions

In the activity concerned with canonical form, subjects did connected it with the developed one regarding a parameter, after they visually acknowledged that parabola concavity changes due to its variation. The software was an essential tool for this observation, more than that; it acted as the *formal aspect* (instead of *intuitive*).

Concerning vertex co-ordinates, we can say that a new knowledge was added to the old ones but not necessarily related to them since, once more, just the visual aspects were enough for subjects.

Although subjects were involved in the solution of activity 1, expressions as "Wonderful!" (A.); "One doesn't need to do; one already knows. The vertex is (u, v)" (O.); "I thought this movement (parabola opening and closing) was related to x value and is really independent of it" (C.) show us that they do not see the needing of a certain *formalism (formal aspect)* to explain what they have acknowledged and that visualisation made easy by the software is just enough (*intuitive aspect*).

A. did another important connection between the new knowledge and the old one when he decided to solve a second degree equation, presented to him in canonical form, without using the more usual development of the square and subsequent Baskara's formula (*algorithmic aspect*):

$$\left(x-\frac{5}{2}\right)^2-\frac{21}{4}=0 \qquad \left(x-\frac{5}{2}\right)^2=\frac{21}{4} \qquad x-\frac{5}{2}=\pm\sqrt{\frac{21}{4}} \qquad x=\frac{5}{2}\pm\sqrt{\frac{21}{4}}\;.$$

We may say that, to A., the example is sufficient and he does not look for any kind of formal justification for the algorithm above (*formal aspect*).

The *algorithmic* aspect was not explored by subjects as well when they needed to transform canonical form into developed one or vice-versa, as they have used the software given data.

Summing up, as conclusion, we can say that the use of dynamic software was decisive factor to subjects to *"perceive" (intuitive aspect)* the role played by parameters and/or coefficients variation onto the parabola's graph. Besides, we could comprehend that for the observed subjects *"formal validation" (formal aspect)* is not necessary when they *"see" (intuitive aspect)* certain facts on the screen.

We then understand that these subjects do not merge parabola canonical form into their practice, since they apparently did not do interrelations between *formal, intuitive and algorithmic aspects* of this mathematical knowledge.

For further research, we highly suggest that researchers look for ways to make subjects to accept and promote the necessity of formal aspects "acting as active components in the reasoning process" (Fischbein, 1993). We also recommend formal aspects to be strongly interrelated to *intuitive and algorithmic aspects* all the way during any learning and teaching situation.

References

[1] CREEM: 1992, *Activités Mathématiques avec Imagiciels. Fonctions Numeriques*, Ministère de L'Education Nationale et de la Culture, France.

[2] Fischbein, E.: 1993, "The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity" in Biehler, R., Scholz, R.W., Strasser, R., Winkelmann, B. (Org.), *Didactics of mathematics as a scientific discipline*, Kluwer, Deutschland, pp. 231-240.

[3] Hoyles, C. : 1999, "Steering between skills and creativity: a role for the computer?" in Igliori, S. B. C., Maranhão, M. C. S. A. (Eds.), *Educação Matemática Pesquisa*, v.1, n.1, EDUC, São Paulo/SP, pp. 99-116.

[4] Silva, B.A., Manrique, A.L., Bianchini, B.L., Dubus, M.T.G., Souza, V.H.G.: 2001, *Atividades para o estudo de funções em ambiente computacional*, IGLU, São Paulo.

Annexe 1 - Activity 1

- Let us work with 2nd degree canonical form ($y = a(x - u)^2 + v$, with a, u and v real numbers, $a \neq 0$).

I.

1. Choose values for a, u and v in such a way that $f(x) = x^2$. What are they?
2. Observe the parabola.
3. Change a value to obtain other parabolas.
4. For a > 0, observe parabola movement when a value increases. Describe this movement.
5. For a > 0, observe parabola movement when a value decreases. Describe this movement.
6. Describe what happens when a < 0 varies.
7. Represent on paper, in the same co-ordinate axes, three parabolas. In each case, write down parameter a value and vertex co-ordinates.
8. Get, with software, the graph of $f(x) = \frac{x^2}{2}$. Draw this graph.

   a = Vertex co-ordinates:

9. Get, with software, the graph of $f(x) = -\frac{x^2}{3}$. Draw this graph.

   a = Vertex co-ordinates:

II.

1. Choose values for a, u and v in such a way that $f(x) = 2(x-1)^2$. What are they?
2. Observe parabola.
3. Change u values many times to obtain new parabolas (parabola movement is called ***horizontal translation***).
4. In each situation, observe parabola and its algebraic expression.
5. Represent, on paper, three different situations. Write down, in each case, u values, algebraic expressions and vertex co-ordinates.

III.

1. Choose values for a, u and v in such a way that $f(x) = 2(x-1)^2 + 1$. What are they?
2. Observe parabola.

7

3. Change v values many times to obtain new parabolas (parabola movement is called ***vertical translation***).

4. In each situation, observe parabola and its algebraic expression.

5. Represent, on paper, three different situations. Write down, in each case, v values, algebraic expressions and vertex co-ordinates.

IV. In each case, determine vertex co-ordinates.

(a) $f(x) = 3(x+1)^2 + 3$.

(b) $f(x) = -2(x-3)^2 - \frac{1}{2}$

(c) $a = 4; \quad u = -\frac{3}{2}; \quad v = 4.$

(d) $a = \frac{1}{2}; \quad u = \frac{1}{3}; \quad v = -2$.

V. Relate u and v with vertex co-ordinates.

Annexe 2 - Activity 2

- Let us work with the 2nd degree developed form ($y = ax^2 + bx + c$, with a, b and c real numbers, $a \neq 0$).

1. Choose a, b and c in such a way that $f(x) = x^2 - 2x + 1$. What are they?

2. Observe parabola.

3. Determine parabola's vertex co-ordinates.

4. Get canonical form $f(x) = a(x - u)^2 + v$. What are a, u and v values?

5. Complete the table.

Function	Vertex abscise x	Vertex ordinate y	u	v	Canonical form
$f(x) = x^2 - 2x - 3$					
$f(x) = 2x^2 - 4x + 5$					
$f(x) = -x^2 + 5x - 1$					

6. From $ax^2 + bx + c = a(x - u)^2 + v$ we can get u and v as functions of a, b and c.

Developing second member of the equation we get:

$$bx + c = (-2au)x + (au^2 + v).$$

By Polynomial Identity Principle we obtain

$$b = -2au \quad \text{and} \quad c = au^2 + v.$$

Determine, from these two equalities, expressions for u and v as functions of a, b e c. (Remember: $\Delta = b^2 - 4ac$.)

7. Using these expressions, test your answers for item 3.

8. Given $f(x) = 2x^2 - 5x + 3$:

(a) determine parabola vertex co-ordinates;

(b) give an algebraic expression for the function, in canonical form;

(c) determine the points where parabola intersects the co-ordinate axes;

(d) draw function graph.

9. Let us use software to observe coefficients a, b and c role on parabolas´ graph configuration and position.

Choose a parabola: ____

(a) When a and b values are fixed and c varies which movement describes parabola's graph?

(b) When c and b values are fixed and a varies which movement describes parabola's graph?

(c) When a and c values are fixed and b varies which movement describes parabola's graph? (Try to follow vertex movement.)

10. Complete the table.

Canonical form $f(x) = a(x - u)^2 + v$	Developed form $f(x) = ax^2 + bx + c$	Factored form $f(x) = a(x - x')(x - x'')$
		$f(x) = 2(x - 1)(x + 3)$
	$f(x) = x^2 - 4x - 5$	
$f(x) = 2(x - 5/4)^2 + 1/8$		

PERCEPTUAL SEMIOSIS AND THE MICROGENESIS OF ALGEBRAIC GENERALIZATIONS[1]

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Abstract: *This paper deals with the problem of algebraic generalizations of elementary geometric-numeric patterns. It focuses on understanding the role played by the various semiotic systems mobilized by students in the progressive process of perceptual apprehension of a pattern and its generalization. The microgenetic analysis of the mathematical activity of two small groups of students in a Grade 9 class shows how making recourse to semiotic resources, such as gestures, language, and rhythm, allows the students to objectify different aspects of their spatial-temporal mathematical experience. The analysis also shows some connections between the syntax of the students' algebraic formulas and the semiotic means of objectification through which the formulas were forged, thereby shedding some light on the meaning of students' algebraic expressions.*

Keywords: generalization, gestures, meaning, objectification, rhythm, semiotics, semiotic-cultural approach, signs, syntax.

INTRODUCTION AND THEORETICAL FRAMEWORK

Resorting to a small number of characters to form an expression, algebraic symbolism allows us to convey an idea that, usually, in natural language, may take one or several lines. Algebraic symbolism does not possess the rich arsenal of resources such as adverbs, adjectives and noun complements that play a crucial role in written and oral languages. Instead, it offers to its users a precision and succinctness governed by a few syntactic rules. However, the ability to grasp how this precision and succinctness works often becomes difficult for students of different ages, as is reflected by the large amount of research devoted to the understanding of students' errors (see e.g. Matz, 1980; Kaput and J. Sims-Knight, 1983). Regardless of their theoretical orientation, the research results agree on this: algebraic syntax is not transparent.

[1] This paper is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

In our previous research, we have focused on the investigation of the meaning with which students endow elementary algebraic expressions. Our research has been encompassed by a semiotic-cultural perspective that rests on the idea that learning is accomplished through the use of diverse semiotic systems. Indeed, even accurate discourse is unable to lead students directly to the object of learning, for learning is entailed by meaning and interpretation. Thus, to learn to generalize geometric-numeric patterns amounts to learning to see and to interpret a finite number (usually very few) of concrete objects or signs in a different way. To learn to generalize means to "notice" (Mason, 1996) something that goes beyond what is actually seen. Ontogenetically speaking, this act of noticing unfolds in a gradual process in the course of which the object to be seen emerges progressively. This process of noticing we have termed a process of *objectification*. To make something apparent (which is the etymological sense of objectification) learners and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators, and so on). These artefacts and signs used to objectify knowledge we call *semiotic means of objectification* (Radford, 2003).

One of the basic principles of our methodological approach to the investigation of the students' algebraic generalizations can be stated as follows. Our comprehension of the meaning with which the students endow their algebraic expressions can be deepened by investigating the semiotic means of objectification to which the students have recourse in their attempt to accomplish their generalizations.

This methodological principle is interwoven with the theoretical tenet of our research approach mentioned above, namely, that learning is essentially a social process of objectification mediated by a multi-systemic semiotic activity.

In previous works we have discussed the prominent role of gestures and language in students' processes of knowledge objectification. We have provided evidence of the key role of deictic activity, both at the level of gestures (like in pointing) and at the level of language (e.g. when students use terms such as *this* and *that*)[2]. In more precise terms, in our study of students' semiotic mechanisms through which the mathematical structure of a pattern is revealed, we have found a rich process of objectification in which the mathematical structure of the pattern is ostensibly asserted by gestures and linguistic key terms (Radford 2002, 2003).

Often, the students' objectification of the conceptual categories required in the generalization of patterns takes the form of a process of *perceptual semiosis*, *i.e.* a process relying on a use of signs dialectically entangled with the way that concrete

[2] By deictic activity we mean the activity embedded in social communicative processes where actions (e.g. pointing gestures), linguistic units (e.g. 'top', 'bottom'), etc. allow one to refer to the objects in the universe of discourse. It is the contextual circumstances which determine their referents. As such, deictic terms depend heavily on the context (see Nyckees 1998, p. 242 ff.) and have a particular function in dialogical processes.

objects become perceived by the students. In this paper we want to deepen our analyses in order to better understand the students' processes of perceptual semiosis. We are interested in understanding the dialectical relationships between the various semiotic systems mobilized by the students in making sense of generality as expressed through algebraic symbolism. We will focus on the work of two small groups of 3 students each, during a regular Grade 9 mathematics lesson. In the next section, we describe the task and some elements of the mathematical lesson and of our methodology.

METHODOLOGY

The data reported here comes from a 5-year longitudinal study, collected during classroom activities. The activities are part of the regular school teaching lessons, as framed by the Ontario provincial Curriculum of Mathematics. In these activities, the students spend a substantial period working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts a general discussion allowing the students to expose, confront and discuss their different solutions. In addition to collecting written material, tests and activity sheets, we have three or four video-cameras each filming one group of students. Subsequently, transcriptions of the video-tapes are produced. Video-recorded material and transcriptions allow us to identify salient short passages that are then analyzed using techniques of qualitative research in terms of the students' use of semiotic resources (details in Radford, 2000).

The mathematical problem on which we will focus here was the first one of a three-problem math lesson. This problem dealt with the study of an elementary geometric sequence (see Figure 1). The students were required to continue the sequence up to Figure 5 and then to find out the number of circles on figures number 10 and number 100. Subsequently they were asked to write a message indicating how to find out the number of circles in any figure (*figure quelconque*, in the original French), and then to write an algebraic formula for the number of circles in figure number *n*.

Figure 1

In the next section, we discuss two examples of perceptual semiosis and the role played by the latter in the students' elaboration of their algebraic formulas. While one of the processes of perceptual semiosis led to the formula "(n+1) + (n+2)", the second process led to the formula "n×2+3". As we shall see, the study of the microgenesis of students' generalizations provides us with rich information about the meaning of students' algebraic symbolism. It will become apparent that the students' apprehension of the pattern and the building of generality are underpinned by a complex articulation of written signs, words and gestures.

PROCESSES OF PERCEPTUAL SEMIOSIS

First example:

The first group is formed by three students, Doug (left), Alice (center), and Mireille (right)[3]. After drawing figures 4 and 5, Doug says:

"So we just add another thing, like that" *(when he utters the last word he makes a sequence of gestures*).

Figure 2. Excerpt of Doug's sequence of rhythmic gestures.

Since the word 'thing' does not have a clear referent, Doug immediately adds the expression "like that". Interestingly, the deictic 'that' does not refer to something concrete on the sheet where the figures have been drawn, but to something else, something that is ostensibly shown by a rhythmic sequence of six gestures iconically suggesting inclined lines (see Figure 2). Doug's ostensible mechanism serves two purposes: (1) to orient the process of perceptual semiosis in a certain direction (here, emphasizing the last two circles on each row), and (2) to convey a sense of generality through the rhythm of the gestures. In fact, the six diagonals virtually drawn by Doug with his rhythmic gestures not only refer to the last two circles diagonally disposed at the end of each figure but also express the idea of something that spatial-temporally continues further and further.

When solving the problem of finding the number of circles in figure number 10, the regularity of the pattern is reformulated: what was previously perceived as a unique object (the couple of two circles) is now atomised (two separated circles). While *drawing* figures 5 and 6 did not require knowing the number of circles in each of these figures, this knowledge became essential for solving the next question that the students solved by *computation*.

This atomisation is then soon refined by Alice, who suggests another way of expressing the regularity, based on another perception of the figures. Now the figures are seen as divided into two rows:

[3] Names have been changed for deontological reasons.

1. Alice: No, you just have to always add one on the top and one on the bottom (*inclining her head towards the right when she says "bottom"*).
2. Doug: Umm. OK. So it's ... [...] How many... How many circles will figure number ten have?
3. Alice: OK. It would be (*pointing with her finger the rows of figure 2*) eleven on the top and then...and then... twelve on the bottom.

Alice mobilises two semiotic resources to objectify her perception of the figures and, by the same token, to refine her understanding of the regularity of the pattern. When talking in general terms ("you (...) always", line 1), the distinction between the two rows is made by the inclination of her head, meant to clearly distinguish the circle added on the top from the one added on the bottom. Later on, when tackling the problem for figure number 10 (line 3), the distinction between rows is made by pointing at the top row of figure 2. This figure indeed provides the students with a *metaphoric* way of talking meaningfully about figure 10; it is a concrete support for them to *imagine* figure 10.

The shift towards Alice's perception of the figure (*i.e.* in two rows) is not problematic for Doug, who soon agrees with her point of view ("Umm. OK", line 2). This point of view allows the students to easily find the number of circles in figure number 10, as well as the ones in figure number 100.

When asked to write a message in French, describing how to explain to another student what she/he should do in order to find the number of the circles in any figure, Doug says:

4. Doug: Each... For each figure… You take the number of the figure...of the...of the... The number of the figure (*balancing nervously back and forth on his chair*) […] (*then, without balancing anymore he says*) let's say that the figure's number is three. You would say one plus three for the top row (*moving his pencil in the air from left to right*) and two plus three... […] No, plus two for the bottom row (*pointing with his finger at one of the figures*) and plus one (*pointing directly to one of the figures*) for the top row. On …of the number... the figure (*stressing the words "on" and "of" by pointing his finger towards the table)*.

Doug does not seem to be comfortable dealing with the problem of "any figure". He is not comfortable in this layer of generality and expresses himself hesitantly, moving on his chair nervously. After the early unsuccessful attempts, Doug abandons this path to generality and uses figure 3 as a crutch. The concreteness of figure 3 allows him to express the general intended computations. As soon as he finishes explaining the computations based on figure 3, the reference to a particular figure fades away (Doug even says "no", as if he were making a mistake). In actual fact, he is not talking about figure 3 specifically. In Kantian terms, the counting process undertaken on figure 3 serves as a way to objectify the *schema of counting*. Doug's effort shows us at least this:

the presence of the general is made apparent by its <u>*absence*</u> *at the discursive level*: "plus two" and "plus three" do not have an explicit linguistic referent and the gesture in the air signifies that the referent is not located on the drawn figures either. By omitting to name the referent, the referent becomes general.

Although Doug' utterance evokes a certain "struggle" (Doug has some troubles, at the end, to make his sentence coherent by trying to include the reference to the figure again), the written message in plain French is quite clear:

> "# de fig +1 pour la rangée du haut et # de la fig +2 pour le bas.
> Additionne les deux pour le total"[4].

The written message tells us more than the sole description of the procedure that one has to perform in order to find out the number of circles in any figure. It also states the geometrical meaning of the operations, intimately related to the students' perception of the figure into two rows. The need to refer to the geometrical meaning can also be found elsewhere in their answers, more specifically when the students answer the second and third questions: "23 circles (11 on the top, 12 on the bottom)" and "203 circles (101 on the top, 102 on the bottom)," respectively.

The algebraic formula that they provide at the end of the problem (*i.e.* "(n+1)+(n+2)") still follows the course of this geometrical explanation, where the brackets delimitate the computations made on the two rows of the figure. Brackets are organizers of the way in which the formula tells us the story of the students' mathematical experience.

Second example:
The group is formed by three students: Jay, Mimi (sitting side by side) and Rita (sitting in front of them). The students begin counting the number of circles in the figures, realizing that the number of circles in the figures increase by two each time. Then, their attention focuses on the geometrical structure of the figures:

1. Mimi: *(Talking about figure 4)* So, yeah, you have five on top *(she points to the sheet, sketching a horizontal line with her hand)* and six on the... *(she points to the sheet, making anther horizontal gesture, lower that the previous one)*.
2. Jay: Why you're putting... oh yeah yeah yeah, there will be eleven, I think.
3. Rita: Yep.
4. Mimi: But you must go six on the bottom … and five on the top.

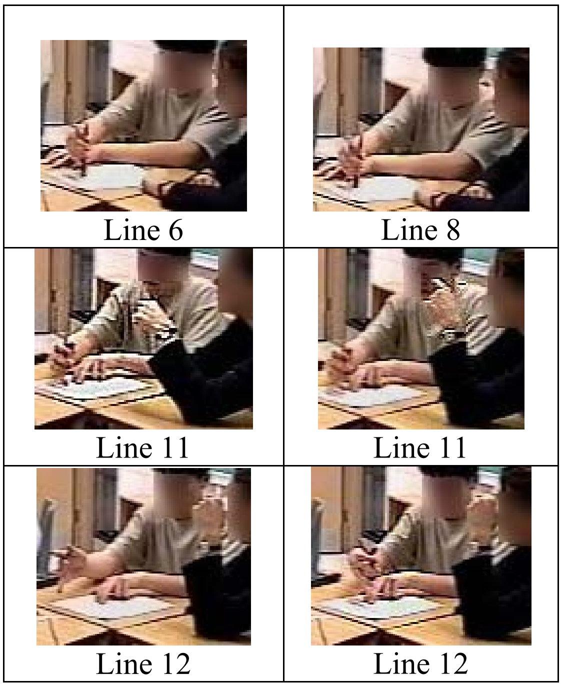
The spatial deictics "top" and "bottom" (lines 1 and 4) used by Mimi offer her group-mates a particular way to apprehend the figures in the ongoing process of perceptual semiosis. Jay's utterance (line 2) reminds us that, despite what is often thought, perceiving is not a simple and direct process. In line 4, Mimi insists on the geometric structure of the terms of the sequence. Her intervention amounts to shifting from blunt

[4] Transl: "# of fig +1 for the top row, and # of the fig +2 for the bottom. Add the two to get the total".

counting to a scheme of counting. To notice this scheme is the first step towards the general.

Mimi's spatial deictics are accompanied by two corresponding gestures. These gestures accomplish a twofold role: they depict the spatial position of the rows in an iconic way and also clarify the reference of the uttered words.

The students' work was interrupted by an announcement made to the class about a forthcoming social activity. While Mimi and Rita paid attention to the announcement, Jay kept on working, writing 23 and 203 as the answers for the question concerning the number of circles in figures 10 and 100. At this point, the announcement was finished and the girls returned to the task and asked Jay for an explanation:



Line 6 | Line 8
Line 11 | Line 11
Line 12 | Line 12

5. Mimi: (*Talking to Jay*) I just want to know how you figured it out.
6. Jay: Ok. Figure 4 has five on top, right? *(he points to the top row of figure 4)*.
7. Mimi: Yeah…

8. Jay : *(Continuing his sentence)* and it has 6 on the bottom *(he points to the bottom row)* […].
9. Mimi: *(She points to the circles while she counts)* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. (*Pause*) […] Oh yeah. Figure 10 would have …
10. Jay: 10 there would be like…
11. Mimi: There would be eleven *(she makes a quick gesture that points to the air)* and there would be ten *(same quick gesture but higher up)* right?
12. Jay: Eleven (*similar gesture but with the whole hand on the paper*) and twelve *(same gesture but lower)*.
13. Mimi: Eleven and twelve. So it would make twenty-three, yeah.
14. Jay: [Figure] 100 would have one-hundred and one and one-hundred and two *(same gestures as in line 12)*.
15. Mimi: OK. Cool. Got it now. I just wanted to know how you got that.

Figure 3. Some gestures occurring in the lines of the dialogue.

Developing Mimi's initial idea, elaborated in lines 1 and 4, Jay here attains a structural apprehension of the figure through which he solves the problem for figures 10 and 100. Let us notice that, to explain his strategy (lines 6, 8), he refers first to figure 4. In line 7 agreement is obtained. Moreover, in his explanation, he uses the same discourse genre as Mimi's: a discourse genre that interweaves spatial deictics (top, bottom) and iconic and deictic gestures. In particular, by pointing gestures he touches the two horizontal rows in which figure 4 can be divided. Mimi then turns to figure 10 (end of line 9) and accompanies her utterance with gestures that keep certain specific aspects of those of Jay: the fact of having one gesture for each row. But whereas Jay's gestures point materially to the rows of figure 4, Mimi's are made in the air (line 11): indeed, figure 10 is not in the perceptual field of the students, so new mechanisms of semiotic objectification have to be displayed. This, we suggest, is the role of gestures here. Of course, Mimi could have simply reached the answer using words. The fact that she did not, and that she used gestures is right to the point that we want to make here: gestures do not merely carry out intentions or information. They are key elements of the process of knowledge objectification. This point becomes even clearer when the students address the question of figure 100. The gestures are again made in the air, and this time at a higher elevation from the desk.

In their path towards generality, students need to mobilize both language and gestures in a coordinated and efficient way. This coordination takes place in particular segments of the students' mathematical activity where knowledge is objectified. These segments of mathematical activity characterized by the crucial coordination of various semiotic systems constitute what we have previously termed *semiotic nodes* (Radford et al. 2003).

In this particular semiotic node, which goes from line 5 to 15, we see how the students merge the geometrical and numerical components of the problem: the former is taken in charge by gestures and the latter by words.

We shall now discuss the part of the mathematical lesson where the students had to face the problem of writing a message to explain how to find out the number of circles for any figure (*figure quelconque*).

16. Mimi: Add. Add three to the number of the figure! (*pointing to the results 23 and 203 on the paper*).
17. Jay: No! […].
18. Mimi: I mean like … I mean like … You know what I mean, like, for figure 1 […] (*pointing to figure 1*) it would be like one, one, plus three; this *(pointing to figure 2)* would be two, two, plus three; this *(pointing to figure 3)* would be three, three, plus three.

As suggested by her gestures (line 16), Mimi seems to have observed that the number of circles in figures 10 and 100 ended with the digit 3 and considered it as a key to look for a general method, something which led her to a new apprehension of the figure. Jay does not understand (line 17). Mimi then explains the idea in more detail (line 18). Here, the gesture with which she pointed to each figure was made up of three indexical gestures. In the case of figure 1, she pointed successively to the top left circle, then the bottom left circle and finally she sketched a small triangle surrounding the three left circles on the right (see Figure 4).

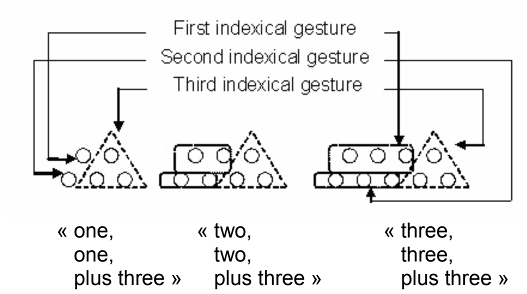


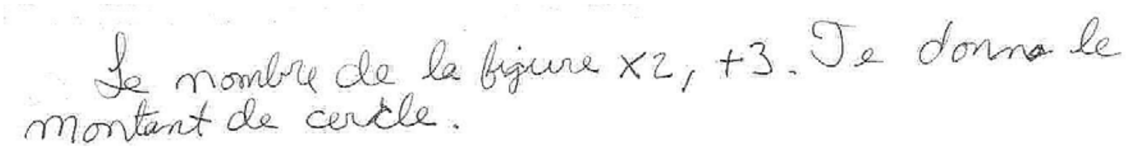
Figure 4. On the left, Mimi making the indexical gestures on figure 1 on the sheet. On the right, the new apprehension of the figures as a result of the process of perceptual semiosis.

The process of perceptual semiosis leading to the new apprehension of the structure of the figures included not only gestures and words, but also rhythm. In fact, the expression "one, one, plus three" is uttered with the same cadence as the expressions "two, two, plus three" and "three, three, plus three". We can detect, in this sentence, the embedding of two types of rhythm. The first one helps to make apparent a kind of regularity within each figure, and in conjunction with gestures and the meaning of words, organizes the way of counting. The second rhythm corresponds to the *pause* made *between* figures: “one, one, plus three” [*pause*] “two, two, plus three” [*pause*] “three, three, plus three”. The concatenation of these two rhythms conveys the idea of generality. It also opens new avenues to keep exploring the general. Thus, in the course of the classroom activity, it became apparent that the first two elements in the counting process were related to the number of the figure.

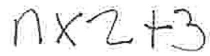
In fact, keeping the numerical example of figure 10, the students soon after manage to express the regularity in natural language:

19. Mimi: You double the number of the figure.
20. Jay: 10+10
21. Mimi: So it will be 20 dots +3. You double the number of the figure and you add three, right. So figure 25 will be 50...3. Right? That’s what it is.

The message was finally refined as follows:



The symbolic formula was:



SYNTHESIS AND CONCLUSION

Our microgenetic analysis of two small groups of students dealing with the generalization of patterns suggests the central role played by spatial deictics, gestures, and rhythm in perceptual semiosis, particularly in the students’ progressive processes of perceptual apprehension of a pattern and its generalization. The analysis also suggests some connections between the syntax of the students’ algebraic formulas and the students’ semiotic means of objectification. For instance, the spatial deictics ‘top’ and ‘bottom’ impressed their mark in the syntax of the formula “(n+1)+(n+2)”. However, the connection may be even yet more subtle. Rhythm, for example, impressed its mark in the message produced by the second group of students, where it appeared under the form of a comma (see above). In the final symbolic formula “n×2 + 3” the comma has disappeared. Rhythm is nevertheless *embedded* in the symbolic expression: it constitutes one of the signifying elements of the students’ formula. In general, deictics, gestures, rhythm, and other semiotic means of objectification do not operate separately from each other. They belong to different semiotic systems whose coordination seems to be crucial in the students’ mathematical experience. This complex coordination of semiotic systems still remains largely unknown in the psychology of mathematics education. This paper does not solve this research problem in its generality. It shows a few elements of it and suggests a research path to be explored.

REFERENCES

Kaput, J. and J. Sims-Knight (1983). Errors In Translations To Algebraic Equations: Roots and Implications. *Focus on Learning Problems in Mathematics* 5: 63-78.

Mason, J. (1996). Expressing generality and roots of algebra'. *Approaches to Algebra.* N. Bednarz, C. Kieran and L. Lee. Dordrecht/Boston/London, Kluwer: 65-86.

Matz, M. (1980). Towards a Computational Theory of Algebraic Competence. *Journal of Mathematical Behavior* 3: 93-166.

Nyckees, V. (1998). *La sémantique.* Paris: Belin.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics* 42(3): 237-268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics* 22(2): 14-23.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning* 5(1): 37-70.

Radford, L., Demers, S., Guzmán, J. and Cerulli, M. (2003) Calculators, graphs, gestures, and the production meaning, in: N., Pateman, B. Dougherty and J. Zilliox (eds.), *Proceedings of the 27 Conference of the international group for the psychology of mathematics education* (PME27 –PMENA25), Vol. 4, pp. 55-62.

MODEL OF A PROFESSOR'S DIDACTICAL ACTION IN MATHEMATICS EDUCATION

PROFESSOR'S VARIABILITY AND STUDENTS' ALGORITHMIC FLEXIBILITY IN SOLVING ARITHMETICAL PROBLEMS[1]

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Abstract: *The paper deals with the issue of problem solving. This was a common theme of two independent projects, which complement each other. One study detects phenomena in graphical models of word problem assignments; the pre-algebraic features of models are discussed. The other gives these phenomena precision by an action model of the problem, focusing on the variability in word problems. Common aspects of both studies are presented.*

Keywords: problem solving, graphical models, variability of teachers, psychological perspective, theory of didactical situations, coding of word problem, reference language, model.

1. INTRODUCTION

In the paper we are presenting two studies originally executed as independent entities; both are dealing with the same topic: problem solving. The first one (J. Novotná) belongs more to the psychological perspective than the purely didactical one, although the didactical concern is not absent. The second one (B. Sarrazy) examines the effects of variability in the formulation of problem assignments on students' flexibility when using taught algorithms in new situations; the research was developed in the framework of the theory of didactical situations starting from various results in the psychological domain. These two studies, although at the beginning carried out separately and on different levels of education, showed themselves to be perfectly complementary. The first one allows the detection of a set of phenomena, whereas the second one gives them precision through an action model of the problem focusing on the variability in word problems. We believe that connecting these two approaches allows us to open interesting perspectives for a better understanding of the role of problem solving in teaching and learning mathematics by giving precision to certain conditions of their use.

[1] Acknowledgement: The research was partly supported by the project GAČR No. 406/02/0829 Student-focused mathematics education.

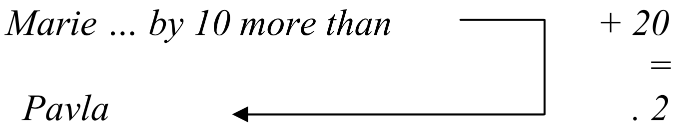
2. MODELS OF WORD PROBLEM ASSIGNMENTS

In this part of the article we investigate the ways that students are modelling word problem assignments when grasping the problems' structure -see e.g. (Novotná, 1999). The following terminology is used: *Coding of word problem* is the transformation of the word problem text into a suitable system (*reference language*) in which data, conditions and unknowns can be recorded in a more clearly organized and/or more economical form. The result of this process is called a *model*[2] (in both cases – models taught by teachers or models as results of the inner need of the solver). The reference language contains basic symbols and rules for creating a model. There exist different reference languages for any one type of word problem.

From a student's solution (Jakub, 13 years, individual experiment)

Problem to be solved: Marie and Pavla each had some money but Marie had 10 CZK more than Pavla. Pavla managed to double the amount of money she had and Marie added 20 CZK more to her original amount. They now found that both of them had the same amount. How many crowns did each of them have at the beginning?

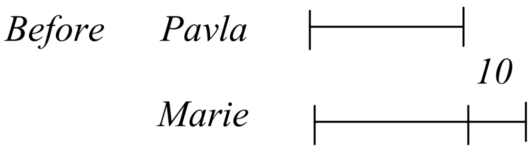
Jakub writes:



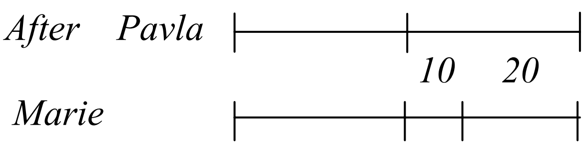
This record of the assignment did not allow him to find a suitable solving strategy. The experimenter recommends him to use the visualisation with the help of line segments, see (Novotná, 1998):

Experimenter (E): *Try to record the situation at the beginning.*

Jakub (J):



Experimenter E: *And after the change?*
J starts to draw a new line segment.
E: *Would it not be better to record it in the same schema?*
After a short discussion, J's graphical representation is



J: *Aha … I do not need to construct an equation!*

This simple example illustrates the influence of an appropriately chosen reference language on the quality of grasping the text and on constructing a mathematical

[2] Our use of the word model corresponds with the representation in (Pierce, 1987).

model of the word problem.[3] In the example, two reference languages were used: verbal and graphical ones.

The solver's choice of one of the reference languages is influenced by several factors: internal ones, e.g. by his/her previous experience, preferred information processing style, personal preferences, and external ones, mainly the demands of the teacher or school system.

From the point of view of impulses for creating a model we distinguish three types: *spontaneous independent* model creation, *externally managed* model creation, and creation of a figure in the *role of a signal* (Novotná, 1999). In real life situations, it is rare to have to solve standard problems with the help of known solving algorithms. To prepare children for dealing with life situations in a successful way, the spontaneous case of model creation is crucial.

The solver's goal when creating a model is to get a better understanding of the problem structure (except in the cases when the reasons for model creation are fully external, e.g. teacher's demands, no intrinsic motivation). A model can have different forms: from detailed rewriting of the assignment to more clearly organized forms, from a verbal description of the assigned conditions to their symbolic record. In this context we can speak of non-algebraic, pre-algebraic or algebraic model forms. In this perspective, a spontaneously created model can indicate the level of pre-algebraic/algebraic thinking of its author.

2.1 Our research

The original aim of our experiment (Novotná, Kubínová, 1999) was to analyse and classify spontaneously created graphical models of word problems. The experiment was conducted with pupils from the 3rd (age 9-10) to 8th grades (age 14-15) in basic schools in both Prague and České Budějovice. The sample consisted of 25 3rd graders, 21 4th graders, 24 5th graders, 22 6th graders, 28 7th graders and 23 8th graders, all of them were from non-specialised classes. The word problem dealt with was a non-standard problem that is not presented in currently used Czech textbooks and the participating pupils have not solved similar ones before. It had the following structure:

A packing case full of ceramic vases was delivered to a shop. In each case there were b boxes, each of the boxes contained k smaller boxes with p presentation packs in each of the smaller boxes, each presentation pack contained m parcels and in each parcel were v vases. How many vases were there altogether in the packing case?[4]

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[3] Duval (1995) states that without distinguishing between object and its representation it is not possible to understand mathematics. In order to separate object from its representation, the student must be able to represent a mathematical concept at least in two semiotic systems. Duval (1999) studies *auxiliary representations* as a tool that helps the solver understand formulations and reformulations in mathematics.

[4] The number of "unpacking levels" and the numbers labeled *b, k, p, m, v* were modified according to the age of solvers. For the 7th/8th grades, the mixed arithmetic-algebraic assignment was used (*v*/*v* and *k* were not substituted by numbers).

The traditional reference language in Czech schools used already at the primary level is a verbal one (Fig. 1a). The use of grahical models (Fig 1b) was the spontaneous decision of solvers. Spontaneously and independently created reference languages were used by solvers of 6th to 7th grades. The younger solvers tried to apply the verbal model copying the one presented by the teacher for other word problem types. Only one of the 3rd to 5th graders used any form of pictorial representation. The probable reason is that at this age, pictures are always connected with the real situation that they represent. The lack of pictures in the solutions indicates that children do not connect the word problem presented to them in school with real objects/situations or found the creation of diagrams for the problem too difficult.

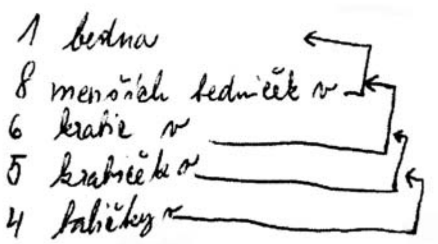

Fig. 1a

Fig. 1b

The larger amount of figures and schemes in the solutions of 6th graders and older students is connected with the use of teaching strategies supporting students' development of the ability to visualise situations. There occured a rich variability in graphical models used spontaneously by individual students. Graphical models were of two main types: procedural and conceptual. We call a model: *procedural* when it clearly expresses the process in time how it is described in the assignment (Fig. 2a); *conceptual* when all pieces of information are recorded as a whole not showing the changes in time (Fig 2b). As to the shape similarity, both *iconic* (consisting of real shape record, Fig. 2b) and *symbolic* (keeps the structural similarity only, Fig. 2a) models were found. Students used various types of accompanying explanatory means (arrows, words, …, Fig. 2b). Big differences were identified also in the completeness of the records.

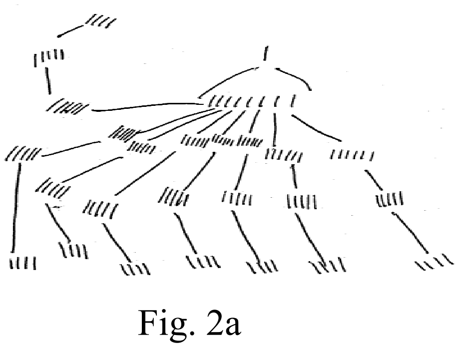

Fig. 2a

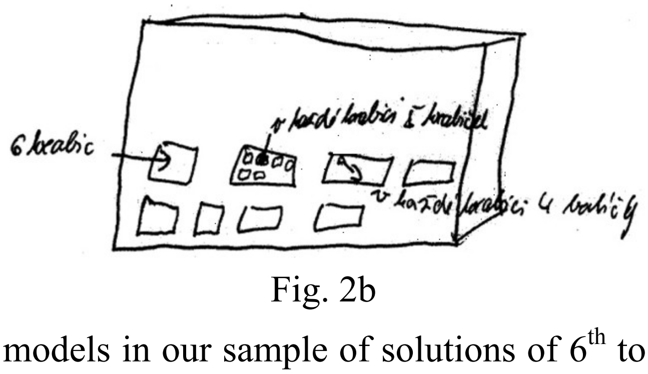

Fig. 2b

When analyzing the use of assignment models in our sample of solutions of 6th to 8th graders we identified difference in the performances of the groups of students from different classes (taught by different teachers): Either the majority of students kept using the externally managed model (mostly verbal, non-algebraic) or most of models were individualized spontaneously created models (of all three “algebraic levels”).

Traditionally, the level of pre-algebraic/algebraic thinking is characterized by use of letters (or other symbols, e.g. ∗, □) when solving mathematical problems. Students’ ability to operate with algebraic symbols in a systematic way is often based on very formalistic knowledge and most often it is taught transmissively. Visualization is seldom used (Novotná, Kubínová, 2001). In the (spontaneous) use of individualized models, we can observe echoes of a conscious transition towards pre-algebraic/algebraic thinking. There are several variables that might be considered as indicators of pre-algebraic/algebraic level of a model from which we present here those that are clearly present in the models in Fig. 1, 2:

- Level of *revealing the structure*[5]: Higher for the model in Fig. 1b than in Fig. 1a.
- Level of *thinking in symbolic language*[6]: Higher for the model in Fig. 2a than in Fig 2b.
- Level of *the reference language abstractness*: Higher for the model in Fig. 2a (small lines in the model represent different real objects even if they have the same form – for the solver it is an abstract “universal” symbol) than in Fig. 2b (the solver tried to distinguish symbols for various real objects).

2.2 Results

As mentioned in (Malara, Navarra, 2001), “… the difficulties in the approach to algebra are rooted in the scarce attention paid to the relational or structural aspects of arithmetics which constitute the basis of elementary algebra. A longitudinal study of the use of different reference languages when solving word problems indicates that the spontaneously created models produced by students have pre-algebraic features. Referring to (Drouhard, 2001), it is closely related to the fact that “(students) have to learn how to read, write, understand, speak, and above all how to use this particular language in order to solve problems and to ‘think algebraically’ “.

The individual differences in the form of graphical models could be explained by the internal students’ cognitive processes, (Novotná, 1999), (Novotná, Kubínová, 1999). By this approach we were not able to explain the striking difference “spontaneity versus copying” in the students groups. The psychological perspective did not offer any explanation of the observed fact. It was necessary to search for it outside the psychological approach. We found a suitable tool for the explanation in the scope of the Theory of didactical situations (Brousseau, 1997), namely in the notion of variability of teachers introduced by B. Sarrazy (Sarrazy, 2002).

[5] (Arcavi, 1994): “Many students who manage to handle the algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing and communicating generalizations, for revealing structure, …”

[6] (Crawford, 2001): „… three broad indicators are defined as essentials to algebraic thinking:
1. Ability to think in symbolic language, to understand algebra as generalized arithmetics and to understand algebra as study of mathematical structures. …”

3. VARIABILITY OF TEACHERS

To place the question studied by the second research it is necessary to return to an episode well known among French educationalists: In 1979, the researchers from IREM in Grenoble presented to students of 9-10 years the following problem: "On the boat, there are 26 sheep and 10 goats. What is the age of the captain?" More than three quarters of students used addition of numerical data from the assignment. This phenomenon is well known to educationalists as one of the apparent effects of the "didactical contract" (Brousseau, 1997; Chevallard, 1988; Sarrazy, 1995). But there remains one question and this is the explanation why, at the same level of competence, students from certain classes are more sensitive to the formal aspects of the problem assignment (answering that the age of the captain is 26) and that others are more flexible in the solving process - rejecting the validity of such problem type (answering: "It is impossible to find the answer to this question.").

3.1 Our field of interest

Learning mathematics is not restricted to learning algorithms only but it is manifested by identifying conditions for their use in new situations: by this criterion it is possible to admit that the child learned something. But these conditions are not present in the algorithms itself and cannot be explicated by the teachers. This is one of the reasons why the "didactical contract" is in large part implicit: the teacher cannot tell students what he is expecting from them without relinquishing the ability to determine what the students learned (Brousseau, 1997). How could it be explained that certain students show that they are able to use the taught knowledge in new contexts, while others, although "knowing" the taught algorithms, are not able to re-contextualise their knowledge? If there is no satisfactory explicative model, these differences are attributed to the charisma of individuals, to their cognitive skills …, or simply to the mysterious mental properties for which teachers do not have any didactical tool for transforming them or letting them develop. The central hypothesis of our research is to consider these inter-individual differences of the sensibility on the didactical contract (measured by an index), as an effect of the teachers' didactical variability in the domain of setting arithmetical problems.

The obtained model is based on the following idea which can be formulated simply: the more the *same* form of didactical organisation presents the modalities of different realisations, the more uncertainty attached is added to this form. To satisfy the teacher's expectations, such a student has to 'examine' the domain of validity of his knowledge much more than a student who is exposed to a strongly ritualised (repetitive) teaching and therefore a much reduced variability. In other words, a strongly ritualised teaching would allow the student to know in advance what he has to do and thus, to adopt a behaviour *ad hoc* (adapted). On the other hand, by the interruptions of introduced routines, a strong variability makes the following strategies futile (controversial): the students cannot rely only on the indicators of introduced routines (semantic indicators, triggers …) and therefore cannot either anticipate or master the liaison of sequences which allow him to discover the behaviours expected by the teacher. This model was not refuted by our results.

3.2 Theoretical model: degree of variability

We will study the problem assignments invented by the teachers.

We asked 7 teachers of the 4^{th} year of elementary school (age of students 9-10), all with the same length of experience at this level, to write down 6 arithmetic problems (without consulting any documents such as textbooks): 3 with a solution requiring addition and the other 3 requiring subtraction. The problems were to be different from the point of view of their difficulty. The 42 obtained problems were analysed using the classification presented bellow.

Consulting a certain number of researches allowed us to identify 14 variables; each of them could explain the small or larger difficulties of a problem. We grouped them into 3 categories:

A. Numerical, grouping the variables which relate to the numerical values of the problem: the type of numerals used; presence of irrelevant data;

B. Rhetorical, relating to the organisation forms of the presentation of the problem (story): organisation of the assignment of the problem: presence/absence of an semantic indicator in the assignment; theme of the assignment; presence/absence of a trigger in the assignment; syntagmatic organisation and temporal organisation; position of the question; vocabulary used; type of formulation: classic and written forms.

C. Semantic-conceptual: This last cluster groups together at the same time, the variables connected with certain rhetorical aspects (the presence of a trigger in the question as for e.g. “altogether”) and certain logical-mathematical variables (the operation that should be used): type of the additive structure; nature of the unknown; correspondence between the syntagmatic order and the operative order; correspondence between the trigger and mathematical operator; correspondence between semantic indictor and mathematical operator.

3.3 Results and conclusions

The procedure for calculating the variety index for additive problems (*IVa*) consisted in counting, for each of the 14 variables, the variations observed (*Vo*) over the 3 problems as a whole in relation to the number of possible variations (*Vp*) (*IVa = Vo/Vp*). If we only calculated the sum of the variations (*Vo*) we would underestimate the value of the index in cases where certain variations are formally excluded by the choice made on other variables. The same calculation procedure was used to measure the variety index in relation to the subtractive problems (*IVs*). The arithmetic average of *IVa* and *IVs* was retained as the measure of variety index.

The 42 problems were assigned to 27 8-9-year-old students. We observed a strong correlation[7] ($r = -.97$; s.; $p. < .001$) between the average of success in the problems solved by students and the variety index calculated for these problems. Thus, it can be affirmed that the higher the variety index, the more difficult and contrasted the problems (although this is not significant, we nonetheless observe a positive correlation between the dispersion of success and the variety index). In other words, teachers with a high variety index produce variations that have highly significant effects on the difficulty of the presentations of the problems. As a result, the variety index constitutes a faithful summary of the “ability” of the teachers to make relevant variations in the wording of the problem – indeed, variability greatly reduces the average score (by half), which shows that the pupils are responsive to the contract and that their success does not resist variation. We can thus assume that variability is a variable that might explain the phenomenon of responsiveness to the contract.

The observed correlation ($r = -.74$; s.; $p. < .05$) between the values of the variety index of the 7 teachers and the average scores of the flexibility (formalism) degree allows us to validate our initial hypothesis: the more the teacher shows an important variability, the more the students show the flexibility in the solution; vice versa, the weaker the variability of the teacher, the more the students are formalists and rely more on the formal aspects of the assignment than on their comprehension when producing the answer.

So, repetitive teaching can guide the students more easily to adapt themselves to the educational situations by determination of indicators (e.g. triggers) only to answer the situations; thus, students may adopt an appropriate behaviour without being in need to understand the meaning of the mathematic knowledge mobilised by the situation. A developed variability invalidates these strategies: the student cannot rely only on these indices any more, and correlatively, the student’s involvement is more probable.

4. CONCLUDING REMARKS

In our experiments presented in Part 1, the variability of teachers proved to be the variable explaining the significant differences in the number of spontaneously created models by students in some groups. It confirms our conviction that students’ results differ when they are asked to reproduce only the reference language presented by the teacher or when they get acquainted with several reference languages or may even use their own reference languages. In the last two cases their results are better. In addition to that, these cases support the development of the student’s personality, mainly his/her ability of critical analysis and consciousness of their responsibility for their own activity.

Moreover, we believe that analysis of models created by students enables the teacher to help them in case that their effort to solve the problem correctly is not successful

[7] The linear coefficient of Bravais-Pearson was used.

(mainly in determining the type of obstacles the student faced). This point is elaborated in more details in (Novotná, 2003).

If the teacher decides to get his/her students acquainted with several types of reference languages, he should be aware that there are not only positive consequences, but also negative ones. One of the most important dangers is the increased uncertainty in less able students who, besides the uncertainty concerning their ability to solve the problem correctly they are also facing the uncertainty, which reference language enables them to solve the problem.

One question remains to be examined in the following work: the origin of teachers' variability. The theoretical framework currently worked on in DAEST (Laboratory of Didactics and Anthropology in the Teaching Sciences and Techniques in Bordeaux) promises to provide a consistent framework for examining such a question, which is understandably interesting primarily for teacher training.

REFERENCES

Arcavi, A. (1994). Symbol Sense: Informal sense-making in formal mathematics. *For the Learning of mathematics*, 14, 24-35.

Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*, [Edited and translated by M. Cooper, N. Balacheff, R. Sutherland & V. Warfield], Dordrecht: Kluwer Academic Publishers. (French version: Brousseau, G. (1998). *Théorie des situations didactiques*. [Textes rassemblés et préparés par N. Balacheff, M. Cooper, R. Sutherland, V. Warfield]. Grenoble: La pensée sauvage.).

Chevallard, Y. (1988). *Sur l'analyse didactique: Deux études sur els notions de contrat et de situation* (doc. Ronéo), Aix Marseille: IREM *14*.

Crawford, A.R. (2001). Developing algebraic thinking: Past, present, and future. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 192-198.

Drouhard, J.-P. (2001). Researches in language aspects of algebra: a turning point? In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 238-242.

Duval, R. (1995). *Sémiosis et pensée humaine: Registres sémiotiques et apprentissages intellectuels.* Bern: Peter Lang.

Duval, R. (1999) *Conversion et articulation des représentations analogiques*, IUFM Nord-Pas de Calais.

Malara, N., Navarra, G. (2001). „Brioshi" and other mediation tools employed ina teaching of arithmetic from a relational point of view with the aim of approaching algebra as a language. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 412-419.

Novotná, J. (1999). Pictorial representation as a means of grasping word problem structures. *Psychology of Mathematical Education*, 12, http://www.ex.ac.uk/~Pernst/.

Novotná, J., Kubínová, M. (1999). Wie beeinflusst eine Visualisierung der Aufgabenstellung den Prozess der Lösung einer Textaufgabe. In: *Beiträge zum Mathematikunterricht 1999*, Vorträge auf 33. Tagung für Didaktik der Mathematik vom 1. bis 5.3.1999, Bern. Ed. M. Neubrand. Hildesheim, Berlin: Verlag Franzbecker, 397-400.

Novotná, J., Kubínová, M. (2001). The influence of symbolic algebraic sescriptions in word problem assignments on grasping processes and on solving strategies. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 494-500.

Novotná, J. (2003). *Etude de la résolution des «problèmes verbaux» dans l'enseignement des mathématiques. De l'analyse atomique à l'analyse des situations*, Bordeaux: Université Victor Segalen Bordeaux 2.

Peirce, C. S. *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by Charles Hartshorne and Paul Weiss, 1931-1935, Volumes VII-VIII, ed. by Arthur W. Burks, 1958, quotations according to volume and paragraph). Cambridge: Harvard University Press.

Sarrazy, B. (1995). Le contrat didactique. *Revue Française de Pédagogie, 112*, 85-118.

Sarrazy, B. (2002). Effects of variability on responsiveness to the didactic contract in problem-solving among pupils of 9-10 years, *European Journal of Psychology of education*, XVII. 4, 321-241.

EARLY ALGEBRA – PROCESSES AND CONCEPTS OF FOURTH GRADERS SOLVING ALGEBRAIC PROBLEMS

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Abstract: *Secondary students' difficulties with elementary algebra are well documented and discussed in the research literature. Since the mid nineties, new approaches have focus on the teaching of early algebra in primary school based on the assumption that it is useful to learn algebraic thinking in conjunction with arithmetic. The aim of the study introduced in this paper is the investigation of primary students' thinking and solution processes with respect to algebraic problems in a classroom setting. Fourth graders working in pairs on a selection of tasks were videotaped to document their collaborative strategies, their concepts and thinking, followed by a semi-structured interview with each pair. Focus of the qualitative data analysis is their understanding of patterns, generalisations and their use of symbolic language. In this paper selected data from a pilot study conducted in June 2004 will be discussed.*

Keywords: early algebra, algebraic thinking, elementary algebra, algebra, qualitative research.

BACKGROUND AND INTRODUCTION

Since the 1970s many international studies highlighted the fact that many secondary students experience difficulties with beginning algebra (see e.g. BOOTH 1988; KIERAN 1992).

With regard to these findings a variety of different approaches were pursued at reforming the teaching and learning of algebra, e.g. by integrating more problem solving tasks and tasks to improve arithmetic skills, "slower" teaching approaches, and the use of new technologies (FREIMAN & LEE, 2004). While research as well as curriculum based approaches showed various degrees of success, difficulties with learning algebra in secondary school remain an important topic in the research literature (MALLE 1993, WARREN 2003) as well as in classroom instruction. In the past decade, teacher educators and researchers in different countries argue that it is necessary to develop algebraic thinking already in primary school (e.g. WARREN 2001). The central assumptions of corresponding international research are that pupils who developed algebraic thinking already in primary school have better arithmetic skills as well as a better understanding of underlying structures and rules, which it is hoped prevent the identified difficulties with the learning of algebra in secondary school. DOUGHERTY (2003) for example, who is conducting a long-term research project in a primary lab school at the University of Hawaii where mathematics is taught algebraically since the first year of school, highlights the chance of providing young children with a wider view of mathematics at a stage of their cognitive development when they are able and motivated to learn these concepts very well.

Most research studies on the teaching and learning of algebra however, have been conducted with secondary students at middle or high school level, frequently focussing on students difficulties and misconceptions, while little is known about primary students understanding of algebraic concepts and its effect on their learning of algebra in secondary school.

THEORETICAL FRAMEWORK

Early algebra, algebraic thinking or pre-algebra are terms used to describe a preliminary stage of elementary algebra prior to secondary school instruction. The idea is not to teach children formal algebra at a younger age, but to enhance special aspects of algebraic reasoning, which are supporting mathematical thinking beyond calculation skills at primary school level.

To date, an exact definition of *early algebra* and its implications for the classroom does not seem to be possible. Recent publications discuss the relationship and transition between early algebra and arithmetic, but the question where arithmetic ends and early algebra begins is not yet resolved. According to MALARA and NAVARRA (2003) one difference is that algebraic thinking refers to the process, whereas arithmetic refers to the product, i.e. finding the answer. CARPENTER and LEVI (2000) in addition, identify two main aspects of early algebra: 1. generalising and 2. the use of symbols to represent and solve mathematical ideas and problems.

> "Young children are capable of making generalizations such as 'when you add 0 to a number, the sum is always that number' – 'when you add three numbers, it does not matter which two you add first' and constructing ways of representing them […]. These generalizations make powerful mathematical ideas accessible to students to solve problems and to deepen understanding." (p. 2)

Hence, one concern of primary mathematics is to develop insight in the structure and properties of our number system and of the operations. Researchers seem to agree that early algebra involves more than the generalisation of arithmetic structures. BLANTON (2004) for example describes an understanding of early algebra that involves generalised arithmetic, functional thinking and modelling, while ZEVENBERGEN, DOLE and WRIGHT (2004) emphasise three main aspects of algebraic thinking: equality, change and generalisations. WARREN (2003) extends the understanding of early algebra by identifying four central aspects: 1. relationships between quantities, 2. group properties of operations, 3. relationships between the operations, 4. relationships across the quantities. These are clearly linked to algebraic approaches and topics at secondary school level. However, the teaching and learning of algebra at secondary school level is traditionally based on the assumption that arithmetic has to be learned before the introduction of algebra because arithmetical understanding is seen as the basis for the development of algebraic understanding. More recently, studies into early algebra suggest that students would benefit from learning algebraic thinking in conjunction with arithmetic.

Comparing pupils algebraic reasoning in the earlier grades (year 4) with their reasoning in secondary school (year 7) concerning the commutative law, ANTHONY and WALSHAW (2002) found that leaving primary school, thinking remains for many students at a procedural level and that this restricts their ability to reason algebraically. WARREN (2001) conducted a study comparing pupils difficulties in primary and secondary school and showed that they go through the same types of difficulties with regard to the commutative law. She suggests that "it will be important for researchers to explore whether well-chosen examples and experiences in the primary school can facilitate this progression" i.e. "the transfer of understanding from arithmetic through generalised arithmetic to algebra" (WARREN & PIERCE 2004, 295).

RESEARCH INTEREST

With respect to acknowledging and starting the described transfer from arithmetic through generalised arithmetic to algebra at primary school level, two "meta-questions" have been identified by the MERGA Special Interest Group "Early algebraic reasoning in the primary years" at the 27th Annual Conference of the Mathematics Education Research Group of Australasia in July 2004 chaired by T. COOPER & E. WARREN:

- What exactly should be the focus of teaching and learning mathematics following this paradigm?
- Which models and representations are effective and suitable in this context?

These "meta-questions" guided the development of the specific research interest of the author which is to describe and better understand the processes, strategies and possible difficulties of early algebraic learning in a primary school setting.

More specifically, the study in progress seeks to address the following research questions:

1. How do primary students approach and solve algebraic tasks? Which processes and strategies can be identified?
2. Which specific difficulties do they encounter? What are reasons for these difficulties? How can they be overcome?
3. Which ideas and concepts of variable do pupils hold?

METHODOLOGY

The study in progress reported in this paper focuses on the work of a number of selected tasks addressing the following aspects of algebraic thinking and understanding identified in the research literature: generalisations, continuing and representing patterns, the use of symbolic language, and different aspects of variables (variable as a certain unknown number and as a description for a general statement, see MALLE 1993).

The design of the study includes a *pilot study* (conducted in June 2004) and a *main study* (to be conducted in March 2005). Both, pilot and main study follow a qualitative research paradigm focussing on the interpretative analysis of children's early algebraic processes and concepts. The data collection involves videos of pupils while solving a range of algebraic tasks with a partner (both partners have a similar level of ability in mathematics) followed by a semi-structured interview with selected pairs which is also videotaped.

Since it was one of the goals of the pilot study to develop an optimal design for the main study, at this stage details of the data collection and analysis entirely focus on the pilot study.

Design of the Pilot Study

A selection of algebraic tasks was given to the teacher of a grade 4 classroom (17 students) in a primary school in a larger city in north-western Germany, who over a week (five lessons) asked the students to work in pairs on these tasks. Each day two groups were videotaped to ensure video-recordings of at least one episode (45 minutes) per group. In addition, one of the two pairs was interviewed and asked to explain their solution processes and the difficulties they had encountered as well as to solve a similar task on the following day. These interviews lasted about 20 minutes.

Comprehensive transcriptions with respect to either the full document or selected parts of the recording relevant to the specific research questions were made.

The interpretative analysis of the data involves both – generating categories as well as being guided by categories identified in the literature (e.g. on different aspects of variables and student difficulties and misconceptions).

Examples of Tasks Selected for the Pilot Study

Another goal of the pilot study was to identify tasks that induce pupils to work and think algebraically and student interaction during the solution process.

In the following three examples of the ten different types of tasks selected for the pilot study. Task 1 addresses translation processes and the use of symbolic language. The variable in this context refers to a certain unknown number. The term "calculation language" (German: *Rechensprache*) was already introduced in the class prior to the data collection as a term to describe symbolic language. The demands of task 2 relate to generalisations, the use of symbolic language and the understanding of variable as a description for a general statement. Task 3 focuses on recognising, continuing and representing patterns as well as generalisations. In addition to the card with the written task the students received a packet of counters.

Task 1

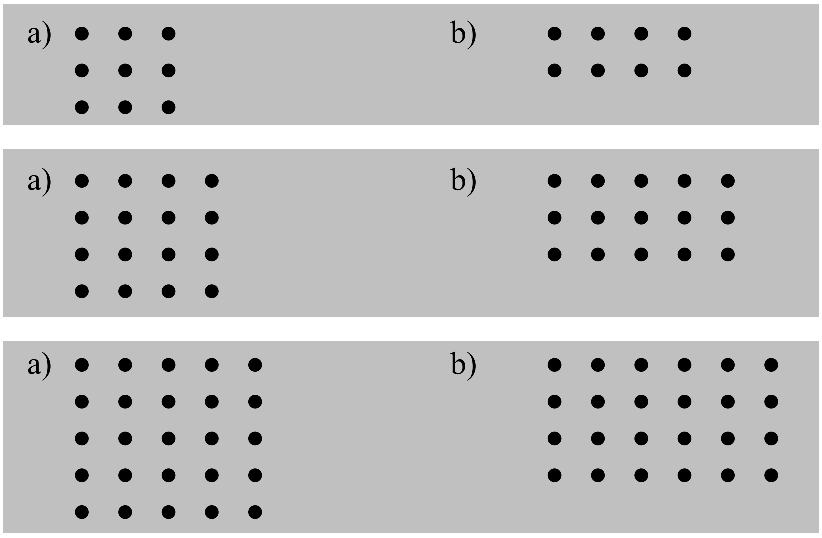
Translate the following word problem into 'calculation language': *I think of a number. I take my number times 2, add 7 and then subtract 9. The answer I get is 16. Which is my number?*

Task 2

a) I have a number and I add zero. What happens to this number?
b) I have a number and multiply it by one. What happens?
c) Which of the following equations describes the statement in a) and which one belongs to b)?

1) $x \cdot 1 = 1$
2) $x \cdot 1 = 0$
3) $x \cdot 1 = x$
4) $x + 0 = 0$
5) $x + 0 = x$

Task 3



Arrange the counters in the patterns shown above. First arrange the counters in pattern a) then in pattern b). What do you notice? Explain your observation and then arrange another pair and describe the structure.

DATA ANALYSIS AND DISCUSSION OF INTERIM RESULTS FROM THE PILOT STUDY

With respect to the presentation and discussion of first interim findings from the pilot study it should be noted that the text book used in the grade 4 class chosen for the pilot study includes tasks that are designed to foster the development of early algebraic thinking. The teacher is a highly experienced and has been involved in several mathematics education research projects in collaboration with the university and contributes as an expert teacher to methods courses at the local university's teacher training program.

The initial analysis of the tasks that follow the format of task 1 suggests that simplifications seem not to be obvious to the children. In the task "*I think of a number, I take it times 30, divide it by 30 and then subtract 7. The result I get is 13. Which is the number I was thinking of?"* even the high achieving pupils did all the calculations to get the answer. Only one child respectively made an ironic comment that indicates her understanding that it was unnecessary to first multiply by and the divide by 30.

Furthermore, general relationships were recognised when presented in a verbal form. With respect to task 2a) the immediate response of the observed children was "the number remains the same". However, when symbolic language was used as in task 2c) all students, including the high achievers, encountered difficulties. These difficulties primarily relate to the comprehension of symbolic language and the required translation processes. According to MALLE (1993) one can distinguish between different aspects of variables:

- a variable as a certain unknown number (unknown) and variable as a description for a general statement (undefinite),
- a mark for a place where one can put a number
- a formal aspect (as a formal sign without meaning with which you do calculations).

However, other authors additionally highlight further aspects of variables, e.g. variable as a statement about relationships between two quantities, functional and dynamical aspects.

In general, the students did not experience problems with the first type of tasks mentioned above (task 1), i.e. recognising variables as certain unknown numbers or to write an x for such a number. But the use of variables to describe general rules or relationships (indefinite) did not appear to be obvious for them. They did not seem to understand that particular meaning of variable as their solution strategies indicate that they thought that the task was not solvable because they did not know what to calculate (see e.g. in the following transcript (00:09:40), (00:12:35)).

The transcript of the video-recording of two high achieving girls dealing with task 2 (see below) suggests, that they are struggling with the different aspects of variables, especially with the aspect of variables as indefinites (versus unknowns) (see e.g. (00:03:35), (00:09:45), (00:12:00). (00:12:35) etc.). The following segment of the transcript starts immediately after the two students have read question 2a) *"I have a number and I add zero. What happens to this number?"*:

(00:03:23):	Leonie	it remains the same.
	Anne	the number remains the same, exactly
	Leonie	wait …
	Anne	no
	Leonie	hm, oh
(00:03:35)	Anne	x times zero, the result is zero
	Leonie	yes, I have a number, it can't be zero, because you don't know which one
	Anne	no, now if we, if we now for example, it could also be 1792
	Leonie	and now times zero, then I still have two thousand four hundred nine something

They immediately found the answer to question a), afterwards they tried to write the answer in symbolic form – which was their own idea and not required in the task – and continued their initial discussion. During working with this task they always came back to the aspect of variables as certain unknown numbers and tried to find a solution with numbers.

(00:04:33)	Anne	shit, and *add* zero, the number remains the same
(00:05:12)	Anne	you can…, you can write the corresponding equation as well, write just, lets say, three pl…, write x plus zero equals zero
	Leonie	x plus zero equals zero, no, not zero, equals y.
	Anne	ok, just write any number instead of x, write three plus zero equals three.
	Leonie	no, I leave it now at that
	Anne	no
	Leonie	why
	Anne	that's foolish
	Leonie	why that's foolish, anyway you can't calculate it
	Anne	no, because, hm, don't know, somehow that's wrong like this
	Leonie	hello?!
	Anne	yes, because then one can see, hm
	Leonie	then I make here now… *[writes]*
	Anne	because then you can see, that this number remains the same
	Leonie	yes, then we have to make it as an *example*

The following segment documents the discussion of the two girls when they work on task 2c).

(00:09:27)	Anne	I think, hm, x plus zero equals zero and here again x plus zero equals zero so this are *[indicates on the paper which equations she thinks are corresponding to one another]*
	Leonie	no, wait, you don't know what x is
(00:09:40)	Anne	I know that's why this task doesn't mean anything to me
	Leonie	do you know, do you know what x is?
	Anne	no, I don't know and that's why this task doesn't mean…
(00:09:45)	Leonie	zero, x is zero, that's for sure look, x plus zero equals zero

	Anne	right
(00:12:00)	Anne	which, which number is x?
	Leonie	zero!
	Anne	zero?
	Leonie	yes, because it says x plus zero equals zero
(00:12:25)	Anne	I'm going to ask Mrs. K. *[the teacher],* I'm going to ask Mrs. K. what x is
	Leonie	*[whistles]*
(00:12:35)	Anne	or wait, we can easily calculate it
	Leonie	we can't calculate that.
	Anne	zero divided by one equals…
	Leonie	one divided by one equals one one times one is one
(00:15:02)	Leonie	yes, five, I would take five look, because you don't know what the result is
	Anne	yes, so, describe, so, everywhere where it belongs to a) we write a) behind it
	Leonie	here is a), this is a)
(00:16:14)	Anne	so, let's say x here is zero, then it will be right
	Leonie	hm, let's make this now
	Anne	which number actually means x?
	Leonie	what?
(00:16:33)	Anne	we calculate backwards, now God, that's also x so, zero plus zero equals zero
	Leonie	no, no, that's plus you have to forget these now, you have to forget these now I would say this is this, because again you don't know the result
	Anne	one divided by one equals one, therefore x is here one hm, zero divided by one…
(00:16:55)	Leonie	I would take x here as well *[indicates the right side of the equal sign]*, because you don't know the result isn't it? *[to the person behind the camera]*

Leonie obviously had the idea to write $x + 0 = x$ for the first statement i.e. using an x on the right hand side of the equation and mentions this idea twice (see (00:15:02) and (00:16:55)). However, her partner, Anne, did not react to this suggestion because she was probably looking for a number as the answer. Hence, she was probably not satisfied with a symbolic term as a result.

	Birte	I don't say anything
	Leonie	please! you know how this works you, you, you don't know which number it is they say, they just say I have a number
	Anne	but then, otherwise you can't solve the question I have a …
	Leonie	yes, the number there can even be zero
(00:17:47)	Anne	yes, it doesn't matter

	every number to which you add zero remains the same, so it doesn't matter what the results is
Leonie	that's right
Anne	every number to which hm you add zero remains the same and that's why this belongs…
Leonie	right, therefore every
Anne	…belongs to…
Leonie	therefore every
Anne	... *[pattern]* a
Leonie	yes, so every equation.

They finally agreed and expressed that all equations with $x + 0$ on the left hand side of the equal sign belong to statement a), because "every number to which you add zero remains the same, so it doesn't matter what the results is".

Overall, the transcript segment of the pair interaction regarding task 2 above indicates four domains of difficulties related to the understanding of variables:

- to recognise and understand that the symbol x can be used in order to describe general rules and relationships (and hence is not restricted to being used for a certain unknown number),
- to accept a symbolic term as a result for a task,
- to understand the meaning of 'x can be every number'
- to develop a concept of the equal sign that is not limited to the understanding that the equal sign means 'give the answer'

These domains will be explored further in the main study.

With regard to task 3 some students immediately realised that pattern b) could be generated simply by moving one row of pattern a) and using this row as a column in pattern b). They also discovered quickly that pattern a) always requires more counters than pattern b). While the number of counters was not asked in the task, some children calculated the number, because they wanted to know how many counters were needed. Some children stopped here, satisfied having found that for the pattern a) always one more counter than for pattern b) was needed. Others recognised further that the first shape is always a square and hence the number of counters is always a square number – "a number multiplied by itself", while the term that describes pattern b) is characterised by the fact that the first factor is always one smaller and the second factor one larger while the total number is always one smaller.

However, none of the children who participated in the pilot study in any way expressed insight in the general relationship $n \cdot n - 1 = (n-1) \cdot (n+1)$ during the work in class. Only during one of the subsequent interviews two girls (Anne and Leonie) discovered that relationship, but they were not sure whether it is true for all numbers.

REFERENCES

Anthony, G., Walshaw, M.: 2002, 'Swaps and switches: Students' understanding of commutativity', in B. Barton, K. Irwin, M. Pfannkuch, M.O.J. Thomas (eds.), *Mathematics education in the South Pacific, Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia*, MERGA, Sydney, Australia, pp. 91-99.

Bills, L., Ainley, J., Wilson, K.: 2003, 'Particular and general in early symbolic manipulation', in N.A. Pateman, B.J. Dougherty and J. Zilliox (eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Honolulu, HI, Vol. 2 , pp. 105-112.

Blanton, M.: 2004, in the lecture from M.L. Blanton and J.J. Kaput, 'Elementary grades students' capacity for functional thinking', during *The 28th International Conference for the Psychology of Mathematics Education,* PME 28, Bergen, Norway.

Booth, L.: 1988, 'Children's difficulties in beginning algebra', in A.F. Coxford and A.P. Schulte (eds.), *The ideas of algebra, K-12. Yearbook*. National Council of Teachers of Mathematics, Reston, VA.

Carpenter, T. P., Levi, L.: 2000, *Developing conceptions of algebraic reasoning in the primary grades*. University of Wisconsin-Madison, Report No. 2.

Dougherty, B.: 2003, 'Voyaging from theory to practice in learning: Measure up', in N.A. Pateman, B.J. Dougherty and J. Zilliox (eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Honolulu, HI, Vol. 1, pp. 17-23.

Freiman, V., Lee, L.: 2004, 'Tracking primary students' understanding of the equality sign', in M.J. Høines, A.B. Fuglestad (eds.), *Proceedings of the 28th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Bergen, Norway, Vol. 2, pp. 415-422.

Kieran, C.: 1992, 'The learning and teaching of school algebra', in D. A. Grouws (ed.), *Handbook of research on mathematics teaching and learning*. Macmillan, New York, pp. 390-419.

Malara, N., Navarra, G.: 2003, *ArAl Project: Arithmetic pathways towards favouring prealgebraic thinking*. Bologna, Italy: Pitagora Editrice.

Malle, G.: 1993, *Didaktische Probleme der elementaren Algebra*, Braunschweig, Germany: Vieweg,

Warren, E.: 2001, 'Algebraic understanding and the importance of operation sense', in M. van den Heuvel-Panhuizen (ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education*, PME Program Committee, Utrecht, The Netherlands, Vol. 4, pp. 399-406.

Warren, E.: 2003, 'The role of arithmetic structure in the transition from arithmetic to algebra', Mathematics Education Research Journal Vol. 15, No 2, pp. 122-137.

Warren, E., Pierce, R.: 2004, 'Learning and teaching algebra' in B. Perry, G. Anthony and C. Diezmann (eds.), *MERGA – Research in Mathematics Education in Australasia 2000-2003*, Post Pressed Flaxton, Sydney, Australia, pp. 291-312.

Zevenbergen, R, Dole, S., Wright, R.J.: 2004, 'Patterns and algebra', *Teaching Mathematics in Primary Schools*. Allen & Unwin, Sydney, Australia, pp. 233-249.

FROM TABLES OF NUMBERS TO MATRICES: AN HISTORICAL APPROACH

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Abstract: *The present study focuses on the teaching and learning process of the concept of matrix in the transition between the secondary school and the university. In particular, two interesting aspects arise in the study of this concept:*

- *The first aspect is the spatial disposition of their elements, which differs from the linear disposition usually used in primary and secondary school.*
- *The second one is the notation associated with matrices, i.e., the usage of subscripts and superscripts, usage of other symbols, etc.*

As a first step in the study of these effects in the secondary school - university transition, I present in the an historical analysis of the concept in order to look at the evolution of the idea of matrix and the different notations used over its history. This analysis provides a basis to the identification of possible epistemological and ontogenetic obstacles in the learning of this concept. Another analysis consists in looking at the syllabuses and textbooks of secondary school and the textbooks usually used at the University in France. A comparison with the historical evolution is also presented. In a future study, I will analyse the instrumentality and the semioticity in the notation used in matrices.

Keywords: matrix, high school – university transition, notation, instrumentality valence, historical approach.

MOTIVATION (STARTING POINT)

In observations carried out at the University, I focused on a classic task in first year undergraduate:

For two square matrices A and B, proof that tr(AB)=tr(BA) where tr(A) is the trace of the matrix defined as the sum of the diagonal elements.

To resolve this task, the students were faced to different kinds of difficulties (symbols handling problems, arrangement of elements in arrows and columns, identification of diagonal elements in the notation, etc.). In order to analyse the origin of these difficulties, I looked at the historical evolution of the idea of matrix. The beginnings of matrices and determinants arise from the study of systems of linear equations (but in different ways).

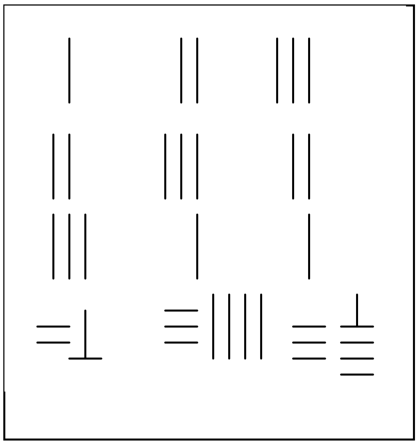
SYSTEMS OF LINEAR EQUATIONS

When one studies the evolution of the concept "matrix", two aspects emerge in order to resolve a system of linear equations: matrix (as a table) and determinants.

FROM TABLES OF NUMBERS...

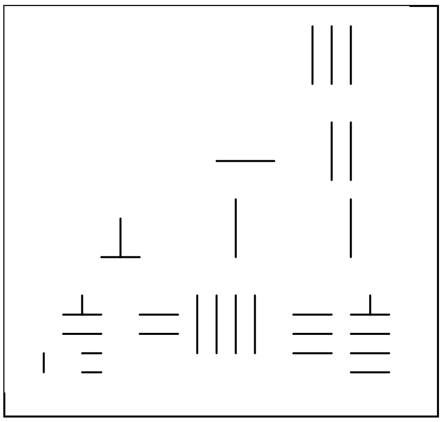
✓ The Chinese and the 'counting board' (200 BC)

The Nine Chapters on the Mathematical Art holds the first known example of a matrix as a table. For example:



This table shows the coefficients of the system of three linear equations with three unknowns.

Multiplying one column by an appropriate number and then subtracting this product as many time as possible from another column, the Chinese rewrite the table as:



from which the solution can be found for the third unknown, followed by the second, and then the first via substitution. This method would not become well known until the early 19th Century.

FIRST IDEAS ABOUT DETERMINANTS AND DIFFERENT NOTATIONS

✓ Leibniz (1693): the system of equations

10 + 11x + 12y = 0
20 + 21x + 22y = 0
30 + 31x + 32y = 0

"12" denotes the coefficient matrix element a_{12}

has a solution because

$$\begin{array}{ccc} 1_0 \,.\, 2_1 \,.\, 3_2 & & 1_0 \,.\, 2_2 \,.\, 3_1 \\ 1_1 \,.\, 2_2 \,.\, 3_0 & = & 1_1 \,.\, 2_0 \,.\, 3_2 \\ 1_2 \,.\, 2_0 \,.\, 3_1 & & 1_2 \,.\, 2_1 \,.\, 3_0 \end{array}$$

Here, he uses the subscript to indicate the unknown's number

which is exactly the condition that the coefficient matrix has determinant 0.

✓ **Cramer (1750): let the system of equations**

$$A^1 = Z^1 z + Y^1 y$$
$$A^2 = Z^2 z + Y^2 y$$

Cramer uses the superscript to indicate the equation's number

the solution is

$$y = \frac{Z^1A^2 - Z^2A^1}{Z^1Y^2 - Z^2Y^1} \quad z = \frac{A^1Y^2 - A^2Y^1}{Z^1Y^2 - Z^2Y^1}$$

✓ **Method of Gaussian elimination:** was used by Gauss in his work on the study of the orbit of the asteroid Pallas. Using observations of Pallas taken between 1803 and 1809, Gauss obtained a system of six linear equations with six unknowns. Gauss provided a systematic method for solving such equations.

... TO MATRICES

✓ **Sylvester (1850):** uses the term "matrix" for the first time. He defined a matrix to be an oblong arrangement of terms and saw it as something which led to various determinants from square arrays contained within it.

✓ **Cayley (1855):** gives the first abstract definition of matrix and he extracts the idea of matrix of the determinant.

He writes

$$(\xi,\eta,\zeta,...) = \left(\begin{array}{cccc} \alpha, & \beta, & \gamma, & ... \\ \alpha', & \beta', & \gamma', & ... \\ \alpha'', & \beta'', & \gamma'', & ... \\ \vdots & & & \end{array}\right)(x,y,z,...)$$

to represent the system

$$\xi = \alpha\, x + \beta\, y + \gamma\, z...,$$
$$\eta = \alpha' x + \beta' y + \gamma' z...,$$
$$\zeta = \alpha'' x + \beta'' y + \gamma'' z...,$$
$$\vdots$$

MATRICES IN THE SCHOOL

Secondary school

In secondary school in France, matrices do not play an important role. They are merely introduced as tables containing numbers. Their usage is related to the resolution of linear equation sets, and the notation employed is limited to matrices 2x2 or 3x3 of the type:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the usage of indexes (superscripts and subscripts) is not necessary.

- Matrices as instruments in the resolution of systems of linear equations.

University

- Matrices introduced as objects: formal definition, properties, operations...
- At least two types of notations are used, which have different instrumentalities:

✓ a_{ij} : Example of its instrumentality valence:

In statistics, the variance and covariance matrices content elements squared (in the diagonal) so use the another notation to bring some difficulties:

$$\Sigma = \begin{pmatrix} s_2^2 & \cdots & s_{2j} & \cdots & s_{2p} \\ \vdots & & \vdots & & \vdots \\ s_{j2} & \cdots & s_j^2 & \cdots & s_{jp} \\ \vdots & & \vdots & & \vdots \\ s_{p2} & \cdots & s_{pk} & \cdots & s_p^2 \end{pmatrix}$$

✓ a_i^j : Example:

Tensor is an important concept in physique and we can think the matrices as a tensor of exactly two dimensions. The tensor may have an arbitrary number of indices, each index ranges over the number of dimensions of space.

In addition, a tensor with rank $r+s$ may be of mixed type (r, s), consisting of r so-called "contravariant" (upper) indices and s "covariant" (lower) indices. Note that the positions of the slots in which contravariant and covariant indices are placed are significant so, for example, a_t^{sr} is distinct from a_{ts}^r.

QUESTIONS TO ANALYSE

- High school - university transition? Which semioticity and instrumentality can explain the choice of a notation rather the other one when resolve different kinds of problems?
- Type of obstacles: epistemological, ontogenetic, didactical...?

METHODOLOGY

- Implementation of a given task to Terminal ES students at the end of the academic year.
- Implementation of the same task to some of the previous students once they become students of first year undergraduate (the selection criteria have to be determined) at the end of the first and second terms in the University.

REFERENCES

ANTIBI A. et al. (2002), *Transmath Term ES (obligatoire et option)*, Nathan, Paris.

ANTIBI et al. (2001), *Transmath Première ES (obligatoire et option)*, Nathan, Paris.

BOSCH M. and Y. CHEVALLARD (1999), Ostensifs et sensibilité aux ostensifs dans l'activité mathématique, *Recherches en Didactique des Mathématiques*, 19(1), pp. 77-123.

BROUSSEAU G. (1983), Les obstacles épistémologiques et les problèmes en mathématiques, *Recherches en Didactique des Mathématiques, 4(2),* pp. 165-198.

BROUSSEAU G. (1986), Fondements et méthodes de la didactique des mathématiques, *Recherches en Didactique des Mathématiques, 7(2)*, pp. 33-115.

CAJORI F. (1952), *A history of mathematical notations*, Third printing, Paquin Printers, Chicago.

CAYLEY A. (1855), Remarques sur la notation des fonctions algébriques, *Journal de Crelle*, Vol. 50 ; in *Collected Math. Papers*, Vol II, pp. 185-188.

CHARTIER G. et al. (1999), *Algèbre linéaire : du lycée à la fac*, IREM de Rennes

COLLETTE J.-P. (1979), *Histoire des mathématiques*, Editions du Renouveau Pédagogique, Canada.

DIEUDONNE J. (1978), *Abrégé d'histoire des mathématiques - 1700-1900*, Tome II, Hermann, Paris.

DOUADY R. (1986), Jeux de cadres et dialectique outil-objet, *Recherches en Didactique des Mathématiques,* 7(2), pp. 5-31.

MARTZLOFF J.-C. (1988), *Histoire des mathématiques chinoises*, Masson, Paris.

YABUUTI K. (2000), *Une histoire des mathématiques chinoises*, Editions Belin, Paris.