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**GROUP 1 : THE ROLE OF IMAGES AND METAPHORS IN THE
LEARNING AND UNDERSTANDING MATHEMATICS (INCLUDING
EMBODIED COGNITION)**

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For the fourth time, a working group of CERME was devoted to study images and metaphors in the learning and understanding mathematics. A striking feature of this group is that, although most of the participants change from one congress to another, their number remains about the same, *i.e.* 15 to 20. This can be interpreted as a sign of a constant interest in these questions among researchers in mathematics education involving new (and mostly young, as it appears) people through time.

Eleven papers were submitted to the group. As required by the organising committee, the review process was conceived as a peer review: in our case each paper was reviewed by three members of the group, including one of the four co-leaders, each of them being in charge of 2 or 3 papers. Each reviewer had then to produce a written report, and the co-leader in charge of the paper was asked to make a synthesis of the 3 reports and to decide if the paper could be accepted for discussion in the group, eventually after some changes; the 3 reports and the synthesis were sent to the author(s), who were asked to take into account the changes considered necessary before sending a second version. As a result, all the papers were accepted for discussion.

During the congress the group work was organised in the following way: in the first session, half of the time was devoted to an individual introduction by each member of the group and to a presentation of what had been discussed in the group at CERME 4 and of the research questions which had emerged then, among which are: What are the characteristic metaphors, in use or possible, for different domain of mathematics or different systems of representation? How do metaphors and representations contribute to learning and communicating mathematical concepts? How does the way of using them influence the construction of mathematical concepts? How can we facilitate students' passage from one type of representation to another? How can teaching lead to a change in students' metaphors? What happens when there is a

mismatch between teacher's and student's metaphors?¹. In each of the remaining five sessions two papers were discussed and the final meeting was devoted to the preparation of a 'reporting session', in which the group (in fact, the co-leaders) were to present the work which had been released during the congress. In this session, each author was asked to write down what they thought important to be remembered of their paper as well as the results of their research, and to raise one or two questions which they thought of interest for future research. A PowerPoint presentation was then elaborated on this basis to be presented the next day to CERME participants coming from the other groups.

Ideas and discussion

A metaphor², implicitly compares two domains of experience, giving meaning to elements of one of these domains (the *target* domain) by reference to structural similarities in the other (the *source* domain). The two domains connected by metaphor have similar elements (the *ground* of the metaphor) but also dissimilar elements, which create a *tension* between the two domains (*Presmeg*).

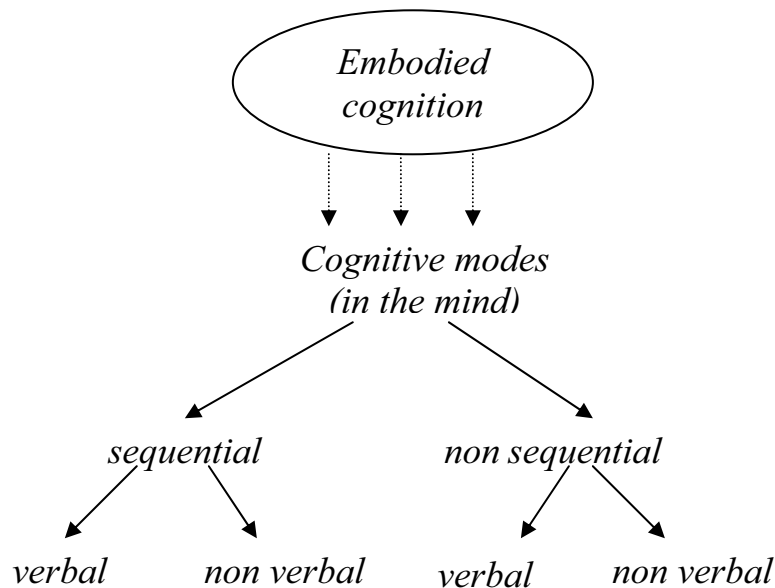
Example : A teacher is a gardener.

- The source domain is gardening, and the target domain is education.
- The ground is the idea of creating suitable conditions for growing.
- The tension is the fact that a student is a human being, contrary to a plant.

The idea that explains how metaphors are linked with bodily experience is the notion of *image schema* (Johnson, 1987, pp 28). A schema is a recurrent pattern which occurs in a person's cognitive activities; image schemata are images associated with such patterns and order in actions, perceptions and conceptions. Cognition may appear under various modes, and a teacher should be aware of that fact; for instance, one can distinguish between verbal and non verbal, or between sequential and non sequential, hence there are four modes when combining them (*Araya*).

¹ See the group presentation by Parzys, B., Pesci, A. & Bergsten, C. in Proceedings of CERME 4.

² '[Metaphor] implies the use of an analogy or close comparison between two things that are not normally treated as if they had anything in common.' (Hutchinson encyclopedia, 9th ed.).



Neuroscience describes the neural organisation which permits the establishment of cognitive modes, and these modes, in turn, structure the output of external registers. One can think that the structure of abstract concepts is communicated more effectively by using the simpler structures of familiar ‘concrete’ concepts; for instance, a rotation metaphor can be useful to construct the ideas of special relativity (*Caines*). The interplay between metaphors and various cognitive modes must be explored, in order to study their relevance for the teaching and learning of mathematics. Multimodal approaches, including different kinds of metaphors, often foster significant understanding. Then a question arises: how can the teacher facilitate connected mathematical learning? (*Soto-Andrade*). This leads to the notion of *reification*. According to Sfard³, “reification – a transition from an operational to a structural mode of thinking – is a basic phenomenon in the formation of a mathematical concept”. The sudden appearance of reification can be considered as the “aha” moment, the moment of real understanding. “It is the birth of the metaphor of an ontological object”⁴. Regarding this question, the ‘written’ (this term including also the drawn) is not merely a visual substitute for the ‘spoken’, but can sometimes show the unspeakable, allowing the emergence of new mathematical ideas. For instance, using ‘operatively’ a diagram, or recognising some similarities between two diagrams, can lead to the solution of a problem (*Kadunz*).

When using metaphors in teaching mathematics, difficulties appear when there is a mismatch between the teacher’s and their students’ focuses. For instance, in the English curriculum, the ‘number line’ metaphor is presented as a ‘key classroom resource’, and highly recommended as a ‘helping tool’. While teachers focus on procedural aspects (and by the way express limited conceptual and structural knowledge), children focus merely on actions and perceptual characteristics, and

³ Reification as the Birth of Metaphor. *For the Learning of Mathematics*, 14, 44-55. 1994.

⁴ *Op. cit.*

express difficulty in reconstructing knowledge about whole and rational numbers (*Doritou & Gray*). Actions using prototypes and embodied objects in interplay with suitable use of protocols, definitions and discourse (*i.e.* interplay between formal and cognitive) is a promising approach for improving the teaching and learning of mathematical ideas (*Rinvold*).

Principal theoretical approaches

Among the eleven papers, two subgroups can be distinguished, the first one being based on the works of Núñez, Lakoff and Johnson on *embodied cognition*, and the second one on the works of Duval on the *representation registers*. In fact, 8 papers quote Núñez, Lakoff and/or Johnson as reference, and 5 papers refer to Duval⁵.

The idea of *embodied cognition*, supported by the convergence of cognitive science, neuro-science, cognitive linguistics and evolutionary anthropology, is based on the assertion that human ideas about such things as number, force, space and time must have their origins in bodily perceptions, and are not disembodied abstractions. They are mental constructions forged out of human experience over an evolutionary time scale. The theory of embodied learning and the conceptual metaphor can be considered as a lens for examining children's informal, intuitive arithmetical knowledge (*Murphy*). Within this framework, a key notion is that of *conceptual metaphor*, which is not merely a figure of speech but is a matter of thought ; it is the mechanism by which the abstract is comprehended in terms of concrete, everyday, sensory-motor experiences such as 'in', 'next' or 'movement'.

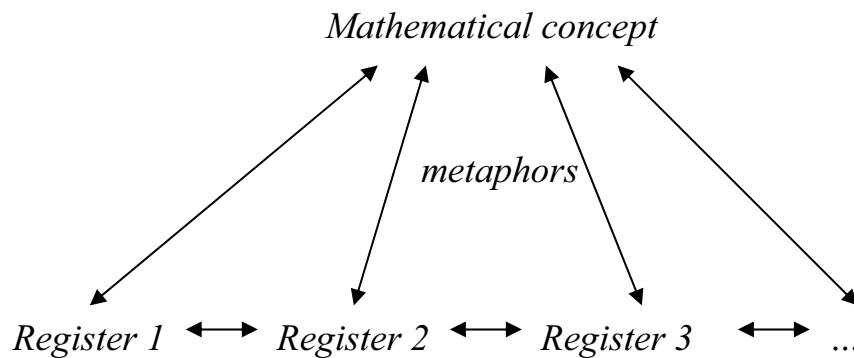
Duval⁶ distinguishes two typical characteristics of the cognitive activity involved in the learning of mathematics: on the one hand, several registers are commonly at play; on the other hand, mathematical objects can never be apprehended perceptively. Two questions arise from this consideration : How is it possible to learn how to move from one register to another ? How do we teach the students not to confuse a mathematical object with its representation ? The origin of many difficulties in the learning of mathematics is linked to these two questions. A classical example is given by geometrical 'figures'. Many students cannot distinguish between the visual 'signifying' information of a geometrical diagram and the geometrical properties of the referent (*i.e.* a theoretical object). Moreover some geometry problems are ambiguous as to the kind of validation wanted by the teacher (*e.g.* May I use measures to justify my conclusion ?). The kind of problem and type of validation are both part of the teacher's responsibility, in order to make the didactic contract (Brousseau) explicit, but most teachers are not aware of this problem (*Gobert*).

Duval defined the notion of a semiotic representation register, which is a system of external modes of semiotic representation, *e.g.* verbal, tables, trees, graphs, geometrical diagrams, *etc.* Problems can arise when several registers can be used to

⁵ As the reader has already guessed, two papers refer to both.

⁶ Quel cognitif retenir en didactique des mathématiques? *Recherche en Didactique des Mathématiques* 16-3, 348-382. La Pensée Sauvage (Grenoble) 1996.

solve a problem. For instance, when solving non-routine problems, students cannot always choose appropriate representations. They often stick to one type of representation (mostly ‘concrete’) and do not (or cannot) move to another one, even when the teacher shows them that another type of representation is more suitable (*Monoyiou, Papageorgiou & Gagatsis*). This lack of flexibility in moving from one type of representation to another can be interpreted as the students’ conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion (*Anastasiou & Gagatsis*). When the research focus is on ways by which students construct connections amongst various mathematical registers, awareness of the role of metaphors in these connections can be a useful research tool.



The discussions that we engaged in during the sessions offered us the possibility of looking for answers to some of the research questions that have emerged from WG1 of CERME 4. We have considered the embodied cognition of Lakoff and Nunez and the representation registers of Duval as principal theoretical approaches when referring to metaphors and representations in teaching and learning practices. Thus, we have tried to give ideas about how metaphors, representations and their ‘links’ can be used by the teachers as a means to communicate mathematical concepts and, by the students, to learn and understand those concepts. We have also highlighted the risks and problems of using metaphors and different representations in teaching and learning practices. One of these risks is that the intended meaning implied by the use of a metaphor may not always be understood by the learner.

For this reason, we have discussed the essential role of the teacher in managing the interplay between metaphors and various cognitive modes. The wide variety of experience from different contexts provided by members of the Working Group helps us to provide more and more detailed answers to the research questions.

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EXPLORING THE EFFECTS OF REPRESENTATIONS ON THE LEARNING OF STATISTICS IN GREEK PRIMARY SCHOOL

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This study aims to contribute to the understanding of the role of the different types of representations and translations in the learning of statistics in Greek primary school. Specifically, this study investigates the abilities of 3rd, 5th and 6th grade primary school students in using representations (verbal, tabular, graphical and symbolic form) of basic statistical concepts and in moving from one representation to another. Results revealed the differential effects of each form of representation on students' performance and the improvement of performance with age. Representational flexibility and associations were also found to vary across grades.

INTRODUCTION AND THEORETICAL FRAMEWORK

In the field of statistics learning and instruction, representations play an important role as an aid for supporting reflection and as a means of communicating statistical ideas. The NCTM's Principles and Standards for School Mathematics (2000) document include a new process standard that addresses representations.

In this study, we revisited the role of representations in an effort further to understand the nature and structure of representations in developing statistical concepts. We investigated the developmental nature of the ability to use multiple representations and translate from one representation to another.

Representations have been classified into two interrelated classes: external and internal (Goldin, 1998). Internal representations refer to mental images corresponding to internal formulations that we construct of reality. External representations concern the external symbolic organizations representing externally a certain mathematical reality. In this study the term "representations" is interpreted as the "external" tools used for representing statistical ideas such as tables and graphs (Confrey & Smith, 1991). By a translation process, we mean the psychological processes involving the moving from one mode of representation to another (Janvier, 1987). Several researchers in the last two decades addressed the critical problem of translation between and within representations, and emphasized the importance of moving among multiple representations and connecting them (Gagatsis & Elia, 2004, 2005; Gagatsis, Elia & Mouggi, 2002; Hitt, 1998; Yerushalmy, 1997). Duval (2002) claimed that the conversion of a mathematical concept from one representation to another is a presupposition for successful problem solving. According to Elia and Gagatsis (2006) the role of representations in mathematical understanding and learning is a central issue of the teaching of mathematics. The most important aspect of this issue refers to the diversity of representations for the same mathematical concept, the connection between them and the conversion from one mode of representation to others. Gagatsis and Shiakalli (2004) and Ainsworth (2006) suggest that different representations of

the same concept complement each other and contribute to a more global and deeper understanding of the concept.

The understanding of a mathematical concept presupposes the ability to recognise the concept when it is presented using a series of qualitatively different representation systems, the ability to handle the concept flexibly in the specific representation systems and finally, the ability to translate the concept from one system to another (Lesh, Post & Behr, 1987). In statistical education, the interest focuses both on the various types of representation and on the translations between them.

This study intends to shed light on the role of different modes of representation on the understanding of some basic concepts in statistics. The study was designed to explore primary school students' performance in using multiple representations of statistical concepts with emphasis on the effects exerted on performance and on the relations among the various conversion abilities from one representation to another by the age of the students.

METHOD

Participants

The sample of the study involved 220 third grade students (age 9), 225 fifth grade students (age 11) and 229 sixth grade students (age 12) from primary schools in four regions of Western Macedonia. These regions were selected because of their diversity in size, population and geographic location. In particular, the four regions vary in terms of geographic location, student population, school size, student achievement, ratio of the number of students to the number of teachers, teachers' methods of instruction, number of schools and proportion of teachers who received a recognized teacher education program at a university in their initial training. Below we briefly describe the content of teaching that students receive in statistics in the third, fifth and sixth grade of primary school according to the Greek curriculum, in order to give some information on students' prior knowledge.

The content of statistics in the third, fifth and sixth grade

According to the curriculum, third grade primary school students are taught to: record data, portray data through the relevant graphic and tabular representations, make assumptions and predictions regarding the results of the relevant actions, and reach the relevant conclusions based on the data. Additionally, third grade students must be able to: perceive the concepts of chance and probability, as well as the relationship between them, detect probable events, calculate the frequency of events and categorise the relevant statistical data and create the relevant tables.

Fifth grade students have often come across the terms "mean value" and "average" in mathematics problems and can perhaps understand their meanings intuitively. According to the curriculum, fifth grade primary school students are taught: the meanings of the terms "average" and "mean value", how to read simple statistical tables and charts, how to use charts in order to present specific statistical data and

how to empirically interpret the meaning of research. Fifth grade students are expected to: understand the meaning and process of finding the average of the numbers provided; know the concepts of research, research population, research sample and research conclusions; know the basic steps that are required for conducting research, which include recording statistical data, sorting data, working out the absolute and relative frequency, graphic representations, calculating the average and formulating predictions and conclusions. Students themselves are required to gather and present statistical data that are drawn from their school and wider social environment.

In sixth grade students are taught to: record data, read simple statistical tables and charts, portray data through relevant graphic and tabular representations, read a table, extract information from it and convert it to a verbal or tabular representation, calculate the average and formulate predictions and conclusions. Additionally, sixth grade students must be able to: work out the absolute and relative frequency, calculate the frequency of events and categorise the relevant statistical data, construct the relevant tables and bar or pie or histogram charts, understand the meaning and process of finding the average of the numbers provided.

Tasks and variables

A test was developed and administered to the students of the three grades. The test consisted of 6 tasks on frequency tables, bar charts and their application to solving everyday problems. These 6 tasks can be divided into three groups of two similar problems about the point of proposed representations nevertheless the content of the problems were different. In particular, the first task gives some information in verbal form and students are required to give the graphic form of this information (bar chart) (V1vg), while the second task gives information of the same kind in verbal form and requires its transformation into tabular form (V2vt). The second task is the following: “The values that follow represent the height of six children: Maria 100cm, Nicos 120cm, Kostas 132cm, John 140cm, Ann 114cm. Represent these data on a table.” The third task involves reading a table (see Table 1) of the frequency of students’ grades, extracting information from it and giving an interpretation in verbal form (V3tv).

Grade	6	7	8	9	10
Frequency	1	3	4	6	5

Table 1: The table included in the third task of the test

The fourth task involves reading a bar chart, extracting information from it and giving an interpretation in verbal form (V4gv). The fifth task involves reading a bar chart, extracting information from it and converting it to a tabular representation (V5gt). The sixth task involves reading a frequency and relative frequency table, extracting information from it and converting it to a bar chart (V6tg).

Right and wrong (from our point of view) were scored as 1 and no answers 0, respectively. Students' responses to the tasks comprise the variables of the study which were codified by an uppercase V (variable), followed by the number indicating the exercise number. Following is the letter that signifies the type of initial representation (e.g. r=representation, t=table, g=graphic, v=verbal) and, lastly, comes the letter that signifies the type of final representation.

Data analysis

For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier & Gras, 2000). This method of analysis determines the similarity connections of the variables. In particular, the similarity analysis is a classification method which aims to identify in a set V of variables, thicker and thicker partitions of V, established in an ascending manner. These partitions, when fitted together, are represented in a hierarchically constructed diagram (tree) using a similarity statistical criterion among the variables. The similarity is defined by the cross-comparison between a group V of the variables and a group E of the individuals (or objects). This kind of analysis allows for the researcher to study and interpret in terms of typology and decreasing similarity, clusters of variables which are established at particular levels of the diagram and can be opposed to others, in the same levels. It should be noted that statistical similarities do not necessarily imply logical or cognitive similarities.

The construction of the similarity diagram is based on the following process: Two of the variables that are the most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables gets very weak. Based on this process, it is evident that the shorter the vertical lines in the diagram the stronger they are. The red horizontal lines represent significant relations of similarity.

RESULTS

Descriptive results

Table 2 presents the success rates of third, fifth and sixth grade students in all types of conversions.

Students' success in each grade varies across the different conversion tasks. Considering the lowest and the highest percentage in each grade, this variation decreases with age: third grade, 22-36%; fifth grade, 48-59%; sixth grade, 75-82%. These findings showed that the success rate improves with age and continued instruction. Continued instruction help students to carry out conversions of statistical concepts more successfully. According to Duval (2002) the conversion of

representations is considered as a fundamental process leading to mathematical understanding and successful problem solving. Thus a conversion between representations to another does ensure the correct understanding of the particular mathematical-statistic concept. Duval (2002) and Even (1998) the ability to identify and represent the same concept through different representations is considered as a prerequisite for the understanding of the particular concept.

Tasks	Type of translation	Third grade success rate (%)	Fifth grade success rate (%)	Sixth grade success rate (%)
V1vg	Verbal - Graphic	33.15%,	54.6%	82.3%
V2vt	Verbal - Tabular	23.5%	62.3%	76.5%
V3tv	Tabular - Verbal	22.2%	60.4%	75.8%
V4gv	Graphic - Verbal	30.16%	48.4%	79.1%
V5gt	Graphic- Tabular	24.4%	53.7%	74.6%
V6tg	Tabular - Graphic	35.56%	59.2%	79.4%

Table 2: Success rates of students in the tasks

In order to examine in a more comprehensive way the differences between 3rd, 5th and 6th grade students with regard to their performance in the various tasks and the interrelation of their responses, a comparison was made between similarity diagrams 1, 2 and 3 concerning the 3rd, 5th and 6th grade respectively.

Similarity analysis results

The similarity diagrams in this study concern the data of each grade separately, and allow for the arrangement of students' responses (V1vg, V2vt, V3tv, V4gv, V5gt, V6tg) to the tasks into groups according to their homogeneity.

Two clusters (Cluster A and B) of variables are identified in the similarity diagram of third grade students' responses as shown in Figure 1. The strongest similarity occurs between variables V1vg and V6tg in Cluster A. It is suggested that students employed similar processes to construct a graph based on information given verbally or in a table. The similarity connection of the variables V1vg and V6tg to the variable V4gv reveals students' consistency with regard to their performance in constructing a graph and their performance in drawing information from the graph and interpreting it verbally. Cluster B consists of the variables V2vt, V5gt and V3tv. It is suggested that students dealt consistently with the tasks that required the construction of a table based on information given in verbal or in graphic form, as well as, with the task involving the verbal interpretation of its data.

The formation of the two distinct clusters indicates that students dealt differently with conversions requiring the construction of a graph or the verbal interpretation of a graph (V1vg, V6tg, V4gv), relatively to the conversions involving the creation of a frequency table or a verbal description of the data given on a table (V2vt, V5gt,

V3tv). This suggests that students in third grade treated the graphic and the tabular representations in isolation. Students' higher success rates at the tasks of the first cluster (V1vg: 33.15%, V6tg: 35.56%, V4gv: 30.16%) relatively to the tasks of the second cluster (V2vt: 23.5%, V5gt: 24.4%, V3tv: 22.2%) indicate their greater difficulty in tackling the second group of tasks and provide further support to the above assertions.

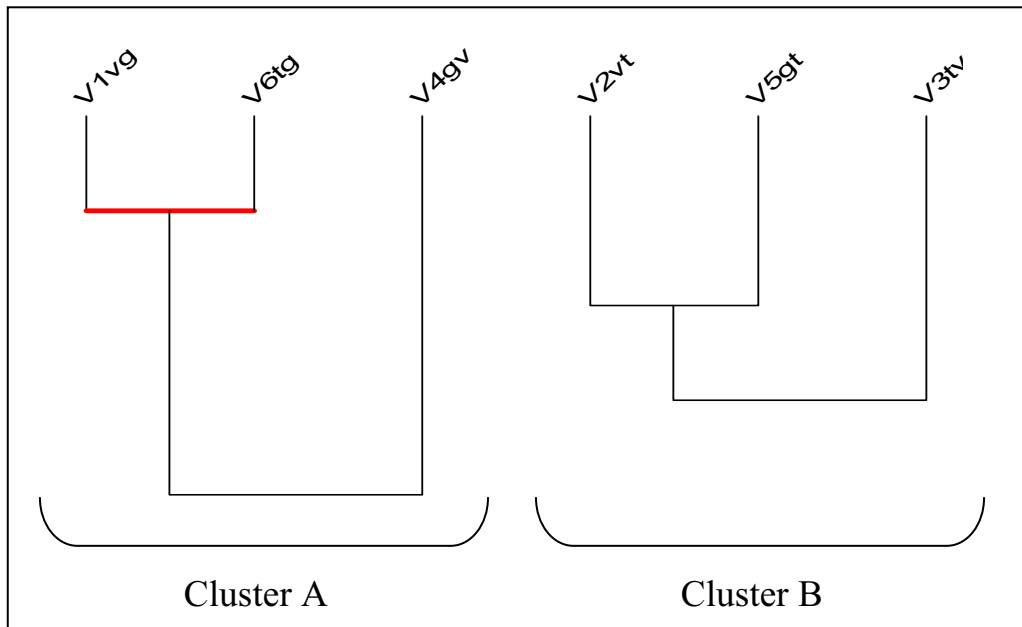


Figure 1: Similarity diagram of third grade students' responses

The similarity diagram of the fifth grade students' responses, illustrated in Figure 2, involves three pairs of variables (V1vg-V2vt, V4gv-V5gt, and V3tv-V6tg). This grouping suggests that students dealt similarly with the conversions involving the same initial representation that is verbal form, graph and table.

Thus, the initial representation of the task had an effect on the conversion or interpretation processes employed by the fifth grade students. The similarity cluster (Cluster B) of the variables including the table as a starting representation (V3tv-V6tg) is disconnected from the other similarity pairs which form a joint cluster (Cluster A), indicating students' compartmentalized ways of handling frequency tables and the other forms of representation, i.e. graph and text.

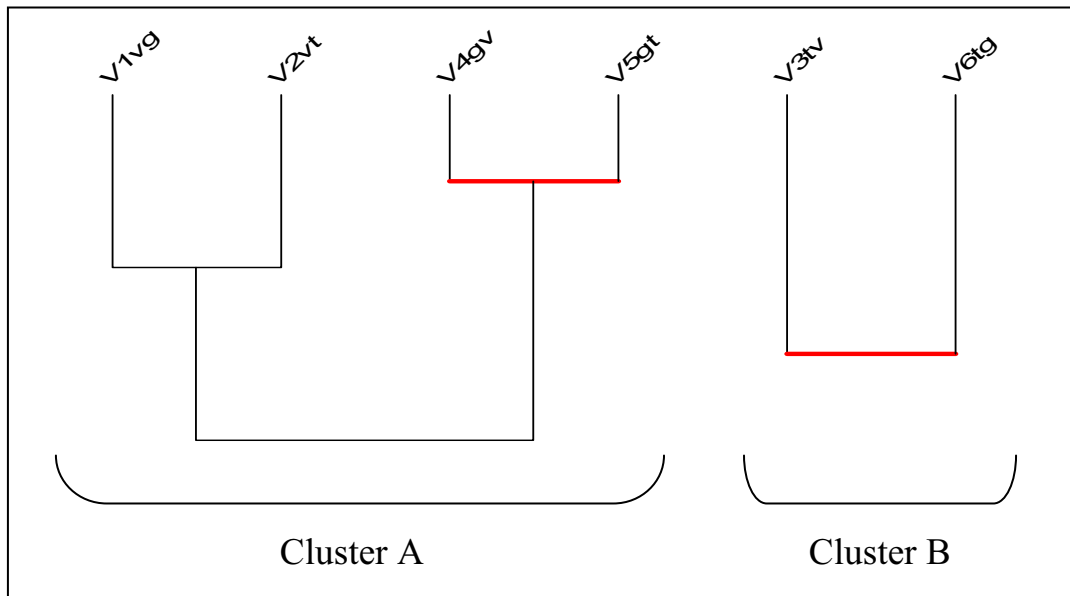


Figure 2: Similarity diagram of fifth grade students' responses

The strongest similarity in the similarity diagram of the sixth grade students' responses, illustrated in Figure 3, occurs between the variables V1vg and V6tg. This similarity reveals sixth grade students' consistency in their processes when constructing graphs on the basis of verbal or tabular representations. Students' responses to the other tasks are interwoven in the similarity diagram, indicating students' coherence in dealing with the corresponding conversions irrespective of their initial or target representation. Students' high success rates at all of the tasks of the test ranging from 74.6% to 82.3% provide further evidence for this assertion.

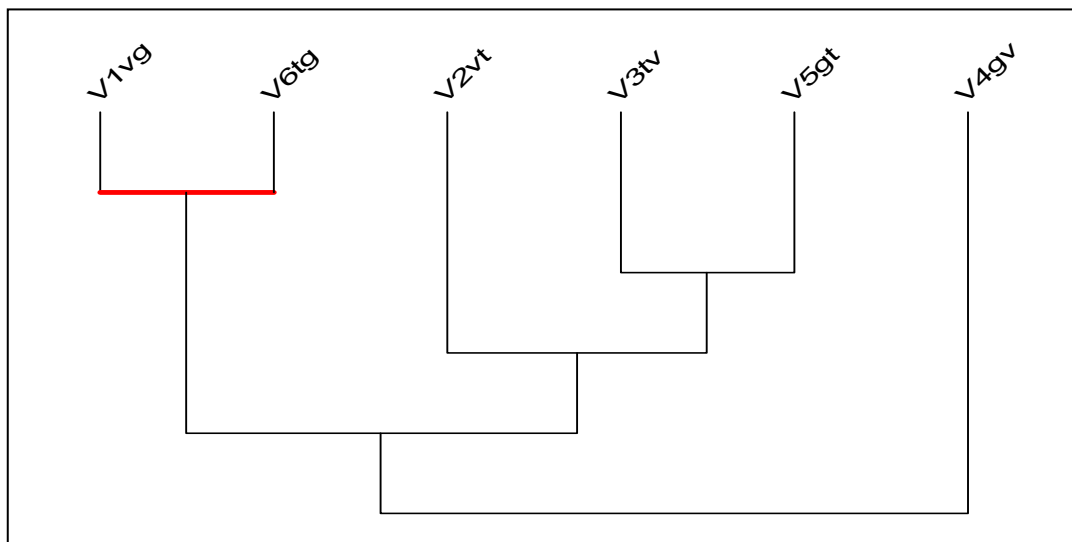


Figure 3: Similarity diagram of sixth grade students' responses

CONCLUSIONS

Representations enable students to interpret situations and to comprehend the relations embedded in problems (Christou, Gagatsis & Zachariades, 2001).

Representations are considered as extremely important with respect to cognitive processes in developing statistical concepts (Kaput, 1987). The main contribution of the present study is the identification of students' abilities to handle various representations, and to translate among representations related to the same statistical relationship across three age levels in primary education. Our findings provide a strong case for the role of different modes of representation on students' performance to tasks on basic statistical concepts such as frequency according to Duval's theory of representations (Duval, 2002). At the same time they enable a developmental interpretation of students' difficulties in relation to representations of frequency.

Students' success was found to increase with age. Moreover, the three similarity diagrams clearly showed the different ways in which third, fifth and sixth grade students dealt with tasks involving different representations of statistical concepts. Third grade students despite showing consistency in constructing graphs and tables separately on the basis of other forms of representation treated these representations in isolation in their conversion processes. Their conversion processes depended on the target representation of the conversions and specifically on their abilities to construct a graph or a table. However, they showed inconsistency in interpreting graphs and tables verbally. Students encountered greater difficulty in analysing and interpreting data given in tabular form rather than in graphic form. This suggests that they have not yet developed the ability to read and interpret-verbally various forms of representation of statistical concepts.

Compared to the younger students, fifth grade students have developed their abilities to construct a table or a graph and to interpret statistical data given in different forms verbally. However, their abilities were found to depend on the initial representation of the conversions. This behaviour yielded consistency in dealing with conversion tasks involving the same initial representation. The different modes of representation though, remained isolated in fifth grade students' processes, as well.

These findings show that despite the improvement of students' performance from third to fifth grade, students in both grades encountered difficulties in the understanding of statistical concepts and more specifically in moving flexibly from one representation to another. Lack of connections among different modes of representations indicates the difficulty in handling two or more representations in mathematical tasks. This incompetence is the main feature of the phenomenon of compartmentalization in representations, which was detected in both third and fifth grade students (Duval, 2002). This inconsistent behaviour can be seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion.

This phenomenon did not appear in the performance of sixth grade students. Their success was found to be independent of the initial or the target representation of the tasks. Their high and consistent outcomes in all of the conversion tasks indicate that they have developed the understanding of the relations among representations and the

skills of representing and handling flexibly basic statistical knowledge in various forms. A possible explanation is that the older students could recognize the mathematical structure and content of the tasks in different representations and deal more flexibly with these components, either by translating them to another representation or by giving an interpretation for them. This suggests that development and instruction generate general cognitive strategies that are increasingly independent of representational factors.

Moreover, in Regis Gras method of implicative statistics, ‘part right’ or ‘part wrong’ is scored as 0,5, but in our research we did not use such a case. For this reason, students who were not sure about the correct answer, and who avoided to answer at all were classified as ‘wrong’.

As we noted above, that statistical similarities do not necessarily imply logical or cognitive similarities and a more detailed investigation of pupils answers and what might be regarded as the ‘equivalence’ of the questions would be useful.

At this point we ought to mention that in Greece many teachers’ supports the lack of necessity for the knowledge of statistics, as much in general subjects as in particular such as average term of a set of data, the frequency, and the construction of charts and graphics that derived from their previous studies. The Greek pedagogical departments do not have statistical education in the corpus until recently. We consider that this important factor that influence instruction in third, fifth and sixth grade.

Based on the findings of this study another important question arises: How could instruction in early grades of primary school help students overcome their compartmentalized ways of thinking and enhance their understanding of basic statistical concepts with the support of multiple representations? It would be practically useful if this question was examined by a future experimental study.

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GUESS WHAT IS INSIDE THIS BOX: LOOK AT THESE OPENED BOXES FOR CLUES

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A game used as a metaphor for some statistics and probability concepts is proposed. It is a pattern discovery game where players guess what is inside boxes. Random variables are considered box features, the classification variable is viewed as the box content, and methods to find the most discriminating variables are viewed as concrete ways of arranging boxes. Based on findings about the source of difficulties with probabilities the game promotes a strategy that uses ecologically valid formats. The verbalization of detected patterns as explicit rules expressed with algebraic language is also encouraged. Teaching is presented as hints to play better: how to store and graph information, and how to describe patterns. Evidence of student's deep understanding of the targeted mathematical concepts is presented.

INTRODUCTION

Learning statistics and probabilities is a major challenge. It is very easy to make conceptual mistakes. They are one of the main sources of cognitive illusions and fallacies (Piatelli-Palmarini, 1994; Gigerenzer, 2000). Gigerenzer (1998) emphasizes that using relative frequencies, probabilities and percentages to represent varying degrees of uncertainty is not a natural format for the mind. It is not ecologically valid. By studying human behaviour in uncertain decision-making situations, Gigerenzer concluded it is not that we are bad or inept at probabilities, but rather that our capacity heavily depends on the format used to represent uncertainties. Using the format of natural frequencies and natural sampling the illusion disappears. Instead of saying that the probability of an event occurring is 0.33, it is psychologically clearer to say that a given event will occur 33 out of every 100 times. This may look odd: although mathematically these forms are equivalent, to the mind they are not.

Brase, Cosmides and Tooby (1998) add two more difficulties. They claim that we are not made to estimate uncertainty for a single event: it is too abstract, we do not run across this type of situation in real life. In other words, psychologically it makes no sense to talk of the probability of heads coming up for a single throw of the coin. What does make sense is to say that for every 100 times the coin is tossed, heads will approximately come up 50 times. In other words, "single events are unobservable" (Brase, et. al. 1998): the probability of a single event cannot be observed by an individual. We don't have the mental mechanisms to handle probabilities of single events, but we do have the mechanisms to observe the frequency with which the events occur. The other difficulty is that of "individuation". We have been made to count complete objects, but not the inseparable aspects of objects. We are not very good at segmenting the world in an unnatural fashion. We are wired to segment and

count whole objects as we find them in nature: under ecologically valid conditions. Our ability to count is not general. It is critically dependent on the specific nature and context of what is being counted. We are only good at counting complete objects that move as a unit, independently of other surfaces. In the case of other kinds of objects or conditions, we tend to select for other more outstanding or natural characteristics and “count” them. For example, toddlers spontaneously count certain objects like teddy bears, but they don’t count parts of objects. However, if the parts of the object have been broken off of the parent object they count them (Shipley & Shepperson, 1990). This capacity for individuating concrete physical objects that satisfy certain specific conditions is part of our innate physics (Leslie, 1994; Spelke 1995; Povinelli, 2000) and whole-object bias (Bloom, 2000).

Based on these findings about the sources of the cognitive difficulties, I propose a game to teach some contents of probability and statistics. I argue that this game can be considered as a metaphor and I provide an example of its educational impact.

THE METAPHOR OF GUESSING ON THE BOX CONTENT

There is plenty of empirical evidence (Cosmides & Tooby, 1996; Gigerenzer, 1998; Gallistel, 1990) that in experiential ecologically valid situations we correctly estimate probabilities implicitly (unconsciously) and act accordingly. In this format, our processing algorithms count the occurrence of events and gradually select the characteristics to be predicted as accompanying the events. Some characteristics are more salient and therefore may initially cause us to fix our attention on irrelevant aspects. But as situations are repeated, the relevant and ecologically valid characteristics start to be selected and stored in associative memories. The move to associations with actions and active representations is slower and more gradual (Siegler & Araya, 2005; Tranel et Al., 2000). From the educational point of view, it is very important to seek ways of helping to make explicit these implicit associations.

Based on these ideas, I report on an experience with the “Magic Surprises” game, that was designed in 1996 to help teach some statistics and probability, particularly discriminant analysis, and to make connections with algebra (Araya, 2000). This is a guessing game, a sort of bingo, in which students must elaborate hypotheses based on evidence that accumulates gradually as the game advances. Different boxes of different colours and sizes contain surprises: numbered cells that can be white or black. Based on previously opened boxes (the learning sample) players must guess on the surprises inside the new, randomly selected boxes. The game encourages systematic information collection as in Table 1, its presentation in several graphic forms, and the use of algebra to make explicit the intuitively perceived patterns. The idea of using a random sequence of boxes is to introduce statistics and probability concepts using the innate capabilities for natural frequencies and natural sampling.



Figure 1 : Magic box and its surprise with 12 cells that was inside the box.

Box				Face				Cells											
Colour	Length	Thickness	Width	Colour	Length	Width	Expression	1	2	3	4	5	6	7	8	9	10	11	12
Blue	8.5	2.23	4.9	Pale	2	1.63	Sad	W	W	W	B	B	W	W	B	B	W	W	B
Blue	5.8	2.16	5	Pale	1.84	1.28	Sad	W	W	B	W	B	W	B	W	W	W	B	W
Blue	5.9	2.6	3.9	Pale	1.53	1.69	Sad	B	B	B	B	B	B	B	W	B	B	B	B
Blue	5.8	2.2	3.5	Pale	1.72	1.24	Sad	?	?	?	?	?	?	?	?	?	?	?	?

Table 1: After the first three outcomes, player n has to guess the colour (Black or White) of cell n for the box features written on the 4th row.

Additionally, the box is simple for our innate physics modules to understand and handle. It satisfies the Spelke conditions (Spelke et Al, 1995): cohesion, continuity, solidity and contact. Furthermore, boxes and surprises facilitate not only counting and individuation strategies, but also confirmation of frequency estimates, and the use of perceptual pattern recognition skills for discriminating between different factors (figure 5).



Figure 2. Screen to input guesses and screen with outcome shown below the guesses.

The teacher or game coordinator can define different relationships between the visible external characteristics of the boxes and the surprises inside. For example, she may choose two variables (i.e. box-length and box-width) and establish that a given cell (cell number two, for example) is white or black, according to whether these variables are over a given straight line (in the plane formed by the box-length and box-width variables). In the example that follows, the teacher has selected a discriminating region defined by two second-order inequations for the box-width and face-length variables. Therefore in the x-y graph the black cells, marked with crosses, appear in the middle of a region defined by two parabolas (figure 3).

Slowly, the distribution of black cells (crosses) and white cells (square dots) begin to form a pattern as the game advances. The first image corresponds to the outcomes of the first ten surprises, the second to the first 100 surprises. The gradual emergence of a clear pattern easily captures students' attention and helps to develop classes that are both engaging and educative.



Figure 3 (a) Graph of the first ten surprises. (b) Graph of the first 100 surprises. A cross represents a black surprise, the other mark represents a white one.

Points are not only awarded to correct guesses but also extra points are awarded if the player can produce rules expressed in algebraic terms that when applied to the present data generates correct guesses. Examples of such rules are:

IF box-width > 5

THEN cell_5 is black

ELSE cell_5 is white

IF box-width + 2*face-length < 8

THEN cell_3 is black

ELSE cell_3 is white

Rules can also be expressed graphically as decision trees.

In the game the teacher not only promotes the processes of counting, estimating frequencies, making tables and graphs, but also establishes a connection between statistics and algebra, and stimulates the search for multiple representations.

When rules are used the game can be played not only to guess the outcome of the next box, but alternatively it can be played to guess for a sequence of 10 or 100 boxes. This way the teacher can make the distinction between one event and several events, avoiding the difficulty of assigning probabilities to single events. For single events, the guess has to be if the cell is going to be white or black, but for multiple events the student can also give rules such as:

IF box-width > 5

THEN 7 out of 10 times cell_5 is black

ELSE 1 out of 10 times cell_5 is black

The mathematical concepts are taught during the game as hints to play better. The teaching strategy has three parts. First, the rules of the game are explained and practised. After a few guesses and checking the outcomes, the teacher explains how to record the information in a table and why in this way it is easier to find patterns.

Then the game continues and after a while the teacher explains how to represent information on number lines. This means, that each measurement of a box feature (for example, box length = 6.1) is represented as a mark located on the position 6.1 of the box length line (Fig 5). Then the teacher analyses the advantages of using our visual processing capability to find interesting patterns on the generated learning sample. Then the game continues. After another couple of turns, the teacher explains how to display the information as x-y graphs. This way the students read the graphs where each cross is the location where a dart thrown by a (black) machine arrived, and each circle is the location where a dart thrown by another (white) machine arrived. Then, the game continues. After a another couple of turns the teacher explains how to describe regions that discriminate between crosses (black cells) and circles (white cells) using rules with algebraic expressions and decision trees.

WHERE IS THE METAPHOR?

There is a long history of games associated with probability, from the first book on the subject, written by Cardano in the 1560s, to practically all standard textbooks used today. The use of games as metaphor is also common in natural language (“play your cards right”, “the ball is in your side”, “life is a game”) and in science (theory of games is used by Lakoff & Johnson as a metaphor for rational action, board games are considered by Holland as one cornerstone metaphor of science). In mathematics metaphors are used everywhere (Lakoff & Nunez, 2000; English,1997; Holland, 1998; Araya, 2000; Richland et Al, 2004; Soto-Andrade, 2006), as in probability (Lakoff & Johnson, 1999), even though most students and teachers are unaware of their use. A metaphor is as a map between abstract ideas and ones more innate, closer to concepts and procedures better known or hard-wired. By the metaphor mechanism a kind of vestigial cognitive organ is invited (Pinker, 1997). There is the source domain of the more innate or better known concepts (Holland, 1998; Lakoff & Johnson, 1980; Holyoak & Thagard, 1995; Gertner et. al., 2001; Finke et. al., 1992) and the target domain with the more abstract or less innate concepts. The target objects do not have to resemble their corresponding source objects, but the structural properties of both domains must be similar (Falkenhainer et. al. 1989).

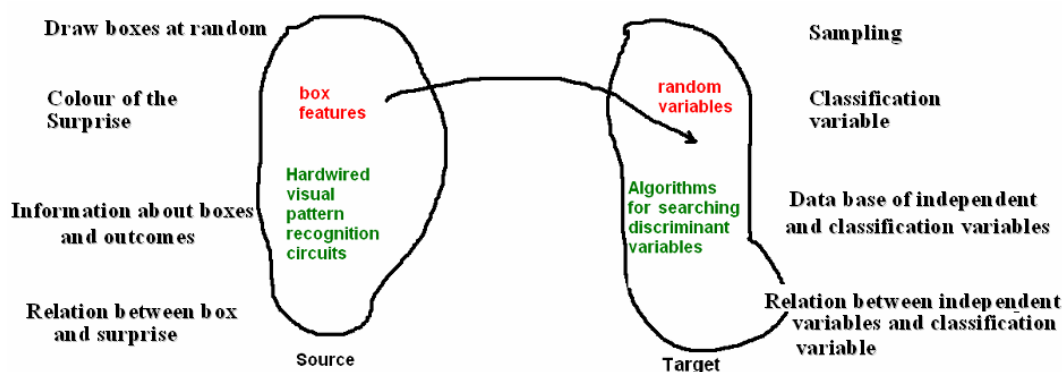


Figure 4: The Magic Surprise game as a metaphor for statistical pattern finding and discriminant analysis.

Here the source is the guessing game. It is a way of thinking about random variables and discriminant analysis (the target). The classification variable (typically with two values) in the target domain is imagined as the content inside the box: a white or black cell. The independent variables in the target domain are seen as different features of the box that contain the cell. These are evolutionary meaningful features such as colour, shape, and characteristics of faces drawn on one side of the box. Furthermore, the notion of some unknown connection between the independent variables and the classification variable is illustrated as the connection between the box characteristic and the colour of the cell. In each game such a connection is designed by the teacher but is unknown to the players. There are powerful hard-wired pattern recognition circuits that we can use to search for clues on the source domain. One way is to form two groups for the previously opened boxes according to the colour of the respective cell inside. This way the player's visual system will automatically find the discriminant variables (see Figure 5). It will detect if there is a box shape, its colour or an expression on the faces painted on the boxes that discriminate between the black or white cells. Once the player has seen this relation then we can ask her to express it in mathematical form. Also, we can ask her to look for other ways to use our perceptual systems to help find or make the findings more precise. The x-y graphs for example, help to find better rules that discriminate between black and white contents, and describe them mathematically.

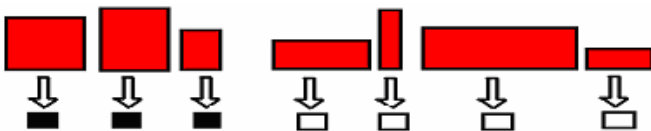


Figure 5: Three boxes contain a black cell and four contain a white cell. Note that it is very easy for the visual system to detect that the boxes closer to a square contain black cells.

Arranging boxes is a smart strategy for using the external world to store information. This way some part of the pattern finding is computed by external world procedures as well as perceptual brain circuits, facilitating enormously the computations that the higher cognitive centres have to do to detect patterns. In this embodiment view of cognition (Ballard 2002), the external world is an important computational resource that frees resources otherwise needed by the internal cognitive system. Furthermore, the body, that serves as an interface between the world and brain, provides additional computational resources that also frees additional resources. The body reacts emotionally on the early implicit detection of patterns, signalling the possible emergence of a pattern. One of the main objectives of the game is to help the translation of the implicit findings to more explicit forms. The transition from the initial implicit learning to explicit learning is a dynamic process very similar to the discovery process on other tasks (Tranel, et.al., 2000; Siegler & Stern, 1999; Siegler & Araya, 2005). The game metaphor and the different graphical tools taught, help the transition since the geometric features of the boxes and faces are easy to describe explicitly using algebraic notation.

EVIDENCE OF LEARNING AND GENUINE UNDERSTANDING OF THE TARGETED MATHEMATICAL CONCEPTS

The game has been played by elementary, middle and high school students and teachers. I report here one illustrative case that has been video recorded. It is a 9 year old girl playing alone with a computer version of the game and where each box has only one cell. After having made 15 guesses and obtained as feedback the 15 outcomes she has all the information in a table and she has marked the previous results on the number lines associated with each variable as shown in figure 6.

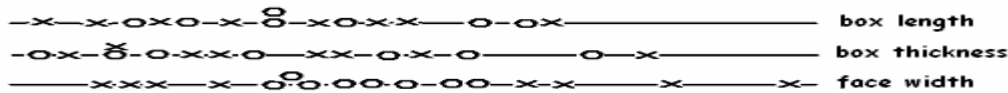


Figure 6 : The first 15 surprises (8 blacks and 7 whites) graphed in three lines. The first line contains the face width of each of the first 15 surprises as a location in the line and marked as cross if the surprise was black and circle if it was white.

At that point of the game the subject is being interviewed:

Researcher: Which of all the components you think is more important?

Subject: face-width

Researcher: and (this case) where is it located?

Subject: here (pointing with the pencil to the corresponding location on the face-width number line)

Researcher: and according to that, what (colour) should it (the cell) be?

Subject: it should be white colour

Researcher: OK, make your guess

She introduces her guess and the computer simulates the opening of the box and the computer announces that her guess is correct.

Researcher: Why didn't you select box-length?

Subject: Because it doesn't help me much

Researcher: Why it doesn't help you?

Subject: Because it is very entangled (she points to the box-length line)

Researcher: OK, and box-thickness?

Subject: It is more entangled (she points to the box-thickness line)

Researcher: and box-width?

Subject: ...also (she points to the corresponding line)

Researcher: and look to face-width. Why face-width is less entangled?

Subject: Because it has the balls more closed together (pointing with a pencil to the zone marked with circles on the face-width line), the crosses are more closed together (pointing now to another zone where the crosses are concentrated) and it is more easy. And if it is in this zone it is black, and if it is here (pointing to a zone) or here (pointing to another zone) it is white.

Not only has the subject quickly learned to plot the data in several lines, marking with circles or crosses according to the colour of the cell, but she has understood that to make a good guess she has to find the variable that is most discriminating between white and black cells, and that this fact is represented graphically on the different lines. She has understood that this representation is very helpful and that a discriminating variable has the face width line marked with crosses and circles concentrated on clearly different zones so that there is not much superposition between crosses and circles. She has also understood how to express her findings as simple rules, that her guesses are only probably correct, and that this probability can improve as she gathers more data from new outcomes, and consequently adjusts her rules. She also can assign probabilities to different guesses, recognizing different conditional probabilities according to partial information expressed as belonging to certain particular regions. Experiences with solving discrimination problems in different contexts, show that students can use this game as a way to think about the problem to solve.

CONCLUSIONS

The pattern discovery multi player game described in this paper has been a successful strategy for teaching some statistics and probability topics and to express the findings with the use of algebraic language. Several “ecologically valid” teaching strategies are used in the game: frequencies and not percentages or probabilities (Gigerenzer, 1998), one event versus several events distinction and, counting of complete objects instead of trying to count inseparable aspects of objects (Brase, et. al., 1998), segmentation as trees, and multiple representations. The game has been played by primary, middle and high school students and teachers. It has been played with or without computers, and the learning of each player has been monitored as the game is being played even if hundreds of individuals are playing. Combined with a strategy of teaching that is presented as several hints to improve the play, the whole activity produces a rapid learning of mathematical concepts and algorithms such as efficient recording of categorical and numerical information, production of relevant tables and graphs, localization of numbers and intervals on the number line, plotting linear and quadratic functions, describing and visually representing linear and quadratic systems of inequalities, estimation of probabilities and conditional probabilities, building and comparing segmentation trees, and computing some simple statistical discrimination measures. The game has been shown to be a good metaphor for discriminant analysis and conditional probability concepts. It activates innate pattern recognition modules that facilitate the understanding of the concepts and its translation into mathematical

language. For example, fourth graders are able to represent the data in graphic form, and search and find discriminating variables, express their findings in algebraic language, and make predictions in probabilistic terms. The game is easily used as a source domain when thinking in other discriminating problems in different contexts. For example when tested on a problem analysing data to find symptoms that better discriminate certain medical conditions, they are able explicitly to establish a map between different symptoms and different features of the boxes, between the presence of the medical condition with the colour of the cell, and between the strategies to seek for the most discriminating symptom and the strategies to seek for the most discriminating box features. This type of evidence of learning and deep understanding of the intended mathematical concepts indicates that this is a didactic strategy worth pursuing.

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Acknowledgements

Thanks are due to Leo Rogers, who provided insightful comments on an earlier version of the manuscript.

THE NUMBER LINE AS METAPHOR OF THE NUMBER SYSTEM: A CASE STUDY OF A PRIMARY SCHOOL

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The number line has been characterised as a metaphor for the number system, although in its completeness it is a very sophisticated one. Using such a metaphor to teach can, however, lead to a learning paradox — its structure needs to be understood by both the teacher and the pupils. This paper examines the way the number line is presented, used and developed by the teachers and children within one school. The evidence indicates that the teachers emphasise its use as a tool with little consideration given to its structure. This appears to have a direct impact on the children's construals and leads to misconceptions, errors and fragmentation. Having acquired only a limited appreciation of what it offers in terms of understanding the number system, the children eventually reject it.

INTRODUCTION

If learning is seen as an active construction process that grows, in part, from the assimilation of new knowledge with existing knowledge, then analogical reasoning associated with metaphors may provide a mechanism for linking the two (Gholson, Smithers, Buhrman, Duncan and Pierce, 1997). However, teaching through the use of analogy and metaphor, presents us with a 'learning paradox' — the structure needs to be understood both in the existing knowledge and in the new knowledge (Bereiter, 1985). If the metaphor is active for the teacher but at the best dormant, or at the worst extinct for the learner, then the desired communication of ideas is unlikely to occur.

Metaphors can allow large chunks of information to be transferred or converted from one domain of knowledge to a new one provided the similarities between the two are established but, even if they are, the metaphor is only as useful as the knowledge of the initial domain allows it to be. Black (1979) suggested that metaphors can help generate new knowledge when the known domain and the new domain interact in the mind of the learner so that each is enriched by the other — a process achieved if the metaphor is 'active' so that both the promoter of the metaphor and the recipient can identify the similarities between the two knowledge forms.

Using evidence drawn from the way in which teachers and children in an English primary school used the number line, this paper indicates that understanding based on solely on the use of the number line for descriptions and actions associated with whole number did not provide a basis for handling its increasing complexity or provide a foundation for the reconstruction of knowledge necessary to cope with fractions.

THEORETICAL FRAMEWORK

Herbst (1997) suggests that the number line is a metaphor of the number system. He defines it as the consecutive translation of a specified segment U , as a unit from zero, that can be partitioned in an infinite number of ways (i.e. fractions of U) to form a number line:

one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997; p. 36)

All kinds of numbers can be represented; the natural numbers (1, 2, 3,...), integers (... -2, -1, 0, 1, 2,...), rational numbers (p/q , where p and q are integers), and real numbers (having infinite decimals). This quality enables Herbst to write about what he calls the “number line metaphor” and its “intuitive completeness” (Herbst, 1997, p. 40). This is a view enhanced by Lakoff & Núñez’s (2000) identification that the points on a line correspond to real numbers — the ‘points on a line’ metaphor.

It is possible, therefore, to build a variety of number lines to introduce different numbers of the number system. Consequently, there can be a one-to-one correspondence between numerical statements and number-line figures and it is these features that would appear to suggest its use as a pedagogical tool.

Within the curriculum material recommended for English schools, (usually identified as the National Numeracy Strategy (NNS) (DfEE, 1999) designed to support the National Curriculum for Mathematics (DES, 1991), the number line is identified as a “key classroom resource”. However, within the documentation there is no explicit reference to the conceptual knowledge associated with its form and use. The number line is not explicitly defined, but seen as:

... a means of showing how the process of counting forward and then back works. It can also be a useful way of getting children to visualise similar examples when working mentally. (QCA, 1999; p. 31)

Additionally, actions associated with the use of the number line are frequently ambiguously associated with a number track and a hundred square but, as Skemp (1989) argues, differences between track and line do not lie simply in the perceptual sense that one has the “spaces numbered” and the other has the “points numbered”:

The differences between a number track and a number line are appreciable, and not immediately obvious. The number track is physical, though we may represent it by a diagram. The number line is conceptual – it is a mental object, though we often use diagrams to help us think about it. The number track is finite, whereas the number line is infinite. However far we extend a physical track it has to end somewhere. But in our thoughts, we can think of a number line as going on and on to infinity. (pp. 139-141)

Within the NNS the number line appears not only as an alternative version of the number track but it also is frequently fragmented to emphasise particular features of

the number system such as whole number and fraction. Expanding knowledge from the former to the latter may present difficulties and obstruct the reconstruction of the child's development of the number system — compatibility of the number line with a pupil's already existing knowledge of whole number may be inconsistent with the new knowledge required for fractions (Davis, Alston & Maher, 1991).

METHOD

The results within the paper are part of a wider study on teachers' and children's use and understanding of the number line representation within the mathematics classroom (Doritou, 2006). The study took place in a school located in a deprived area of the English West Midlands. Following the 2004 inspection, the overall assessment for mathematics was identified as 'Good'— teachers had good subject knowledge and the teaching and mathematical development of the children was improving although the overall standards were slightly below average.

The study embraced the period 2003 and 2004 and reports on outcomes derived from the analysis of a series of classroom observations and interviews with 22 pupils with median ages ranging from 6.5 to 10.5 (four from each of the year groups Y2, Y3 and Y4 and five from each of the Years 5 and 6). The sample represents children of mixed ability in mathematics, as identified through their dominant frames of reference (Pitta, 1998) triangulated with their achievement within Standard Attainment Tasks (SATs), and the teacher's personal assessment. The observations focused on the use of the number line by teachers in their lessons, while the interviews considered children's learning as a result of these lessons.

RESULTS

The illustrative examples within this paper reflect a small proportion of evidence illustrating the use of the number line metaphor in the development of whole numbers and fractions. It is drawn from across the year groups and demonstrates the teacher's indications of meaning associated with the number line and the way the number line was used and interpreted by the children.

Within each year group, use of the number line was frequently associated with lessons dealing with whole number — 13 of the 24 observed — and fractions and decimals. It was associated with ordering number (Year 1), elementary addition and subtraction (Year 2), the addition and subtraction of two digit numbers (Year 3) and following revision of these approaches extended to three digit addition and subtraction (Year 4). In Year 5 the number line was used to focus upon multiplication and division of numbers by 10 or 100 whilst within Year 6 it was division. Additionally, within Year 4 the main emphasis was fractions whilst within Year 6 it was fractions and decimals.

Whole Number Considerations

Teachers within Years 1, 2 and 3 were frequently ambiguous in their use of the terms number track and number line and made little, if any, distinction between these terms and that of ‘hundred square’:

If you look against my board, you can see a big number line on the top that goes from zero all the way up to ten and then a smaller number line [*actually a number track*]¹ underneath that goes from one to twenty. (TY1)²

A ruler is a bit like a number line. (TY2)

Really, they [*the number line and number track*] are sort of similar things, but this [*number line*] goes zero to one hundred, this [*number track*] goes from one to one hundred, so it’s the same really... (TY3)

This ambiguity was later expressed by some of the children during their interviews. In response to the question “What did you do in class today?” responses included:

Using a number line [*means hundred square*], coloured it with pens, got numbers on it and the number line don’t go up to one hundred, but the square does. (Child 2.2)³

The teacher done a sum. He had to jump on a number line to get the answers. (Child 3.4)

We had to find the difference between the take-away sums. Thirty-nine take-away thirty-six are close together, so you get a number line or a hundred square and count down to the highest number. (Child 3.2)

These reflections on the lessons not only suggest that the children missed the conceptual differences between the number line as a representation of continuity and the number track (or hundred square) as a representation of counting numbers, but also tend to focus on procedures.

The emphasis of teaching was clearly on the use of the number line as a tool to order numbers, develop forward and backward counting and generally support addition and subtraction. In the early years the usual approach used throughout all the observations when performing addition or subtraction was to start from the largest number, and then add on (or count back) to the next number or the next ten etc.:

Just put the largest number first. (TY2)

The largest number has to come first on a subtraction. Twenty-two take-away two. Use the number line — take-away two and you move back. (TY3)

Later subtraction was introduced as complementary addition:

Smallest (number goes in the beginning) coz the number line goes up. ... The first thing I want to do is to get to the next multiple of ten... (TY4)

¹ [*Italicised*] comments within quotes identify the observer’s/interviewer’s comments added for clarification

² TY1 Represents the teacher (T) observed in a particular year Y — in this instance the Year 1 teacher.

³ Children are identified by year group and designated within that group. Thus Child 2.2 is from Year 2 and is the second child interviewed

Subsequent actions were then associated with a “jump” from the first placed number to the second. Teachers of the older children increasingly demonstrated a tendency to provide visual references to jumps without associated number line segments.

In one particular lesson within Year 2, the teacher used an unmarked line (“the empty number line”) to demonstrate the addition of 11 and 37. She proceeded to place 37 near the left hand side of the line and with her index finger, traced a jump of 10 and then a jump of 1, stressing the partition to demonstrate how children should first ‘add ten’ and then ‘add 1’ thus starting at 37, ‘jump ten’ to 47 and then ‘jump one’ get the sum 48. After a second similar example, the teacher concluded:

This is the way we can use our number line to help us add numbers. (TY2)

This theme also formed the basis of the first two lessons observed within Year 4. After “revision” of addition, the emphasis moved to the use of the empty number line (later superseded by the simple illustration of jumps) for subtraction. The notion of “jump ten” was generalised as “bridging ten” and in essence, the children were encouraged to use the same approach for subtraction that they had used for addition:

Smallest (*number at the beginning*) coz the number line goes up. (TY4)

The children were then to establish the difference between the smaller and larger numbers. What is particularly interesting is that the conceptual differences between take-away and finding difference through complementary addition was not made explicit in any of the lessons. The generic term subtraction was used without operational clarification.

The selected children from within Year 4 were interviewed after these first two lessons and presented with a series of addition and subtraction problems. Since the number line had been the central resource used within the observed lessons, it was assumed that all children would refer to it. However, only two of the four children attempted to use it and both of these arrived at incorrect solutions. The other two children used partition and when asked to use the number line were unsuccessful.

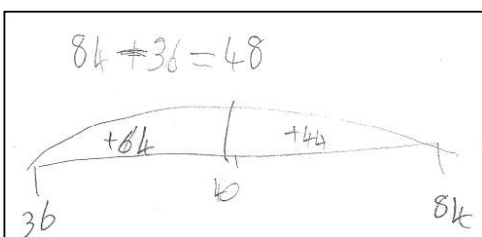


Figure 1: Subtraction instead of addition (Child 4.3)

One reason for error was confusion in the use of a similar procedure for both addition and subtraction. Child 4.3 (Figure 1), for example, attempting to find the solution to 84 and 36, replicated the procedure used by the teacher during the lesson. Though she made some procedural errors, these were corrected. Recognising that 48 could not be the correct answer she casually changed the plus sign to a minus sign.

Child 4.2, who represented amounts by a jump, also made errors in the calculations. When asked why he did not use a number line he explained:

Coz I don't like drawing lines (*marking the points*)... that's where you make a mistake when you draw out lines. In Year 3, coz I used to get told off... drawing lines. (Child 4.2)

Child 4.4 marked points ("lines") on her empty number line to support counting up to 52 from 17 when dealing with $52 - 17$ and, giving the correct answer 35, explained:

Coz there were thirty-five spaces... The little lines [*small marks between 17 and 52*].

Child 4.1 indicated one of the issues she perceived when using a number line:

Coz it ain't the same answer as when I partitioned it... I've done the number line wrong... The partitioning way [*is better*].

When Y5 children were asked to describe the substance of their lesson, none associated partition with use of the number line, but one indicated that an alternative way of doing the addition was to use a number line, but her attempt to do so was unsuccessful and she indicated that:

The vertical [*standardised*] way is the easiest. The number line is the hardest. (Child 5.5)

Within Year 5 addition was also reinforced during one of the observed lessons. Here the teacher used empty number lines to illustrate the way the bridging process may be used to solve two digit addition combinations. The lesson then proceeded to focus on addition with the standard algorithm. During the subsequent interview, only one child referred to the use of the number line whilst the others referred to the use of the standard algorithm and gave illustrative examples using this algorithm.

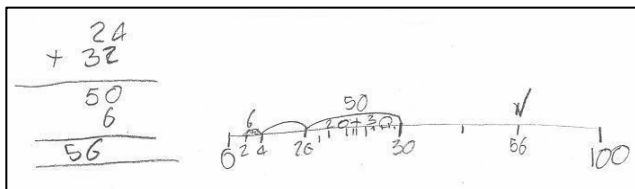


Figure 2: Child 5.1 using partition

Child 5.1 demonstrated an interesting variant on the use of the number line and on partition to obtain a solution to $24 + 32$ (Figure 2 refers). She first marked 0, 100, then 20 and 30, then 4 and 2 and then made jumps to add them:

You have to do little bits here [*in between 20 and 30*]. Put fifty-six there... Twenty-four add thirty-two... two and thirty and twenty and four... two add four [*jump from 2 to 4*] equals six. Twenty add thirty [*jump from 20 to 30*] equals fifty. Then we add them together. Fifty and six, fifty-six. (Child 5.1)

From a perceptual perspective, the child indicates that the line has numbers and marks ("little bits"), but apart from being the indicators of the position of a number, there appears to be no conceptual sense of why such mark may be there or of what the relationship between each may be. Note for example that there are twelve almost randomly placed 'bits' between 20 and 30 and that the interval between the 'bits' appears to have no relationship (note the interval 0 to 56 and the interval 56 to 100). From an operational perspective, the partitioning process guides the jumps made on the number line, but these jumps have no relationship between what is happening and the eventual outcome of the process (see the sum of $2 + 4$ and the sum of $20 + 30$).

It is interesting to note that after 4 years relatively sustained experience with the number line in the development of addition and subtraction, the children's conceptual understanding of its relatively sophisticated nature appears to have progressed very little. They reiterate the ambiguities expressed by their teachers and much prefer the final compressions represented by the standard algorithms to solve addition and subtraction combinations. The relation between the initial use of the number line, the "jumps" and subsequent bridging processes, partitioning and the compression of this into a standard procedure seem to have been overlooked.

Consideration of Fractions

Using the number line as a resource to develop understanding of fractions was only seen within Year 4 where five of the seven observed lessons dealt with this topic. Number line related tasks associated with fractions involved, for example, an unmarked stick approximately one metre long, empty number lines, a segmented line with the ends marked 0 and 1, a segmented line with the ends marked 0 and 5 and a variety of other lines with the left end marked 0 and the right marked with various numbers such as 60 or 100.

The unmarked stick was frequently referred to as a number line. The ends were then identified and the children asked to find particular fractions between the end points. Though the stick carried the implication that it was a unit interval partitioned into fractions this was not made explicit. The focus was upon correctly naming, ordering and establishing equivalences between particular points on the stick. In this sense, the development carried the same features as lessons observed within Years 1, 2 and 3 the correct naming and ordering of points in sequence on a line or path.

During one phase of a lesson, the teacher presented the children with a line:

We're gonna call it a line [*line marked on a sheet of paper*], it's gonna be our number line. Put zero at this end [*left end point*] and one at this end [*right end point*]. Where would you put a half? Hands up if you put this [$1/2$] as well. Did you also find that it was in the middle? If I fold this piece of paper see if I have a half in the right place. Now find a quarter and three quarters. (TY4)

This particular feature of folding a number line was assimilated by at least one of the interviewed children:

I've got a way of checking the half! You could fold the sheet in half... (Child 4.3)

The emphasis on "middle" and "half" was emphasised by the teacher,

A half of anything will be exactly in the middle. (TY4)

and clearly remembered by the children:

There's only one half in the middle... the proper half. (Child 4.3)

Anything that's in the middle equals a half. (Child 4.4)

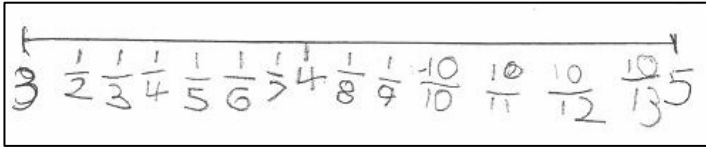


Figure 3: Applying whole number operations to fractions (Child 4.2)

However, whilst half appeared relatively easy to pinpoint on a number line, other fractions were problematic. On the line segment from 3 to 5, Child 4.2 pinpointed number 4 (the middle) successfully,

but failed to pinpoint other fractions correctly (Figure 3):

One and a half [*wrote 1/2*], one and three quarters [*wrote 1/3*]... one over four [*1/4*], one over five [*1/5*]... one over nine, ten tenths, ten over eleven, ... , five. (Child 4.2)

The clearly marked number 4 was ignored when other fractions were inserted. There appeared to be no attempt to pinpoint the position of these fractions and they appeared to be the child's perceptions of the way fractions were ordered. Clearly the child's knowledge of fractions was influenced by whole number considerations — the larger the number the larger the fraction. However, 10 seemed to be of particular significance. After completing $1/9$, the child wrote $10/10$. When asked “Why should ten over ten come after one over nine?”, the child replied “Coz nine comes before ten...” It appears that Child 4.2 forgot the sequence he was writing, 10 now became the numerator although the denominator continued to reflect natural number order.

Some of the confusion of the children may have arisen from the presentation of the teacher. Presenting one example, the teacher drew a number line segment with only the ends 0 and 10 marked. She proceeded to mark the points usually representing the

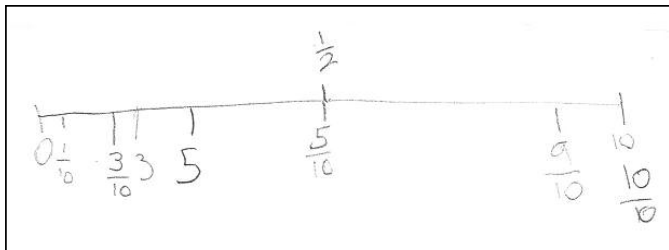


Figure 4: The relative positions of fractions and whole numbers (Child 4.4)

number 1 to 9 by placing $5/10$ in the middle, and then, in order $1/10$, $9/10$, $3/10$, $4/10$, $7/10$, $1/2$ at the places where numbers 1, 9, 3, etc. would normally go. There was no indication that the fractions represented partitions of the number line segment. Child 4.4's interpretation is in Figure 4.

Throughout the series of interviews, only one child (Child 6.2) indicated signs he appreciated the conceptual underpinning of the number system by the number line. It was more usual for the children whether within Year 2 or 6 to see the number line through its visual characteristics — numbers in order with little marks to show where the numbers are. Such features were implicit in the teacher's representations and the acts which eventually became “jumps” without any explicit reference to the number line. The underlying continuity inherent within the number line was not established for these children who saw whole number and fraction as two discrete systems.

Child 6.2 proved to be the exception however. When asked what he would put on a line segment marked 0 in the middle and 1 on the right hand side, he recognised the symmetry that could be applied through the use of positive and negative integers, the

equivalence classes associated with decimals and fractions and appreciated that no matter how small the number represented by a partitioned unit or a partition of this partition, there could be further partitions. He could identify the relationship between the different forms of numbers — wholes, fractions, decimals and negative numbers — and use this relationship to give a sense of his understanding of continuity. During the observed teaching nor during interviews with other children was the potential richness expressed by this child apparent. An illustrative example is seen through the explanation associated with the freedom with which he interpreted a request to identify some numbers on a line with the 0 to 1 segment marked towards the right hand side:

Child 6.2: I could [*mark more numbers*], but that would go into eighths and things like that. Between that [0] and that [-1] I'd do it into tens. That's one, two, three, four, five, six, seven, eight, nine [*marking notches for the negative decimal numbers*]. Minus zero point one, minus zero point two and so on.

Interviewer: Could you put a quarter on this line?

Child 6.2: Yeah! That's the quarter there [*points at 0.25*], that's the half [*points at 0.5*] and that's three quarters [*points at 0.75*].

DISCUSSION

It is clear from the evidence that both the teachers and the children within this school interpreted numbers as locations on a path and any associated operations were seen as acts of moving along the path. The number line provided the path and the jumps, or the bridging processes, the acts. Lakoff and Núñez (1997, 2000) suggest that the similarities that arise from such an interpretation of an arithmetical metaphor can be interpreted as “Arithmetic is Motion”. Such a conception would seem to satisfy the objectives outlined within the National Numeracy Strategy (QCA, 1999) but it does not appear to have provided a sense of the conceptual structures outlined by Herbst (1997). Indeed, Foxman & Beishuizen's (2002) have suggested that the more successful calculation strategies of British pupils, when dealing with simple arithmetic, are based on sequential jumping methods rather than partitioning or splitting methods. Indeed, the evidence from this school seemed to be on emphasising the validity of the metaphor and not on the strengths it may possess to clarify the properties of the number system. The potential of the metaphor is simply used as a pedagogic representation that supports the development of procedures without the structural basis that could support its meaningful understanding and, particularly in later years, a more successful appreciation of its use. Gradually the complexity of ideas presented to the children, immersed as they are in ambiguity, appeared to lead them to conclude that arithmetical operations can be done in an easier way than that which encourages use of the number line. Additionally, the presentation of ideas that required the children to reconstruct their knowledge of the whole number system to include fractions of wholes was not supported by reference to the conceptual links

between the two. The children appear to have learned little from their experience with the number line as a metaphor of the number system. It appears to have evoked a strong sense of ‘Arithmetic as Motion’ in the context of whole number but this was eventually superseded by a preference for a standard algorithm and did not support the extension to fractions. In the latter sense particularly, it did not serve its purpose.

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WAYS OF THINKING ABOUT THE USES OF IMAGES IN LEARNING AND TEACHING GEOMETRY: A MORE THOROUGH INVESTIGATION OF THE LINKS BETWEEN DRAWINGS AND FIGURES

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The aim is to highlight and describe clearly the correspondence or conflict about links between what is “seen” in a geometrical drawing and what is “known” about the geometrical referent of this drawing. Characterization is made with two criteria. On one hand is the visual “signifying information” in a drawing defined as visual information which enables us to see a spatial relation which could be modelled by a geometric property. On the other hand is the “geometric property” of a theoretical object called “the referent”. Considering pupils difficulties in managing these links, called in the paper D-F Links, we propose to question elements which are on the teacher’s side, elements depending on the responsibility of the teacher: the part of the didactic contract about what kind of problem pupils have to deal with: be they spatial or geometric problems; and consequently, the kinds of validation forms which are attempted: deductive, perceptive or by the use of instruments; and the conversion to a graphic register already made (or not) in the wording of a geometric problem.

Introduction

The question about the links between a ‘drawing’ and a ‘figure’ in geometry is an old intrinsic epistemological difficulty which has been studied for some time in didactic research. A good synthesis of the state of the problem can be found in the recent Handbook of Research on the Psychology of Mathematics Education (Gutiérrez and Boéro Eds., 2006) with articles about visualization (Presmeg, 2006) and about Teaching and Learning Geometry ([HKCL], 2006).

In Cerme 3, B. Parzysz (2003) follows the two paradigms developed by Houdement and Kuzniak (1998) in France, to define the *spatio-graphic geometry* and the *pre-axiomatic geometry*. In spatio-graphic geometry, called G1 by Houdement & Kuzniak, “the objects in play are physical (models, diagrams, computer images ...) and the proofs are of a perceptive nature (eyesight, comparison, measure ...).” In pre-axiomatic geometry, called G2, “the objects in play are theoretical (their existence proceeds from axioms and definitions) and proofs are theoretical as well.”

Furthermore, for more than ten years researchers in geometry have distinguished between a “figure” and a “drawing”. I refer to Laborde and Capponi (1994) about using these two terms in geometry: “As a material entity the drawing may be considered as a signifier of a theoretical referent (a theoretical geometrical object). The geometric figure consists of pairing a referent with all of its drawings; then a figure is defined as all the pairs formed by two terms: the first term being the referent,

and the second being one of the drawings which represents it ; the second term is one drawing among all possible drawings of the referent. In accepting this, the term geometric figure establishes a relation between a geometrical object and its possible representations.”

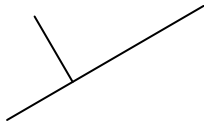
Clarification about what “the referent” is.

To clarify the purpose behind, the theoretical geometrical object as a theoretical referent will be called only “the referent”. Even if the construction of a concept has its roots in all of the representations of the referent, we can possess an autonomous concept of something independently of all of its possible representations. For instance, children construct the concept of “flower” with familiarity with an environment of a great variety of flowers, making bouquets with different sorts of flowers, seeing lots of pictures of flowers in books, singing songs about flowers, ... The concept of “flower” does not exist alone, independently of all possible modes of representations (verbal, design, graphic, photography, ...). But we can talk about and consider this concept without signifying any one of these images. That is what “the referent” means. The beginning of conceptualization of a geometric object bears some similarity to this point above. The first stage in the conceptualization of a referent is constructed by associations with rich variety of representations of this referent (Vygotski, 1934). Even if conceptualization in geometry is more than that, and other stages are needed to elaborate the geometrical concepts, our purpose concerns the early stages of learning (primary school and beginning of secondary school, or between 3 and 12 years old).

Since a referent has autonomy and may relate to an infinite number of semiotic or cognitive representations, it is almost the same for a drawing. A drawing has its autonomy as strokes on the paper or the computer screen. *A drawing alone does not possess any geometric signification. It may possess it only if a geometric referent is specified.* For instance, a drawing with three strokes two of which are equal in length, may represent a happy birthday hat, a Indian’s house, an Egyptian pyramid, a triangle, an isosceles triangle, the two sides of a rhombus and it’s diagonal, When one sees a drawing, what is the referent one will consider? If it is not signified by language, the interpretation depends only of the person who is seeing the drawing. The referent certainly cannot be defined by the drawing. It’s a real problem in geometry because the language involved is sometimes much too complex to define all the considerations about a configuration of geometrical figures. It is specially difficult when we consider topological properties, or intersections and common points from a few different objects which may need considerable care to describe the referent unambiguously. Probably that explains why mathematicians often use drawings with their texts about problems (but not with their text of a proof), or why geometrical problems in textbooks are frequently accompanied by drawings. But we will see below how this phenomenon becomes involved with pupil’s difficulties.

Signifying information and D-F Links

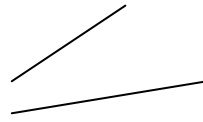
Following the distinction made by Parzysz (1988) about “seeing” and “knowing” we introduce the criteria “signifying information” defined as: visual information from a drawing which enables us to see a spatial relation which could be modelled by a geometric property like parallelism, perpendicularity, lengths, equality of angles or symmetry. For instance, consider the drawings below:



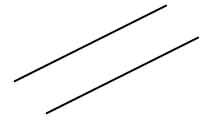
Drawing 1



Drawing 2



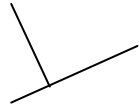
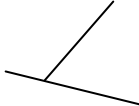
Drawing 3



Drawing 4

Drawing 1 possesses signifying information about perpendicularity which the others drawings do not possess. Drawing 4 possesses signifying information about parallelism which the others do not possess. But all of these drawings possess visual information referring to some kind of spatial relation within the drawing considered as a spatio-graphic object. Thus visual information may be signifying information or it may not. For instance, let us consider the figure defined by drawing 1 and the referent “two perpendicular segments with a common extremity”. Then drawing 1 possesses signifying information for perpendicularity but does not possess signifying information for equality of lengths.

The distinction between visual information and visual signifying information is really essential, and specifies the criteria with which a characterisation about D-F Links may be described. D-F Links can indeed be classified in the following way: Let S stand for the visual signifying information; NS for no visual signifying information; P for a geometric property explicitly given for the referent; NP for a geometric property lacking in the description of the referent. So the four cases are: SP, SNP, NSP, and NSNP; below we have a table which shows these cases illustrated with “two segments perpendicular or not” as the referent, and “two segments perpendicular or not” as the drawing.

D-F Links		REFERENT	
		P : « two perpendicular segments »	NP : « two segments » without a specified perpendicular property
DRAWING	S : visual signifying information 	Case SP CORRESPONDENCE	Case SNP CONFLICT
	NS : visual no signifying information 	Case NSP CONFLICT	Case NSNP CORRESPONDENCE

These four cases can also be described with language: visual signifying information *embodies* a geometrical property of the referent (or not); Non-signifying visual information *embodies* a geometrical property of the referent (or not).

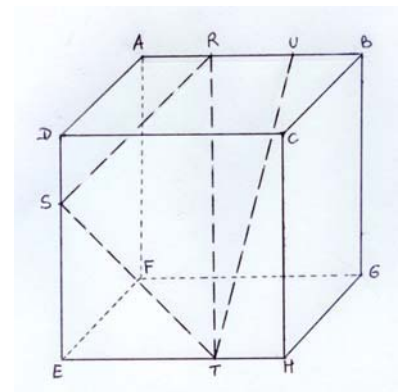
Take us an example where these different cases are well stated:

The figure is defined by:

Referent:

ABCDEFGH is a cube: sides ABCD and EFGH are parallel, with [AF] and [DE] two parallel lines. R, S, T, U are stated respectively on [AB], [DE], [EH], [AB] (R and U are distinct) with the constraint: the lengths AR, DS, HT, BU are equal, and corresponding to $\frac{4}{13}$ de AB.

Here is one of all possible representations of the referent:



Case SP DSC with a right angle at D; AFR with a right angle at A; ...	Case SNP RUT with a right angle at R; RDS with a right angle at D; ...
Case NSP RUT with right angle at U; TBU with a right angle at U; ADS with a right angle at D; ...	Case NSNP SCH; SRC; ...
If we choose another representation of the referent, the cases would not be the same.	

The results of research principally about pupil's difficulties indicate that they deal with drawings as spatio-graphic objects in the spatial space, as shown by Presmeg (2006). They deal with visual signifying information as geometrical properties of the referent. The question often claimed is: how can we help pupils to change their relation to the drawings to move from G1 to G2, the drawing-world to the geometrical world? Certainly these difficulties will always be there for the pupils, because this is an epistemological and phenomenological phenomenon in the intrinsic nature of geometry. But even if pupil's learning depends of them, of course, some elements may be studied which are on teacher's side, elements depending on the responsibility of the teacher. The following paragraphs present some ideas about this.

Kinds of problems and the forms of validation

Texts are often accompanied with drawings. The difficulty in considering a geometric problem for most pupils is, as the research has shown: once the formulation of the problem is represented by a drawing, pupils tend to work mostly with the drawing, treating it as a spatial object and not necessarily as a representation of the geometrical referent given by the text. Therefore, this is an important element to consider in teaching how to choose an exposition of the exercises. Two criteria may be considered: the kind of problem and the form of validation. Let us first consider the kind of problem: spatial or geometric?

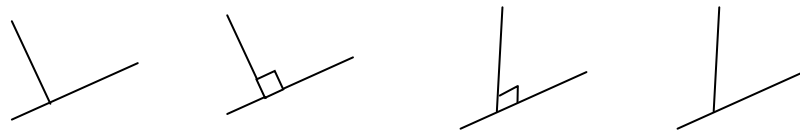
- Spatial problems are problems in G1: a spatial problem is posed in the spatial reality, with spatial objects which possess spatial properties. A solution to a spatial problem concerns only the specific situation of this problem. A spatial problem is unique. It requires a pragmatic solution which is appropriate to this problem and only to this one. A spatial problem is characterized by its specificity. Spatio-graphic problems are considered as spatial problems.

- Geometric problem are problems in G2: a geometric problem is posed in the geometric reality, with geometric objects which possess geometric properties. A solution to a geometry problem concerns all of the configurations generated by the referent. It requires a geometric solution proved with mathematical reasoning as a general result adapted for all configurations. Generality characterizes geometric problems.

There seems to be a relation between the kind of problem and the modes of its validation for a solution. We believe that the kind of problem and its validation forms have to be separated, making clearer the management of D-F Links in problem solving. Remember the three validation forms: deductive, perceptive, or with the use of instruments. Each form has its own rules which enable us to declare valid assertions about objects and relations between them in a particular kind of problem. *The "really" about an assertion refers to a choice of its validation form.* The deductive form corresponds to the assertion which may be deduced with mathematical rules, proof without contradiction, and with language and deductive organization; the deductive form is the typical validation form in G2. The perceptive form is based on the conspicuous character of images for which are clearly without

perceptive ambiguity; if I see such information and I see it clearly without perceptive ambiguity then I may consider it as a data or a possible answer to my question. Validation with instruments is a particular perceptive validation form in which perception is according to some scale and using an instrument appropriate for this scale. For instance, an angle may be right by measurement with a set square on the black-board, but it may be non-right by measurement with a set square on paper. Perceptive or instrumented forms are typical validation forms in G1.

Take an example and consider how the answers may be different according to the choice of validation form. Consider the following spatio-graphic question: “Is the angle a right angle in these drawings below?”



Deductive form of validation	No answer	Yes If we know what the code signifies	No answer
Instrument form of validation	Depending about perceptive scale It may be right for one scale and may not be right for another.		
Perceptive form of validation	Yes	No	

The wording of a problem

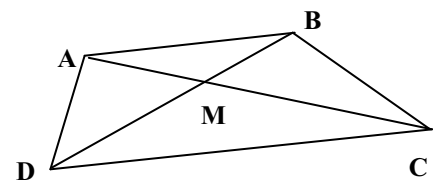
Let consider now some examples of the way some problems are written, and try to make explicit the kind of problem and the validation form.

Ex1: “Draw a rectangle which has one of it’s corners at the centre of a circle.” This is a spatio-graphic problem: the objects are spatial, we can draw them with the hand or with instruments. Nothing is defined in the text as a referent or a validation form to consider this wording as a geometric problem.

Ex2: “ABCD is a parallelogram. ASB and ATD are two equilateral triangles outside the parallelogram. Prove that STC is an equilateral triangle.” Here it is clearly a geometric problem and the deductive form of validation is explicit through the word “prove”.

Ex3: “ABCD is a regular trapezium, M is the intersection point of the diagonal (AC) and (BD). Compare area of AMD and area of BMC. Justify your answer.”

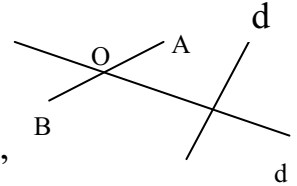
The drawing opposite accompanies this wording.



According to the text alone, this is a geometric problem, but the validation form for the comparison is not really explicit. This ambiguity is strengthened by the presence of the drawing which places the problem in the spatio-graphic environment. We know now through research results that in this case most pupils will

work with this particular drawing. Why should they not have to work with this image? Otherwise why is this image there? The image creates ambiguity about the kind of problem and its validation form. Certainly one will make a drawing to give oneself a representation of the configuration to solve the problem, but it's very different when you have the responsibility to do that, or when this conversion from wording to a drawing is made by someone else.

Ex4: “Construct a figure like this one, with $OA=OB$.
Construct E and F as projections respectively from A and B
on the line (d) in parallel to (d’). What is the nature of AEBF? Justify.”



Here the problem is clearly given in the spatio-graphic environment, as a construction task. So the quadrilateral AEBF is unique. But the validation form is not explicit.

We see the ambiguity of the terms of a geometric exercise, and the importance of thoroughly thinking through the role of images to define the kind of problem the pupils are intended to deal with. The following hypothesis might be stated: a geometric problem can be defined on condition that the characteristic of generality will be respected and the deductive form explicit. The language is powerful enough to generate this possibility and to symbolize generality. But how can this condition be met when a drawing is given with the wording of the problem, focusing the pupil's attention on the specificity of the drawing?

Geometric problems and their conversion

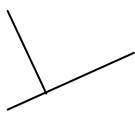
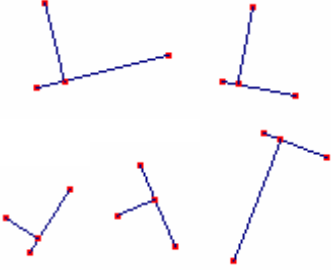
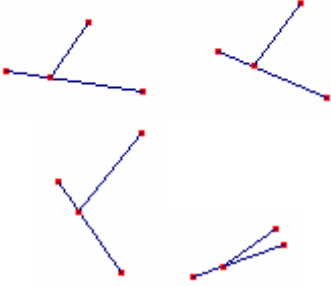
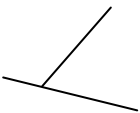
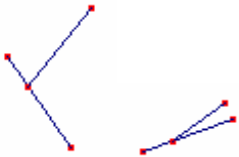
Duval (1998) calls the operation of translating the term from one register (here the text) to another register (here the drawings): “a conversion”. In the examples above 3, 4, or “the cube”, the conversion is already made. Yet the production of a drawing by oneself in order to create one's own visualization (sensitive or cognitive) of the geometric configuration certainly involves comprehension better than reading a particular kind of ready-made image which focuses one's attention on particular spatial characteristics. To make a drawing by oneself helps to develop external and internal representations of the problem. To try to produce your own image to represent “the cube” defined only by the referent given above: you will certainly draw something which has no likeness with the drawing given in the wording. Furthermore, because you are an expert in mathematics, you manage D-F links using your drawing to help you to treat and solve the problem. Because some representations possess a heuristic function, and some others are obstacles for conspicuous indications of properties of the referent, images for conversion or treatment are different and multiple, and are chosen by the expert to help them to access geometric properties or arguments about the configuration.

Thus we deduce that conversion is a specific activity for solving a problem in geometry which leads to the epistemological question about the generality of a theoretical object and specificity of one of its external or internal representations. Conversion is bound to D-F Links. Either the responsibility to make the conversion in a geometric problem is on the solvers side, it is the responsibility of pupils, or the conversion is already made in the spatio-graphic environment, but then the condition

of generality must be met, and yet it is impossible without a multiplicity of drawings. In this last case, Dynamic Geometry Environment (DGE) certainly offers this possibility to see an ‘infinity’ of drawings associated with the same referent. In all cases, teachers have the responsibility to define the kind of problem.

D-F Links and conversion in DGE

How is it possible to clarify explicitly for pupils the different D-F links in order that they can understand some necessary knowledge to enable them to manage the correspondence and conflicts between “seeing” and “knowing”? Perhaps the Dynamic Geometry Environment is a rich and adaptable tool to assist this task, because dynamic geometry software is constructed from an axiom: the invariance by movement of visual information characterizes signifying information about a geometric property of the referent which is associated to a particular drawing by the instructions for its construction. “When the user drags one element of the diagram, it is modified according to the geometry of its construction rather than according to the wishes of the user ... spatial invariants in the moving diagrams represent geometrical invariants.” ([HKLS], 2006). In other words, through the given visual rule-axiom, signifying information which is invariant by “drag mode” is a sign referring to a geometric property of the referent. Refer again to the table for D-F links considered now in this environment.

DF-Links DGE as semiotic mediation of the geometric referent defining the figure.		REFERENT – MEDIA : DYNAMIC GEOMETRY ENVIRONMENT (DGE)	
		P : « two perpendicular segments »	NP : « two segments » without specified perpendicular property
DRAWING	S : visual signifying information 	CORRESPONDENCE 	CONFLICT SETTLED 
	NS : no visual signifying information 	CONFLICT SETTLED by impossible case in 2D	CORRESPONDENCE 

In this environment conflicts are settled; it means only that visual data are always being signified by the drag-mode. Even if the difficulties remain for pupils to access the geometric properties in solving problems, DGE enables pupils to clarify explicitly the D-F Links and manage them easier in order to understand that beyond drawings there is something else they have to deal with. When just one drawing illustrates a geometric configuration, its visual aspects are too pregnant with its spatial specificities. But when many (an ‘infinity’ of) drawings illustrate a geometric configuration, the focus of pupil’s attention is on the common properties of all these drawings: the referent (even if it is difficult to make it explicit).

Suppose now that “the cube” above is given (ready-made) in DGE (in 3D). The drag-mode allows generating an ‘infinity’ of positions in space, and then drawings of those positions, which can be seen by pupils. Conversion is already made, but the movement of the image makes pupils aware of what is “seen” does not correspond to what they are required to know about the referent. Some positions of the image for instance, show RUT with a right angle at U, and others without a right angle at U. Thus drawings do not focus attention about one specific case, but let us think about the diversity. Also drawings may enable the heuristic function. Even if the spatio-graphic mode is used to embody a geometric problem, the dynamic geometry environment might be a tool which can be adapted to clarify D-F links and the kind of problems pupils have to deal with.

Conclusion

Many books about teaching or curriculum treat the pupils’ difficulties in management of relationships within geometric figures as if they were lacking a skill according to the “type of control” for a drawing: perceptive control, instrument control or deductive control. In my paper I have tried to show the necessity of equipping teachers with a tool to think of pupils’ difficulties as an epistemological matter. Clarification of D-F Links, the kind of problem, and conversion as a crucial first stage in problem solving, might be interesting areas to investigate what teachers need to make the didactic contract explicit. Certainly it will not be enough to help children learn only to use geometrical knowledge to resolve geometric problems and to have familiarity with the deductive form for validation in G2.

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MATHEMATICAL WRITING

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Writing about one's own doing of mathematics is a topic regularly found in papers on mathematics education. Fewer authors focus in their research on the act of writing when doing mathematics. The following paper concentrates on this kind of writing. What is the meaning we can give to writing - or preferably inscriptions¹ - when we are learning mathematics. Considering the breadth of the field the statements presented here offer only a short view on the relation of speech and the written when doing mathematics. Building on various linguistic theories between speaking and writing and a case study, the aim of this paper is to stress the suggestion that the written form is more than simply a visual substitute for the spoken word. Using this for the learning of mathematics, I will argue that when doing mathematics new ideas can emerge from the written.

INTRODUCTION

Literature on mathematics education offers a series of research results on the meaning of writing about doing mathematics, “post-process”. Examples of these results and their lively and sometimes controversial discussion can be found in Doherty, 1996, Morgan, 1998, Porter 2000, or Pugalee, 2004. The majority of these studies view the written as an instrument of secondary importance. They investigate the use of texts students write on their own learning of mathematics. Writing about one's own doing of mathematics offers a chance to learn mathematical concepts. Therefore, students create their own texts on mathematics (Eigenproduktionen in German, Maier, 2000). Here we find mathematical diaries and similar extensive descriptions of mathematical activities. With their help the language of mathematics may become part of the student's language. When reading these papers very little information is presented about the written itself as a means of learning mathematics.

In the following I will present a particular – in some sense complementary – view on the written when doing mathematics. I will argue that the written is more than just materialized speech that it is, so to speak, more than something that follows the spoken. In explaining my ideas I will start by presenting a video-based case study, which reports on two students' mathematical activities while solving a geometrical

¹ For the use of „inscription“ see Latour, 1990 or Roth, 2003. I use inscription to describe anything that is written on paper, blackboard, computer screen etc. An interesting research question would be to find similarities and differences between the use of the *inscription* and *image schemata* as presented by Mark Johnson (Johnson, 1987). In my paper I will not investigate this question as I concentrate on *materialized* mathematics being always visible to our eyes whereas “image schemata operate at a level of mental organization that falls between abstract propositional structures, on the one side, and particular concrete images, on the other.” (Johnson, 1987, p. 29).

task. Afterwards I will refer to some valuable suggestions from media theory (linguistics) and its view on writing. So what is the significance of the written² in learning mathematics when new knowledge – knowledge in *statu nascendi* - comes into existence? I will argue that in some cases the written itself can become a source of new ideas when learning mathematics.

METHODOLOGY

To illustrate my view I will use a video-based case study³. It shows two 14 years old pupils solving a problem. Their activities together were captured by two video cameras. For the purpose of supporting the evaluation, the two video pictures were incorporated in a single picture (see figure 1). One camera was fixed in one position, while the observer focused the other on interesting details. The students were given 90 minutes time to answer the question presented to them. The video was taken in the afternoon when classes had finished.

CASE STUDY PART 1

Figure 1 shows both students and the object they had to investigate. The students had been asked to describe the movement of the given object (a surface of revolution) on the table. They were to use their mathematical/geometrical knowledge. The question seems to be formulated in a completely unrestricted way. This openness was intended. The researcher's aim was to establish a context where both students could themselves feel like researchers. The study was designed in such a way that they should write down all their attempts without looking for an algorithmic solution. A more narrowly formulated task would have resulted in such a strategy. The number of tools both participants



Figure 1

of our case study were allowed to use also mirrored the openness indicated above. Besides their tools for doing geometry (ruler and compass), they could also use in addition different measuring tools (a tape measure, vernier calipers which is a tool for measuring the diameter of a circle) or computer software (spreadsheet software and software for dynamic geometry). Observing the video the viewer can easily recognize two courses of action. In both lines inscriptions – i.e. written forms – were invented and widely used. However, these inscriptions differ greatly from each other.

The first line starts from the observation of the movement of the surface of revolution on the table. Several times the participants in our case study, like young children

² Research results on similar questions – inventing and using inscriptions – can be found in diSessa, 2000.

³ I am grateful to M. Katzenberger – mathematics teacher at Gymnasium St. Paul/Carinthia - for making this video available to me. He produced it in spring 2005.

playing with a toy, pushed the object to roll on the table and observed this rolling with great attention. Thereby they focused their interest on the points of contact where the rolling touched the table. Therefore, they could imagine a closed curve of these points. However the question of how to record the history of this rolling remained, and of how to fix the contact points? A clever strategy, invented by both students, was to bring them a step further. They took several sheets from a stock of paper to let their object roll on a “soft plane”. While rolling the object on the sheets one student pressed it down hard onto the paper. The result was the complete and visible trace and impression of the movement on the sheets. In this way the history of the movement was documented as shown in figure 2. All subsequently constructed traces of the objects rolling - in this first action line – were built using this impression technique, which itself is nothing other than a special kind of inscription. It took only a few moments for both students to suppose that both curves are circles with a

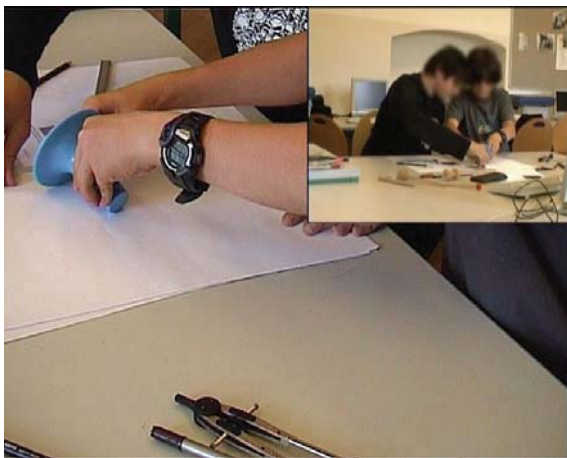


Figure 2



Figure 3

common centre. This point was found by means of elementary geometry. A circumscribed square was drawn around the greater “circle” using the impression. The diagonals of the square immediately led to the centre of the circle. It is worth recording that, although they had learned it in their geometry lessons, our students never used the theorem about the circumcircle of a triangle.

When watching the video one recognizes that many steps on the way to a solution were heavily influenced by inscriptions which the students had already produced or which they invented and drew ‘as they worked’. Starting with a virtually unsystematic playing with the surface of revolution, they followed a strategy which enabled them literally *to feel the curves they were looking for*. To strengthen this first tactile impression and to make it more utilizable for their visual senses one student coloured the impressed curve with his pencil (figure 3). From the seen and the felt, both students conjectured that the curves they were looking for had to be circles. This paved their first way to a solution to the given problem. Thus our students used their inscriptions en route to their goal. The written – in this case study a geometrical construction with all its peculiarities - did not follow the spoken. On the contrary, – at least in my view – the written was the precursor to formulating the next step towards the solution, i.e. the writing comes first then the speech follows.

THEORETICAL APPROACHES

A review of research reports on mathematics education indicates quite different tools for interpreting students activities. Some of them focus on investigating relations between internal or mental representations and external or physical representations (Goldin and Kaput, 1996; Goldin, 1998a, 1998b). Others pay particular attention to the language used when doing mathematics. They offer very thorough interpretations of students activities applying methods from hermeneutics (e.g. Krummheuer, 1997).

In 1987 Latour wrote: “Before attributing any special quality to the mind or to the method of people, let us examine first the many ways through which inscriptions are gathered, combined, tied together and sent back. Only if there is something unexplained once the networks have been studied shall we start to speak of cognitive factors.” (Latour, 1987, p. 258). We can also follow Latour’s suggestion using means from semiotics (Dörfler, 2005; Hoffmann, 2005; Kadunz, 2006b;). In this paper I will offer another approach.

If we look into the history of western thinking, we notice that numerous philosophers, linguists or semioticians did not follow this view on the writing-speech relationship. The linguist Roy Harris (Harris, 2001) describes in detail this relation from a linguistic and a historical point of view. The only example I wish to take from Harris, is his reference to Aristotle’s strict separation of the written from the thought. He argued that the written (*grammata*) is inferior to the spoken as the written is ruled by convention⁴. To think and speak comes first; writing is only in second place. At least during the 20th century several interesting texts were published showing the relation of the written and speech in a new light (Harris, 1986, 2001; Leroi-Gourhan, 1993; Krämer, 2003). Harris as well as Leroi-Gourhan argued that the roots of writing could be found neither in an image like doubling of facts nor in a linear doubling of human speech. One step beyond this is Harris’ claim (Harris, 1986) that before man was able to use letters he learned to use numbers. Man became “numerate” before he became “literate”. Similar ideas on the history of counting can be found in Schmandt-Besserat (1997) or Nissen (1993).

A useful position on investigating writing is presented by authors like Sybille Krämer (Krämer, 2003) or Wolfgang Raible (Raible, 2004). Both of them can be seen in the tradition of Harris and Leroi-Gourhan. One of Krämer’s questions in her paper from 2003 asks whether the only job of writing is fixing the spoken. Furthermore, she asks whether the order of writing follows the order of the spoken. If we answer this last question positively, then “...it is only the presence of the graphic-visual dimension

⁴ “A possible clue lies in the fact that Aristotle was born about twenty years after an important orthographic reform: the official introduction of the Ionic alphabet to Athens (403 BC), replacing the previously used local Attic alphabet. Naturally, documents and inscriptions in the old Attic alphabet did not disappear overnight. Every Athenian of Aristotle’s generation was perfectly well acquainted with the two systems, and therefore with the following facts. ... the possibility of changing alphabets shows that there are no intrinsic links between *grammata* and sounds: *grammata* can be invented, borrowed or adapted to suit any needs.” (Harris, 2001, p.35)

that is admitted to the writing” (Krämer, 2003; p. 159; my translation). It is this graphic-visual property of writing we can recognize in the two dimensions of a written text, which are a source of viewing structural aspects to the reader and the writer as well. Modern literature (poetry) makes use of this. There is no equivalent of this aspect in spoken language. Krämer’s conjecture that, by writing, the order of our thoughts can become visible is significant for me. We find examples in the table of contents of a book. By looking at such a table for just a second, we get an impression about the importance of the parts of the book. We do not need to read the table of contents sequentially. Other examples which present aspects of our thoughts in the written are the use of italics or footnotes.

“What is presented in a text is not the phonetic event (Lautgeschehen) but structural facts such as grammatical categories and relations between thoughts and structures of arguments” (Krämer, 2003, p.160; my translation).

Krämer offers these ideas as a basis for an alternative theory of writing. Thereby she investigates writing as a medium, a symbol system and as *Kulturtechnik*. It would be far beyond the scope of this paper to present Krämer’s ideas in detail. There is only one point to which I wish to refer. In her deliberations about writing as a symbol system Krämer writes about the construction of “cognitive objects“(Wissensdinge).

“A phonographic understanding of writing is based on the assumption that writing refers to speech. In contrast to this position, we presume that the reference for all writings are abstract things, more or less theoretical entities, which are not visible. If this assumption holds then the power of notational iconicity lies in the fact it brings everything we can think, and which is thereby invisible, to the register of perception.” (Krämer, 2003, p. 164; my translation).

I stress that this capacity of writing is in Krämer's view not a capacity with which we can see the invisible – whatever this may be – behind the visible. Rather, she asks whether this bringing to the register of perception is by itself already a form of creation of that which is offered to the visual sense.

Beside the structural aspects as a result of the two dimensions when looking on a written text there is another aspect of the written when doing mathematics. The written can be seen as a means for performing operations or as a system for doing operations. Following Krämer we call it operative writing (“operative Schrift”). This kind of writing does not concentrate on spoken language and so it does not serve communication immediately. What are the profits we can expect from using the written as “operative Schrift”? I mention two: Exploration and cognition. If we take notice of exploration then mathematical writing offers the opportunity to transform mathematical signs following very strict rules. While transforming there is no need for considering the semantics of the signs. The simple addition of natural numbers is an example for using operative writing as we only need to know an algorithm for adding and a multiplication table we have learned by heart in primary school. When solving linear or quadratic equations we also need no semantics. We just stick to the algebraic rules. If there is no need for interpreting the activities done then we are free to concentrate on the beginning and the end of a “calculation”. We can change the given parameters to explore their impact. The following case study part 2 will present

the example of one of our students searching for an error he had made using an algebraic equation.

I mention a second aspect. As operative writing frees us from time consuming interpretation of (mathematical) signs we gain the freedom to interpret the results of a calculation. In front of our eyes - metaphorically speaking - new ideas can come into existence. In this sense operative writing serves our cognition. With these brief indications that writing itself can construct the new, I shall now return to the students and to their activities.

CASE STUDY PART 2

The video data I will present now offers a new solution of a very different kind. The route to this solution can be seen from three positions. From the first we see free hand drawing plays a crucial role. As a second I mention the collaboration between the students where they use one drawing together and from this drawing develop the main solving strategy. As a third we will find in the students' activities different kinds of inventing and using writing and drawing, in particular the rule-governed transformation of an algebraic equation.

After finding their solution empirically, the observer encouraged both students to look for an alternative method of solving the task using the measuring tools offered. They suggested the vernier calipers to be the best tool for this purpose. The diameters of the base circle, the diameter of the top circle, the height and the distance between points of these circles were measured with these vernier calipers. Finally, these measuring activities were the source for the inscription shown in figure 4. To be able to judge the creation of this inscription it is necessary to explain the mathematical-geometrical background of our students. During class seven and class eight both were members of a course named "Geometrisches Zeichnen" which means geometric drawing. Geometric drawing (Technical drawing in England) is a subject of instruction taught in Austrian academic secondary schools (Gymnasium) and in lower secondary schools (Hauptschule) as well. The main topic of this subject is to learn how to draw a plane or spatial object following the laws of geometry. Computer software is widely used. Beside this, they develop drawing skills for producing sketches with and without measured values. Furthermore, our students' mathematics

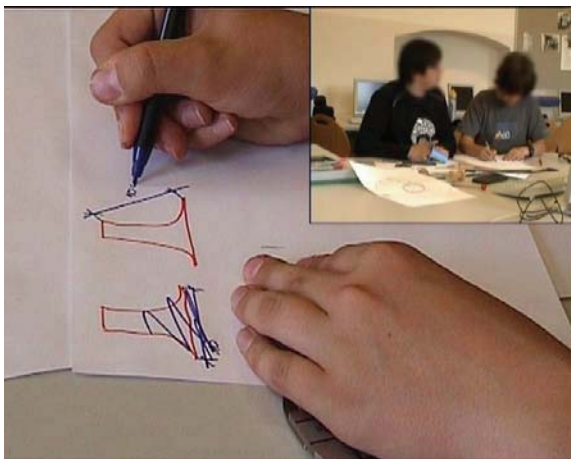


Figure 4

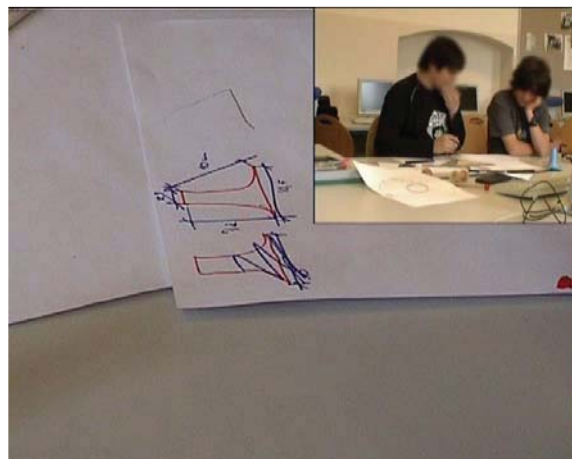


Figure 5

teacher always encouraged them in their mathematics lessons to produce sketches when they had to solve a mathematical problem. Therefore, inventing and using sketches with measurements was part of their mathematical life. The sketch we can see in figure 4 became the starting point for a new method of solving the given task. After this, a student, I will call him B, had finished his drawing he started to label it with measured values obeying labelling rules he had learned in school. During sketching and labelling our three-dimensional object became an object in the drawing plane. A problem from geometry in the three-dimensional space was transformed into a problem in plane geometry (figure 4). After observing the drawings in figure 4 the other student, I will call him A, started to draw a right-angled triangle. We can see the faint drawing in figure 5. Then he stopped and both students compared the given object with their drawings. After several minutes student B started a further attempt. The three sketches – figure 5 – motivated student B to look for similar triangles to calculate the cone which encloses the given surface of revolution. To fulfil this plan he had to calculate the length of a segment from an arbitrary point of the base circle to the unknown top of the cone. In figure 4, we see one part of this segment, which

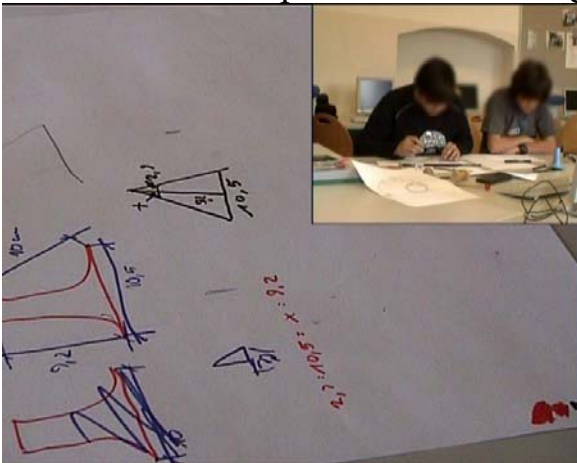


Figure 6

was measured with the vernier calipers. The idea of employing similar triangles developed not only from the sketch in figure 4 but also from the sketch student A had drawn (figure 5). Let's hear what student B said after two minutes of carefully observing all sketches.

B: Just wait a moment.

In the meantime student A had begun to draw his vertical projection.

B: Now let me draw. Do you know what I have thought? It is the intercept theorem that represents the relation!

A: (looks doubtful)

B started his explanation with the aid of his labelled sketch. Then he began to draw a new inscription. He labelled it with all the measurements (figure 6) and used this inscription as a means to establish an algebraic equation.

In his first equation, he made a mistake. As he compared his solution with the already existing “engraving” solution he recognized his error. So he made another attempt using the intercept theorem. He labelled A's faint drawing of a right-angled triangle – not shown in figure 6 – with measurements and obtained a second equation from this drawing. This equation led after some transformations to another numerical solution which fitted the “impressed” solution.

INTERPRETATION

Viewing this video and taking Krämer's ideas into consideration, we can say that “initial” ideas – the ignition so to speak - and their verbal formulation often started immediately after a new sketch was finished. Both the ways of arriving at a solution

that I have presented here support my view on writing and drawing when doing mathematics.

If we remember the first data I presented in part 1 the successful idea for finding a solution started from rolling the object on the table. The students had already used this kind of movement when they investigated other bodies of revolution. Finding their interesting strategy of pressing the object into the sheets of paper emerged from a rather chance observation. The students' achievement was their connecting of the impression and the given task. This impression was just a necessary requirement for finding the first step to the solution. Memorizing the colouring of the impressed curve we can say that the first solution was determined by their senses. Hand and eye the sense of touch and the visual sense organized the student's activities.

Compared with the data given in the case study part 1, the data from part 2 seems to be more profitable for my enterprise as I will show now in this precise description. After having made a series of measurements B started to draw a sketch from the axial section of the object which he labelled carefully. The labelling with all its details was an easy job for student B. This ability has its root in his geometrical socialization. On the other hand this construction of the sketch was in some sense like "mechanical" activities. How did student B invent the idea of using the intercept theorem? To begin with we could suggest that B could read this theorem from his drawing. But B could not, and he needed support from his colleague, student A. Similarly to the impressed solution something unintentional was the source of a successful idea. We can find this source in student B's activities when he labelled his sketch with measurements. Labelling a sketch or any other geometric drawing was a well-known practice for both students. Student A did not see just a section of the given object when he looked at the given sketch. His engagement with the given object and observing the measurement labels led A to have the idea of drawing a right-angled triangle. We can say that A's unorthodox action "abused" these measurement labels. When (ab)using these labels A always had the context in mind as he referred all sides of his right-angled triangle to the given object. Video data show that in the meantime student B had followed A's activities very carefully. Now two drawings were drawn on the paper. There was the right-angled triangle as the result of A's "abuse" and B's own sketch. If one lays the first drawing over the other and additionally knows the intercept theorem then it is conceivable that a person would gain the idea of using this theorem. This is exactly what B did. With Krämer we can say that the idea for solving this problem developed from the drawn and the written.

The remaining activities can also be seen in the light of Krämer's view on the written, as presented, or more precisely on the operative use of the written. Video data shows that student B formulated an algebraic equation. He used it to explore his solving strategy and to prove it empirically. As B had deduced the interception theorem with the aid of two geometrical drawings this theorem had to pass the test. But this did not happen. In his first attempt student B made a mistake when establishing his equation. However, as the calculation of one variable was the only task B had to fulfil he could easily test his calculated result against the already existing "impressed" solution. We can say that the rule-governed transformation, the operative use of the written, supported the exploration.

CONCLUSION

The case study presented has shown the importance of inscription (the written and the drawn) when solving a mathematical problem. Constructing and using drawings and the written as well can be seen as a possible source of new knowledge. By using well-known inscriptions or by inventing new ones, allows mathematics to happen right in front of their eyes. On this basis they may be able to use these writings successfully. In some cases, as illustrated through video recordings, spoken language only came after the written.

My deliberations should not challenge the importance of the spoken when learning mathematics. Similarly as in the introduced view on the written, where inscriptions may bring something – which is not part of the spoken – to the eyes of the learning student, there are elements of the spoken language, which cannot be expressed by the written (e.g. gesture, facial play).

Beside answering the research question it was also of importance for me to offer some arguments that the relation between the spoken and the written is not a hierarchical one. An example of destructing such a relation was introduced by Jaques Derrida (1997).

Further research could follow at least two directions. Following the first one we could compare the results of interpretations of similar - or the same - empirical data using other theoretical approaches (See footnote 1 and Kadunz 2006a). A second direction could carry on Derrida's idea of destructing hierachical relations where the relation between the written and the spoken is just one example. Learning mathematics always means inventing rules and following rules. Is the relation between inventing and following a hierarchical one? Some hints to answer this question may be found in Wittgenstein's deliberation on language and mathematics (Wittgenstein, 1984; Krämer, 2002).

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STUDENTS' AND TEACHERS' REPRESENTATIONS IN PROBLEM SOLVING

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The aim of this study is to investigate the representations used by students as they attempt to solve non-routine problems, and the representations used by teachers for solving and teaching the same problems. Furthermore we examine the relationship between students' and teachers' representations. Participants were 107 primary school students of the 5th and 6th grade and semi-structured interviews with 20 teachers were conducted. It was found that students prefer to use more concrete representations (pictures and diagrams) instead of abstract representations (symbols and algorithms) to solve the problems although sometimes these types of representations lead to incorrect solutions. On the contrary teachers prefer more abstract representations to solve and teach the same tasks.

INTRODUCTION

The role of representations in mathematical understanding and learning is a central issue in the teaching of mathematics. The most important aspect of this issue refers to the diversity of representations for the same mathematical object, the connection between them and the conversion from one mode of representation to others. This is because unlike other scientific domains, a construct in mathematics is accessible only through its semiotic representations and one semiotic representation by itself cannot lead to the understanding of the mathematical object it represents (Duval, 2002).

Representation is any configuration of characters, images or concrete objects that stand for something else (DeWindt-King & Goldin, 2003). Kaput (1987) suggested that the concept of representation involves the following five components: A representational entity, the entity that it represents, particular aspects of the representational entity, the particular aspects of the entity that it represents that form the representation and finally the correspondence between the two entities. Following Kaput's definition, the representation is considered a mental symbol or concept, which represents a concrete material symbol. It takes the place of another element and obtains more capabilities than the object itself. In other words, the representation is autonomous and independent from the represented object and the individual can modify and elaborate it without constraints.

The aim of this study is to investigate the representations used by Cypriot students as they attempt to solve non-routine problems, and the representations used by their teachers for solving and teaching the same problems. Furthermore we examine the relationship between students' and teachers' representations.

THEORETICAL BACKGROUND

Representations in Problem solving

There have been many attempts to define and describe problem solving, but most have fallen short of capturing the sense of what it involves. One shortcoming of most definitions is their failure to acknowledge the essential role played by representations in the process (Cai & Lester, 2005). Successful problem solving in mathematics involves coordinating previous experiences, knowledge, familiar representations and patterns of inference and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity that prompted the original problem-solving activity (Lester & Kehle, 2003, p. 510). Consequently, representation is regarded as an especially important construct in mathematics learning in general and specifically in problem solving.

Among the characteristics of successful mathematics problem solvers is their ability to create and use appropriate representations, both internal and external. In this study, we are interested in external representations used by students to solve non-routine problems, and by teachers to solve and teach the same problems. External representations can have one or more different forms (verbal, symbolic, pictorial, diagrammatic, manipulative objects). The use of representations in mathematics problem solving, especially external ones like diagrams, is very important (Pantziara, Gagatsis & Pitta-Pantazi, 2004).

In solving a problem, a solver needs to establish representations of the problem not only to help her or him organize and make sense of the problem, but also to communicate her or his thinking to others. Initially, the problem solver's representation might include only the "givens" and the statement of the goal of the problem. Usually, the representation used by a problem solver changes. After the problem has been solved, the problem solver may use yet another representation to express her or his solution. Thus, representations are the visible records generated by the solver to communicate her or his thinking about the way the problem was solved. Moreover, these representations, both initial and final, may differ among problem solvers (Cai & Lester, 2005).

Many studies have revealed a striking difference between students' representations, especially cross-national studies which examined thinking and reasoning involved in U.S. and Asian students' mathematical problem solving. Asian students tended to use symbolic representations (e.g. arithmetic or algebraic symbols), while U.S. students tended to use visual representations (e.g. pictures) (Cai & Lester, 2005).

Some researchers (Dreyfus & Eisenberg, 1996; Smith, 2003) have pointed out that concrete representations and strategies have limitations since they are context-or task-specific strategies in problem solving. Concrete representations may limit students' thinking and further learning unless they can shift to more generalized approaches. Therefore, symbolic representations may be considered more advanced and

sophisticated (Dreyfus & Eisenberg, 1996). According to developmental psychologists (Bennett, 1999), children start off representing their world in concrete terms and only later shift to more abstract representations. As they mature, students in their later elementary and middle school years can begin to think and manipulate mathematical objects mentally and represent them numerically and symbolically. However, recent studies suggest that a developmental perspective may not explain students' use of different representations in their problem solving (Cai, 2004). These differences may occur due to different teaching approaches and representations used by the teachers in the problem solving process. Some teachers less frequently encourage students to move to more abstract, conventional representations and strategies in their classroom instruction. One of the common misconceptions held by those teachers is that concrete representations or manipulatives are the basis for all learning since they believe that concrete representations or manipulatives can facilitate students' conceptual understanding (Burrill, 1997). On the other hand, some teachers view only symbolic and numerical solutions as "mathematical solutions". They do not regard pictorial solutions of a problem as "mathematical" (Cai, 2004).

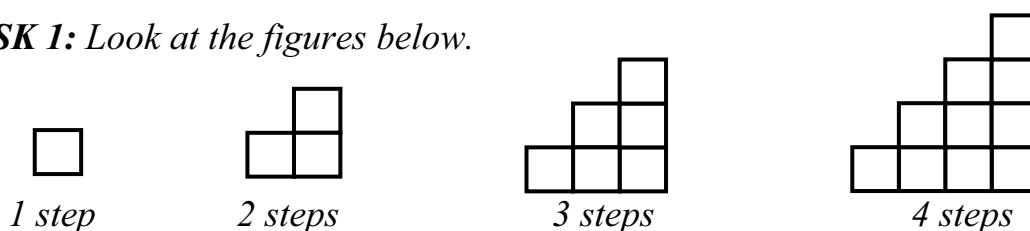
In the light of the foregoing discussion, the research questions of this study were as follows:

1. What kinds of representations are used by fifth and sixth graders as they attempt to solve non-routine problems?
2. What kinds of representations are used by the teachers in these children's schools as they attempt to solve and teach non-routine problems?
3. Why students and teachers use specific types of representations and not others?
4. Is there a relationship between the kinds of representations used by teachers and students?

METHODS

Participants were 107 primary school students of the 5th and 6th grade, eleven and twelve years old respectively, 52 boys and 55 girls from four schools. Students were given about 80 minutes to consider eight non-routine problems taken from Cai (2004). Here we present three sample tasks:

TASK 1: Look at the figures below.



a) How many blocks are needed to build a staircase of 5 steps? Explain your answer.

b) How many blocks are needed to build a staircase of 20 steps? Explain your answer.

TASK 2: Sally is having a party.

The first time the doorbell rings, 1 guest enters.

The second time the doorbell rings, 3 guests enter.

The third time the doorbell rings, 5 guests enter.

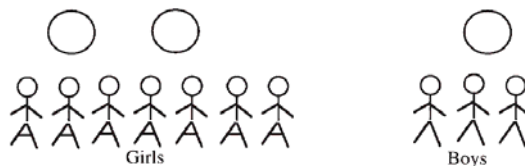
The fourth time the doorbell rings, 7 guests enter.

Keep going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

a) How many guests will enter on the 10th ring? Explain or show how you found your answer.

b) 99 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.

TASK 3: Here are some children and pizzas. Seven girls share 2 pizzas equally and 3 boys share 1 pizza equally.



Does each girl get the same amount as each boy? If not, who gets more? Explain or show how you found your answer.

Each student's response was assigned a numerical score from a five-level (0-4) scoring rubric. To receive a score of 4, a student's explanation or solution process had to show a correct and complete understanding of the problem. To receive a score of 3, a student's explanation or solution process had to be basically correct and complete except for a minor error, omission, or ambiguity. To receive a score of 2, the explanation or solution process had to show some understanding of the problem but be otherwise incomplete. If a student's explanation or solution process showed a limited understanding of the problem, it was scored as 1. If a student's answer and explanation showed no understanding of the problem, the response received a score of 0. If a student omitted a task the student's response would also be scored as 0.

Furthermore the representations used by students for solving each non-routine problem were analysed. Four categories of representations were used: symbolic, pictorial, diagrammatic and verbal. The first category included students' solutions involving algorithms and symbols, specifically numbers and letters. The second category included students' solutions involving drawings (pictures). The third category included students' solutions involving tables. Finally, the fourth category included students' solutions involving verbal explanations or reasoning. Both the representations and the responses given by students were coded by two persons.

Semi-structured interviews with 20 teachers of the participating schools were conducted. First they were given the three tasks presented above and asked to solve them. The solutions given by the teachers were categorised according to the four kinds of representations mentioned before. Later on they were asked to mention what kinds of representations they were going to use while teaching these tasks in their classrooms. They were only asked to talk about their teaching approaches and not actually teach these problems.

RESULTS

Students' responses and representations

Table 1 shows the frequencies and percentages of students' responses to the tasks.

Tasks	1a		1b		2a		2b		3	
	(n=107)		(n=107)		(n=107)		(n=107)		(n=107)	
	f	%	f	%	f	%	f	%	f	%
Correct and complete answer	80	74.8	22	20.6	56	52.3	15	14.0	8	7.5
Correct and complete answer except for a minor error	6	5.6	15	14.0	8	7.5	6	5.6	0	0.0
Some understanding but otherwise incomplete task	3	2.8	9	8.4	5	4.7	4	3.7	3	2.8
Limited understanding of the task	1	0.9	19	17.8	1	0.9	10	9.3	21	19.6
No understanding or omission of a task	17	15.9	42	39.3	37	36.4	72	67.3	75	70.1

Table 1: Frequencies and percentages of students' responses to the tasks

It seems that the task 1a was the easiest because a large percentage of students (74.8%) gave a correct and complete answer. The most difficult task was task 3. Most of the students showed no understanding or omitted this task (70.1%). Tasks 1b and 2b were solved by a small percentage of students, 20.6% and 14.0% respectively, due to the fact that a generalization was necessary. Task 2a was solved by half of the students (52.3%).

Table 2 shows the frequencies and percentages of students' representations to the tasks.

Representation	1a		1b		2a		2b		3	
	(n=107)		(n=107)		(n=107)		(n=107)		(n=107)	
	f	%	f	%	f	%	f	%	f	%
Symbolic	9	8.4	24	22.4	14	13.1	18	16.8	1	1.0
Pictorial	76	71.0	49	45.8	0	0.0	0	0.0	60	56.1
Diagrammatic	3	2.8	9	8.4	63	58.9	29	27.1	0	0.0
Verbal	11	10.3	13	12.2	4	3.7	6	5.6	10	9.3
Omission of the task	8	7.5	12	11.2	26	24.3	54	50.5	36	33.6

Table 2: Frequencies and percentages of students' representations to the tasks

Most of the students solved task 1a using a pictorial representation (71.0%). It is noteworthy, that all the students who used pictorial representation managed to solve the problem successfully. Most of the students tried to solve task 1b using the same type of representation as in task 1a (45.8%). None of them managed to solve the problem correctly. It seems that students were influenced by the picture given in task 1a and tried to solve task 1b in the same way. Some of the students who used symbolic or diagrammatic representation managed to solve the task correctly. Most of the students solved task 2a using a diagrammatic representation (58.9%). Almost all of them managed to solve the problem correctly. The use of a diagram seemed to be the most appropriate way to solve this task, although the problem was given in a verbal form. Half of the students omitted task 2b due to the fact that a generalization was required. The students who tried to solve the task 2b using a diagram (table) (27.1%) gave a wrong response. The diagram was not the appropriate type of representation for solving this task and it seems that the students who used it were influenced by the previous task. Most of the students who used a symbolic representation gave a correct response to task 2b. Most of the students (56.1%) used a pictorial representation to solve task 3. These students failed to solve the problem correctly. Probably they were influenced by the picture already given in the task. The students who used symbolic or verbal representations gave the most correct responses.

Semi-structured Interviews

Semi-structured interviews with 20 teachers were conducted. Table 3, shows the representations used by the teachers to solve the three tasks.

Most of the teachers (13) solved task 1a using pictorial representation. They drew a staircase with 5 steps and counted the blocks. In task 1b most of the teachers used a diagram in order to solve it. They made a table and tried to find the pattern. No one tried to solve the problem using a pictorial representation. Most of the teachers (13)

solved task 2a using a diagram. They made a table and found the pattern. In task 2b all teachers found the general formula and solved the problem symbolically. Nineteen teachers solved task 3 symbolically. They used fractions, made them equivalent and then compared them in order to decide who gets more.

Tasks	1a	1b	2a	2b	3
Symbolic	2	7	7	20	19
Pictorial	13	0	0	0	0
Diagrammatic	5	13	13	0	0
Verbal	0	0	0	0	1

N=20

Table 3: The representations used by the teachers to solve the tasks

Table 4, shows the representations teachers mentioned they were going to use while teaching the three tasks.

Tasks	1a	1b	2a	2b	3
Symbolic	2	2	0	0	16
Pictorial	3	3	0	0	3
Diagrammatic	15	15	20	20	0
Verbal	0	0	0	0	1

N=20

Table 4: The representations used by the teachers to teach the tasks

Most of the teachers mentioned that they were going to use a diagram to teach tasks 1a and 1b. Eight of them argued that they were going to help their students to make generalizations and find a general formula. Specifically, they spoke as follows:

Teacher 1: “They (the students) have to make a table and find the result. They also need to find a general type $n \times (n+1)/2$.”

Teacher 2: “With a table they can find the relationship/general rule and apply it.”

Teacher 3: “In order to answer the second question (20 steps), the students have to discover the general formula with my help. They can not continue the pattern until the 20th step or draw it. Those are time-consuming solutions.”

The three teachers who stated that they were going to use pictorial representation mentioned:

Teacher 4: “I will ask them (the students) to draw different staircases and help them realize that each stair is one block more than the previous.”

Most of the teachers mentioned that is difficult for the children to find a general type.

All teachers stated that they were going to use a diagram in order to teach tasks 2a and 2b, by making a table and find the pattern. Twelve of them mentioned that students have to find a general type. Specifically they argued:

Teacher 6: “...with a table. I will help them understand what is happening and at the end we will find a general type.”

Teacher 7: “In the beginning with a table. Then they will notice that we double the number of the doorbell ring and subtract one ($2x-1$).”

Teacher 8: “We will make a table and try to find the relationship between the number of the doorbell ring and the number of the guests. Each time we add the number of the doorbell ring with the previous number.”

In task 3, sixteen teachers stated that they were going to use a symbolic representation in their teaching. Specifically they said:

Teacher 10: “Definitely with a symbolic representation. It is not easy for the students to compare $\frac{1}{3}$ and $\frac{2}{7}$ using a picture. It is easier to make fractions equivalent and compare them.”

The three teachers, who stated that they were going to teach the problem with the use of a pictorial representation, mentioned that they were going to guide students to separate the circles and compare the pictures.

DISCUSSION

The aim of this study was to investigate the representations used by Cypriot students as they attempt to solve non-routine problems, and the representations used by teachers for solving and teaching the same problems. Furthermore we examined the relationship between students’ and teachers’ representations.

Students’ performance in all tasks except task 1a was moderate. The problems given were not the ordinary problems that students deal with in their textbooks. The three tasks presented in this study could be solved using different strategies. They could not be solved with a simple algorithm and they required more complicated mathematical reasoning. Almost three quarters of the students gave correct response to task 1a. This task seemed to be easy to solve using a pictorial representation.

Students chose the most appropriate type of representation (pictorial), meaning the easiest and less time – consuming one to solve task 1a. On the contrary, the task 1b although it was similar to the task 1a it could not be solved easily using the same type of representation. The students failed to solve the task 1b because they used the same type of representation as in task 1a instead of using a more productive and less time - consuming representation. They tried to build a staircase and count the blocks. During the counting process they reached a wrong answer. The drawing they made would have been useful if the students could see the steps as a sequence of triangular numbers or as an arithmetic series but this did not occur. In task 2a almost two thirds of the students used the most appropriate type of representation for solving the problem (diagrammatic). As in task 1b, students used a less appropriate type of representation to solve task 2b and gave an incorrect response. In tasks 1b and 2b, students failed to use symbolic representation in order to reach a general formula. In task 3 almost half of the students used pictorial representation and failed to solve the problem because it was not easy to compare the two heteronymous fractions with the use of a picture. It was easier to use a symbolic representation, turn the heteronymous fractions into fractions with equivalent denominators and compare them. In general,

students preferred to use more concrete representations instead of abstract representations. In some tasks concrete representations were appropriate and led to correct solutions while in other tasks abstract representations were necessary in order to reach easily a correct solution.

As expected all of the teachers used the most appropriate representations for solving the three tasks, although these representations were different from the ones used by them for teaching the same tasks. Furthermore the representations used by the students for solving the problems were different from the representations used by the teachers for teaching the problems. Specifically, the teachers preferred to teach tasks 1a and 1b using a diagrammatic representation, while students tried to solve them using a pictorial representation. Moreover the teachers mentioned that they are going to help students come up with a general formula but none of the students managed to make a generalization. Although students tried to solve the tasks 2a and 2b in the same way their teachers prefer to teach them (diagrammatic), again they were not able to reach a general formula as the teachers would like. In task 3 students used a pictorial representation while the teachers prefer to use a symbolic representation in their teaching. It seems that students prefer to use concrete, pictorial representations as they think they are easier to handle and help them reach a correct answer. It is obvious that pupils need to go through an iconic stage before they are ready to use symbols. Teachers ought to be aware of this and help their students develop the transition between iconic and symbolic stage. They also ought to use multiple representations in their teaching. Each type of representation can be useful for reaching a correct solution to a problem if students are able to handle the representation appropriately. In some tasks pictorial representations led to incorrect solutions because students did not manage to handle them appropriately.

Students should learn to use easily all types of representations in order to solve a problem correctly. It is also very important to be able to choose the most appropriate type of representation for each problem, meaning using the representation that will lead them easily to a correct solution. In order to achieve this, teachers should engage students in discussions about the appropriateness of different types of representations in problem solving. If a central aim of classroom instruction is to foster the transfer of acquired knowledge beyond the initial learning context to new circumstances, concrete experiences are only effective if they offer support for generalization and conceptual understanding (Presmeg, 1997).

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THE ROLE OF THE CONCEPTUAL METAPHOR IN THE DEVELOPMENT OF CHILDREN'S ARITHMETIC

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This paper explores the relationship between the perceptual and the conceptual in the development of children's informal arithmetic. It compares two major theories- Piagetian abstraction and embodied learning – in order to clarify the building of abstract ideas from perceptual, sensory experiences. The arithmetic principles of commutativity and associativity are examined within these two theories. The theory of embodied learning and the conceptual metaphor is considered as a lens for examining children's informal, intuitive arithmetical knowledge.

INTRODUCTION

This paper is presented as an examination of theoretical issues and existing empirical research that explores how children's formal mathematical ideas can be built from informal, intuitive arithmetic. Although research has examined children's invented procedures and the flexibility of the procedures (Carpenter and Moser, 1984; Steinberg, 1985; Kamii, Lewis, and Jones, 1993; Foxman and Beishuizen, 1999) the research into how children develop the ability to use flexible methods is more limited, although Gray and Tall (1994) have associated success in the use of arithmetic flexibility to the notion of a 'procept' where numbers are viewed as both processes and concepts.

Arithmetic principles, such as commutativity and associativity, can play a role in the development of flexible calculation strategies. There is evidence (Groen and Resnick, 1977) that children develop the use of arithmetic principles without instruction. They come to use the arithmetic principles intuitively from their own informal, spontaneous development of arithmetic.

If there is evidence that children develop an intuitive and spontaneous use of principles such as commutativity, how does this happen? This paper intends to examine two theoretical models - Piagetian theory of abstraction and the more recent theory of embodied learning - that both provide a model for the development of the arithmetic principles.

CHILDREN'S IMPLICIT USE OF PRINCIPLES OF ARITHMETIC

There is evidence that prior to or in the absence of direct instruction young children will devise their own procedures that assume mathematical principles. Groen and Resnick's (1977) empirical work with 4- year olds showed that, even though instruction in addition was limited to the 'count-all' strategy with physical objects, many children soon abandoned this more primitive strategy and initiated the 'count-on' strategy. They also found that many of the children spontaneously chose to start

with the larger number. For example, given the problem $2 + 7$, the children swapped the numbers to $7 + 2$ in order to make the ‘count on’ more efficient. This ‘count-on from the larger’ strategy assumes commutativity in that $2 + 7 = 7 + 2$ or, more formally $a + b = b + a$. Although still relying on a counting procedure there is an implicit use of an arithmetic principle in order to use a more economical strategy.

Further to the more efficient count-on strategy children may use known facts in an innovative way as they invent their own arithmetic procedures. Beishuizen, Van Putten and Van Mulken (1997) and Fuson (1992) have identified two main types of invented procedures. One is termed ‘splitting’ numbers where tens and units are dealt with separately ($23 + 4$: a child may add the $3 + 4$ and then add to the 20). Another is termed ‘complete’ number where one number is kept complete ($24 + 7$: a child may keep the 24 complete but split the 7 into 6 and 1). Such ‘splitting’ number or ‘complete’ number procedures assume associativity in that $(20 + 3) + 4 = 20 + (3 + 4)$ or formally $(a + b) + c = a + (b + c)$.

Children who invent their own procedures would appear to have an intuitive understanding of arithmetic principles such as commutativity and associativity. It is possible that children come to assume that arithmetic operations are commutative as they realise the principle of order irrelevance (Gelman and Gallistel, 1978) and apply this assumption to addition.

“Addition in the child’s view, involves uniting disjoint sets and then counting the elements of the resulting set. According to the order irrelevance principle it does not matter whether in counting the union you first count the elements of one set and then the elements from the other or vice versa” (p. 191).

When extended to three sets, associativity is also used intuitively.

THEORETICAL PERSPECTIVES

The spontaneous development of the principles of commutativity and associativity would suggest that children bring intuitive, informal arithmetic to the classroom. Descartes’ notion of intuition is that of certain and evident knowledge (Lakoff and Johnson, 1999) where we cannot help but see what is before us. ‘Knowing is seeing’ is a tenet of Lakoff and Johnson’s view of embodied learning. Phrases such as ‘I see what you mean’ or ‘Let’s see what is in the box’ are used to convey knowledge of what has been said or knowledge of what is in the box. From an embodied learning viewpoint perception, as a sense-impression from the external world, is seen as a source domain for knowledge. Conceptualisation of abstract ideas can be reasoned about from domains of experience, of which many are sensory-motor. The cognitive mechanism for such conceptualisation is the conceptual metaphor. The conceptual metaphor is not merely a figure of speech but a matter of thought (Lakoff, 1980). It is the mechanism by which the abstract is comprehended in terms of concrete, everyday, sensory-motor experiences such as ‘in’, ‘next’ or ‘movement’.

Lakoff and Nunez's (2000) analysis of mathematical ideas suggests an elaboration of everyday commonplace experiences such as object collection and object construction onto the abstract world of number. Mathematical reasoning is seen as a product of bodies and brains. The notion of embodied learning presents a mechanism to work up from sensory experiences to abstract concepts. Through metaphorical projection abstract concepts are brought into being from the sensory, figurative world (Johnson, 1987).

In the Piagetian viewpoint concepts of number and arithmetic are not seen to be developed through sensory-motor experiences but through reflective or pseudo-empirical abstraction. Whereas empirical abstraction described the unconscious abstractions from the sensory-motor elements and the observable properties of objects themselves (Piaget, 2001), reflective abstraction described an operation on a mental entity that becomes in turn an object for reflection at the next level, allowing for further mental operations (Gray, Pinto, Pitta, and Tall, 1999). Although a two-stage hierarchical process reflective abstraction does not draw its information from the sensory, physical experiences of empirical abstraction but from the coordination of the objects. Empirical abstraction has no parallel hierarchy. That is, there is no empirical abstraction from the results of previous empirical abstraction (Piaget, 2001). There is no projection from perceptual knowledge and so perceptual knowledge cannot be the source of new constructions.

ABSTRACTION AND ARITHMETIC PRINCIPLES

In Piagetian terms abstraction in the development of number and arithmetic is non-empirical. The notion of number is not supplied by the senses so there is a need to attend to non-perceptual properties of the objects. Abstraction of this form is termed pseudo-empirical. It draws its information from apprehending the properties that are presented by an object but where the properties were introduced by previous actions. The focus is on the actions of the objects and the properties of those actions. The child may be 'leaning' on the perceivable results but the perceived properties have been introduced by the child's actions. Such an abstraction entails a level of reflection.

The spontaneous development of the arithmetic principles, such as commutativity, would occur as a form of pseudo-empirical abstraction. The source is drawn from the coordination of the actions of counting and manipulating the objects. The coordination of objects may impress on a child that there is a reason for a particular result, a 'quasi-necessity' (Piaget, 2001), where a child is certain of an event even though the child may not understand the reason for it. The child gains the impression or assumption of commutativity. In Piagetian theory the sensory, experiential world has no direct relationship with the child's assumption of commutativity.

CONCEPTUAL METAPHOR AND ARITHMETIC PRINCIPLES

The notion of embodied learning provides a model where perceptual, sensory-motor experiences are part of the formation of concepts. Abstract ideas can be conceptualised and reasoned about from domains of experience that are mostly sensory-motor (Lakoff and Johnson, 1999). In understanding an idea, we may talk about ‘grasping an object’ (a sensory-motor experience) and if we fail to understand an idea we talk about it ‘going over our heads’ (a sensory-motor experience). Such sensory-motor structuring is apparent in the sense of quantity where we say that ‘more is up’. Here ‘more’ is conceptualised in terms of the sensory-motor experience of verticality, which may have derived from the filling of a glass of water.

Lakoff and Nunez (2000) have provided a model for the development of arithmetic principles from perceptual systems. From the properties of object collections we can determine equal results through different operations in the construction of the collections and see that the same collection results from any order. More specifically, the knowledge that combining object collections A and B in the physical world give the same result as combining B to A can be mapped onto the number world (p.54). This would be similar for three sets.

Other everyday experiences show us that there are various ways to get the same results. Lakoff and Nunez gave the example of shopping for an item by going to the shops, by mail catalogue or over the Internet. These are all different processes that result in the purchase of an item. Such knowledge is represented as an Equivalent Result Frame (Lakoff and Nunez, 2000, p.87). The conflation of these metaphors, Arithmetic is Object Collection and Arithmetic is Object Construction, with the Equivalent Result Frame would explain the emergence of commutativity and associativity in children’s arithmetic. Here the abstract reasoning of commutativity or associativity is based on the perceptual experiences of seeing the identical result of the combinations or the different processes in everyday life.

DISCUSSION

Both theoretical perspectives have provided explanations for the spontaneous development of arithmetic principles that are used in flexible calculation strategies. The Piagetian viewpoint would seem to provide a model for examining the perceptual separately from the conceptual. The embodied learning perspective would seem to provide a model where the conceptual can be built from perceptual, sensory experiences.

Baroody and Ginsburg’s (1987) empirical research provided an example of a boy, Case, who appeared to be uncertain of applying commutativity to addition procedures. When asked if commuted pairs such as $6 + 2$ and $2 + 6$ would add up to the same thing or something different, Case’s response was that the pairs were ‘almost the same but different’. When asked to add $2 + 7$ and $7 + 2$ Case seemed uncertain whether the commuted pairs were equivalent or not and carried out

counting procedures with both pairs to check. Why would he say ‘almost the same but different’? We do not know for certain why Case responded with the statement but it is tantalising to speculate.

One speculation may be that Case is focusing on the perceptual attributes of the object collections. After all a collection of 2 objects of one type, say colour, and 7 of another would appear as a different collection to one of 7 and 2. The arrangement of the objects would give a different pattern. Hence focus on the ‘rich image of the objects’ would not suggest that the commuted pairs are the same. Pitta’s (1998) empirical studies of young children and counting cubes suggested that the more able children attended to mathematical qualities, such as the notion of five when asked to say what was important about the set of objects. The children who did not use efficient strategies would focus on the concrete experiences such as pattern or colour. Thus it is possible that some children did not know what was relevant to focus on. In the same way that Tall (2004) proposed the need to focus on the non-perceptual attributes of objects, maybe Case has not rejected the ‘rich image of detail’ of the objects in order to focus on the structural relations. This speculation would support the Piagetian notion of reflected abstraction that draws on the non-perceptual attributes. The focus is on the coordination of the objects and not the objects themselves.

Take the situation where you have 2 sweets and are given 7, this would be a very different situation to having 7 sweets and being given 2. Even the action of counting out 2 sweets and counting out 7 is different to counting out 7 and the counting out 2. So if the child focused on the ‘rich image’ of the actions the commuted pairs may not be seen as equivalent.

In mapping from the object collection metaphor of the physical world to the world of numbers the physical attributes would not seem to be helpful in making sense of the situation and seeing the equivalence. Sfard (1994) commented on the implausibility of the claim that metaphorical projection from the perceptual to the abstract could be a simple correspondence between a sensory experience and an abstract concept in a similarity relation. Sfard has interpreted the embodied notion of conceptual metaphor as non-comparative. She saw the conceptual metaphor as

“... a mental construction which plays a constitutive role in structuring our experience and in shaping our imagination and reasoning. In other words, rather than being a product of a comparison between two existing things or ideas, metaphor, as conceived by Lakoff and Johnson, is what *brings abstract concepts into being*” (p. 46)

Sfard continued to explore the notion of the embodied schemata as those originally built to put order into our physical experience, which are “‘borrowed’ to give shape, structure and meaning to our imagination” (p. 47). She proposed that this view of embodied learning and the conceptual metaphor does acknowledge the abstract reasoning from the physical world or ‘figurative projection’.

But how might the embodied learning perspective explain Case's uncertainty with the commuted pairs? If I am allowed to take a comment from one of Sfard's interviewees, it may shed some light.

“It is only when you are perfectly certain, without having to check, that things might be exactly the way they are. It's like in the case of an intimate familiarity with a person. With such a person you often know what he is going to do without having to ask ... The (abstract) things have a life of their own but if you understand them, you make predictions and you are pretty sure that you will eventually find whatever you foresaw... The intimacy is exactly what I had in mind: you know what is to happen without making any formal steps...” (p. 49)

This response reflects Johnson's (1987) view of understanding, that it is not just a matter of reflection on pre-existing knowledge but as the “way we experience our world as a comprehensible reality” (p. 102). If, as Sfard suggested, “experiential comprehension gives people an ability to anticipate behaviours of material objects without reflection” (p. 49), a further speculation might be that Case has not yet been able to use experiences to support anticipation and certainty of the result. The conceptual metaphors of everyday experiences of equivalent results may help him feel familiar with intuitive ideas such as commutativity and to make sense of the abstract mathematical notion. A young child's understanding of commutativity may not solely be through operational reasoning and reflection on the process but also through analogical reasoning on the equivalent results of the process as an ontological object, a familiar known experience.

DEVELOPMENT OF AN EMPIRICAL STUDY

This theoretical discussion proposes that the conceptual metaphor can have a role in the construction of children's arithmetic and acknowledges the possibility that mathematical concepts such as number can be built from bodily actions and perceptions. An empirical study would allow further substantiation of such a proposal. In the area of neuroscience empirical evidence would suggest that embodied learning and the notion of conceptual metaphor does play a role in the development of mathematical ideas. As Rogers and Caines (2007) proposed, mental processes can be said to exist by virtue of neural processes where human ideas such as number have their origins in bodily perceptions. Metabolic brain imaging techniques provide evidence that intuitive ideas are part of a functional web that connects primary sensory and motor areas.

But how could a methodology be developed to investigate the role of the conceptual metaphor in children's learning in arithmetic? The development of an empirical study would suggest methodological difficulties, one of these difficulties being how to investigate implicit, intuitive knowledge in young children's solving of numerical problems.

In Sfard's study she had asked the question 'What happens in your mind when you feel that you have understood a piece of mathematics?' to research mathematicians. Responses suggested a notion of familiarity with the mathematics. As mathematicians it is possible that they were able to see 'right to the ground'. In other words they were explicitly aware of the intuitive sense of the mathematics. A brief investigation with prospective primary teachers did not provide the same responses of familiarity. However some of the prospective primary teachers did refer to 'cogs in the brain', 'ideas clicking into place', 'pieces of jigsaw fitting into place' or even 'a flick of a switch'. These in themselves are metaphors for connections so even if they do not mention the notion of familiarity they may be referring to the sense that the mathematical idea is becoming part of a functional web. Whether this question would elicit such responses from children has not been tested.

A further methodology would be to observe children's overt strategies as they carry out trials to solve addition problems and how these relate to their understanding of commutativity or associativity as in Canobi, Reeve and Pattison's (2002) study. Canobi et al's study provided further evidence of the relationship between children's intuitive use of the arithmetic principles and their development of flexible strategies. Children were asked to justify the correct responses and it would seem that the children referred to the equivalence of the results but this is not explored fully. A further examination into the children's autonomous ideas of equivalence as a structural commonality could be pursued.

Other studies have provided evidence that children's analogous reasoning enables them to see the structural commonalities in multiplication and division word problems (English, 1997). It can also be seen that young children may focus on the actions and relationships within different numerical problems. A child may solve a problem through direct modelling where the strategies used reflect the specific actions of the problem and, as such, interpret each problem as a new, individual one (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Young children may not see the 'common thread' or structural commonality that ties the direct modelling strategies together. As children progress to more advanced counting strategies it is considered that they begin to see the common thread. In the same way that English demonstrated that analogous reasoning helped children to see the structural commonalities so it is possible that analogous reasoning enables children to see the 'common thread' or equivalence different contextual problems and even in trials related to commutativity and associativity. Hence as each trial is presented to the children as a problem with different actions and relationships analogy plays a role in allowing them to see the structural commonality of equivalence.

The combined methodologies of observing overt calculation strategies and investigating analogous reasoning could be seen to enable the determination of a relationship between analogy and the development of flexible calculation strategies. A methodology is still needed to determine how the intuitive knowledge is arrived at

and to identify the mechanism that allows the analogous reasoning to take place. If conceptual metaphors are involved, how are they processed?

CONCLUSION

The conceptual metaphor and embodied learning perspective may help to explore how sensory experience is built on and provide a way of exploring the relationship between children's informal, intuitive knowledge and the formal knowledge of the classroom. Sensory experiences with object collections could be built on to develop the intuitive knowledge of the arithmetic principles that in turn could support the development of flexible strategies.

In exploring the example of commutativity it is possible to see on one hand the development as a non-empirical abstraction based on reflection of the child's actions on objects but the dissociation from the perceptual does not provide the opportunity to explore the relationship with children's sensory experiences. On the other hand, exploring commutativity from an embodied learning perspective provides the possibility that the abstract world of number can be built from perceptual, sensory experiences.

The issue of divergence in children's use of arithmetic remains a concern for educationalists. It has been suggested that low attaining children rely on procedural counting strategies whereas more able mathematicians recognise the economy of flexible strategies (Baroody and Ginsburg, 1986; Gray, 1991). Children who adhere to procedural counting strategies may find it more difficult to learn flexibility in arithmetic procedures, even with instruction (Murphy, 2004). As further instruction in arithmetic procedures takes place in school, it is possible that mathematics may become a subject that makes little sense to these children. It would seem worthwhile to examine how children make the mental leaps that allow them to understand the mathematics.

As yet little is known how metaphors are processed (Gentner, Holyoak, and Kokinov, 2001) but conceptual metaphors may provide a lens to investigate children's development in arithmetic. The notion of embodied learning could help examine how children build an abstract notion of number and develop an implicit use of principles that informs their arithmetic from informal, sensory experience. The examination of children's development in arithmetic in terms of conceptual metaphors has inherent methodological difficulties in determining young children's explicit awareness of something that is implicit and intuitive. Other studies have paved the way in indicating the relationship between arithmetic principles and flexible strategies and also in children's progression from the 'rich detail' of informal arithmetic and direct modelling to the more abstract counting strategies that rely on analogical awareness of structural similarities. A review of observation or interview techniques is needed to investigate the mechanism that is happening and to determine the role of the conceptual metaphor.

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THE POWER AND PERILS OF METAPHOR IN MAKING INTERNAL CONNECTIONS IN TRIGONOMETRY AND GEOMETRY

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This paper draws upon data collected in two research projects, on the teaching and learning of trigonometry and geometry respectively, to illustrate the power of metaphor to facilitate students' connected knowledge in these fields. The structure of metaphor is examined. Because the domains connected by metaphor have both similar and dissimilar elements, it is necessary for both teachers and learners to take into account the tension introduced by the dissimilar elements in order to make the most of the potential connecting power of metaphor. Illustrations are provided of how spontaneous idiosyncratic metaphors of learners as well as taught metaphors provided by their teachers may become a powerful shared socio-mathematical force or an individual means for connected learning in geometry and in trigonometry.

ALISON'S WATER LEVEL: A VIGNETTE

Already in the early 1980s, in my doctoral research on visualization in the teaching and learning of high school mathematics, it became apparent that metaphor could be used in the service of generalization in this teaching and learning. All of the difficulties experienced by the 54 visualizers in that study could be related in one way or another to problems of generalization. Visual imagery and inscriptions, by their nature, provide one concrete case of a mathematical principle; yet they have to be understood in a general sense in order to be used effectively. In the course of 188 transcribed interviews over the course of a school year, there were two means of overcoming this one-case concreteness of visualization. One of these was by the use of *pattern imagery*—which depicts pure structure—and the other was by embracing the power of metaphor to connect disparate domains (Presmeg, 1985, 1992, 1997a&b). From this early research, an illuminating illustration of the usefulness of idiosyncratic personal metaphors to learners is provided in the following extract from a transcript of an interview in which Alison was solving a trigonometric problem that required her to find a reference angle in the second quadrant, given the rotation angle (... indicates that time has elapsed).

Alison: Then it would be sine ... second quadrant. So, 180; you take it from 180, 'cause that's your water level.

Interviewer: Oh, is that how you think of it [laughing]? And how does the water level help you to get it?

Alison: Oh, um ... you've got ... It's like a ship sailing: can't sail that way really [gesturing up and down, along the y axis]. There, that, sort of, it can. You

work it from there. You take most of your ... 'cause that's your 360 ... it's there as well.

Interviewer: Did you think of that for yourself or did someone tell you about the water level?

Alison: No, I just thought of it.

Alison's water level metaphor enabled her to make sense of a trigonometric principle in terms of her real life experiences. This encapsulation of the principle also had apparent mnemonic advantages: she could recall the principle with ease.

THE STRUCTURE OF METAPHOR

Metaphor has directionality: if one says, "A teacher is a gardener", one is talking about a teacher, highlighting those nurturing aspects of the field that are shared in teaching and in gardening. It would be far less appropriate to say "A gardener is a teacher", because plants have less capacity to be taught than human learners. "A teacher is a policeman" provides a very different conceptualization of the work of a teacher. The work of a teacher is the *target* domain; gardening or policing are the *source* domains. These domains are also known as the *tenor* and the *vehicle* of the metaphor, respectively, in literary classifications (Leino & Drakenberg, 1993). The elements that are common in the two domains being compared are known as the *ground* of the metaphor. Those elements that are not comparable in the two domains comprise the *tension* that is present in every metaphor.

The foregoing brief analysis of the structure of metaphor (elaborated in Presmeg's 1998 paper) provides the theoretical lens for analysis of the data on metaphor collected in two recent empirical investigations. Both research studies were concerned with *connections amongst registers* or domains in students' learning, thus although the research had a wider focus, metaphor was a natural part of these studies by virtue of its function in connecting domains of experience. Registers are taken as *modes of representation* in this paper. Duval (1999) considered conversions, both within and amongst various registers (such as algebraic and diagrammatic), to be essential aspects of effective learning of mathematics. The research on which this paper is based provided confirming evidence for Duval's claim, although caution is needed in taking students' ability to move fluently amongst registers as sufficient evidence of deep conceptual understanding (Presmeg & Nenduradu, 2005).

TWO RECENT RESEARCH PROJECTS

In the spring semester of 2006, I had the privilege of collaborating with colleagues in two investigative studies. With Susan Brown, enlarging the scope of her doctoral research (Brown, 2005), I investigated elements of her teaching of trigonometry that facilitate learners' construction of connections amongst the various registers of school trigonometry, such as triangle definitions, the coordinate plane, the unit circle, and the graphs of trigonometric functions. In a separate study with colleagues Jeffrey

Barrett and Sharon McCrone, in Jeff's class of mainly preservice elementary school teachers, we investigated connections between understanding of geometric principles using dynamic geometry software and using traditional Euclidean compasses-and-straightedge constructions, in the course *Geometric Reasoning: Geometry as Earth Measures*. The nature of this course, "earth measures", enabled the introduction of some trigonometry along with the geometry, in the topics taught and in the interviews that were conducted with selected students. After all, the meaning of the word "trigonometry" is "triangle measurement," taken from the Greek words *trigon*, triangle, and *metron*, measurement, and trigonometry is widely used in navigation and in land surveying on a global scale. My research question in both projects was the same, namely, *How may teaching facilitate students' construction of connections amongst mathematical registers?* In this paper I shall concentrate on the metaphors that were evident in the 30 task-based interviews (6 interviews in the geometry class and 24 in the trigonometry project).

In the trigonometry part of the course *Enriched Advanced Algebra/Trigonometry* taught by Susan Brown, in addition to observation of the teaching of seven lessons, four students were interviewed six times each, on January 30, February 6, 13, and 27, April 17, and May 9. The four students, Laura, Raj, Jim, and Brian (pseudonyms) were chosen by the two researchers in consultation, in order to achieve a variety of learning styles. Each individual task-based interview of 15-25 minutes' duration was audio-recorded; I transcribed the recorded data immediately following the interview or on the same or the following day, when memories of the context were fresh. In the course of the 24 interviews there were 9 instances of spontaneous metaphors and four references to metaphors that had been introduced by the teacher (Susan). In the geometry course with preservice teachers, one metaphor, which was first mentioned by the teacher, Jeff, became adopted by the whole class and used throughout the remainder of the semester by students and teacher alike. Data collection in this class included video recordings of all class sessions, field notes of one observer (either Sharon or Norma) in almost all sessions, reflective notes on each session by the teacher (Jeff), and transcriptions of audio-recorded interviews. Sharon and Norma each interviewed three students twice (near the beginning and the end of the spring semester), with one student, Mary, in common to both interviewers: thus four students were interviewed twice each, and Mary four times. In this paper I shall draw on field notes in three sessions, February 14, 16, and 21, in which a metaphor became part of the class discourse.

EXAMPLES OF "TAUGHT" METAPHORS

In the trigonometry data from Susan's class, two of the four students initially experienced difficulty in working with rotation angles in the third quadrant. Both Laura and Jim had a tendency in the first two interviews (Jan. 30 and Feb. 6) to drop a perpendicular to the y axis rather than the x axis in the third quadrant, in attempting to find the reference angle. The triangles they constructed introduced an added level

of complexity and hindered their working. Jim in the discussion that ensued even expressed resistance to working from the x axis in that quadrant: he did not like drawing a reference triangle “backwards on itself” in the third quadrant, because it would be “blocking” the rotation angle (see Jim’s transcript data in Presmeg’s 2006 paper, vol. 1, pp. 29-30). After these first two interviews, on February 7, the teacher, Susan, asked her class to perform a computer program in which they created a colourful “bow tie” to illustrate the principle of using the x axis in finding reference angles in the four quadrants (figure 1, without the colour). After this exercise, none of the four students interviewed used the y axis for this purpose.

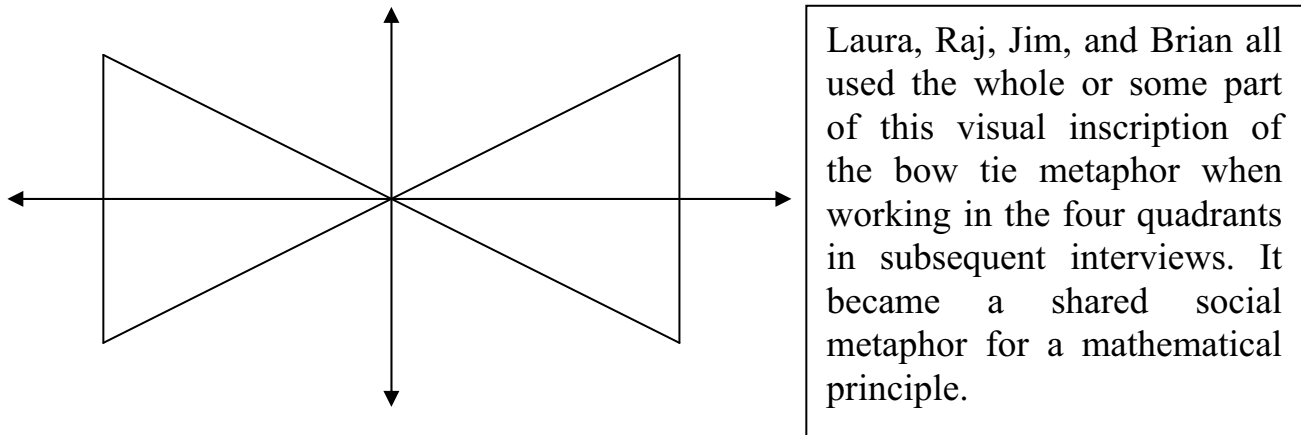
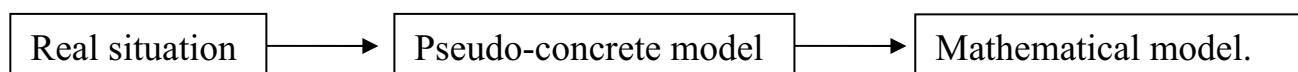


Figure 1. The bow tie inscription.

Susan had introduced another metaphor, that of a boom crane and its movements, in working in a half-plane in the early part of the course. Both Brian and Jim indicated in the first interview (January 30) how helpful this real-world connection was for them in making sense of the extension of the right triangle trigonometric definitions to the coordinate plane. “Mrs. Brown does a good job of just, you know, showing us lots of different ways with real life examples, and that helps”, reported Brian. However, in its connections of triangle trigonometric definitions with the structure of a coordinate half-plane, the boom crane metaphor turned out to serve more than just the purpose of a real life example, because its target domains were both triangle and coordinate plane trigonometric fields. The bow tie also effectively linked these two target domains, while the source domain came from real life. The dual target domains thus facilitated students’ conversions between these two registers of trigonometry in each of these two metaphors. The bow tie inscription is already one step removed from the imperfections of a real bow tie: here lies the tension of the metaphor. The inscription is a *pseudo-concrete model*, in the sense of Parzysz’s (1999) characterization of mathematical modelling from the real world:



In the “earth measurement” course, Jeff introduced a metaphor in the geometric context of constructing various kinds of quadrilaterals, both by means of classical Euclidean constructions and by using dynamic geometry software. In the computer

laboratory (unlike the constructions with compasses and straightedge), students' constructions of quadrilaterals such as a square or a trapezoid could in some cases be "broken" by dragging vertices: the whole inscription could come apart! The metaphor of *unbreakable* was adopted by the whole class and used by students in sessions in the lab on February 14, 16, and 21 as a test of whether or not a construction had been done rigorously. For instance, in attempting to construct a square (February 14), Jenny used Geometer's Sketchpad to draw a circle, then to construct lines that appeared to be perpendicular, through the centre of the circle. She connected the points of intersection of these lines with the circle, in order to create a square – but after dragging, the object was obviously no longer a square: it was not unbreakable. This metaphor became an important test, used in class dialogue subsequently not only in the computer lab but in using classical tools, where dragging was not an option. The metaphor had come to embody a general principle for the class, one that went beyond the initial context of its inception.

The metaphors of the bow tie, the boom crane, and the unbreakable sketch all used source domains from real life contexts, but their target domains were mathematical principles. The bow tie encapsulated the principle of drawing reference triangles to the x axis in the four quadrants of the coordinate plane. The boom crane made a logical connection between the movements of this crane and the principle that one could identify a point in the plane by means of a distance from an origin and an angle measured from the positive side of the x axis. The metaphor of an unbreakable sketch drew upon students' real experiences with objects that could be shattered or come apart. All three of these metaphors were memorable for the students, as evidenced by the fact that talking about them became part of the shared classroom culture. These metaphors were originally introduced by the respective teachers, Susan and Jeff. However, as in the case of Alison's water level metaphor in the opening vignette, interviews also revealed spontaneous metaphors with which students endeavoured to make sense of mathematical principles.

EXAMPLES OF SPONTANEOUS METAPHORS

Of the four students interviewed in the trigonometry project, Jim was the one who thought most visually in his choices of inscriptions and language in solving problems in trigonometry. His terminology was often metaphorical. For instance, the following transcript is from the interview on April 17. I was asking Jim how he knew that the tangent of an angle is the sine of the angle divided by its cosine.

Jim: Because for the tangent it's the y, I guess the y, um, the y coordinate on the unit circle. And the cosine is the horizontal ... um, on the circle.

Interviewer: I know what you mean.

Jim: So, then, on the unit circle the cosine is the horizontal movement and sine is the vertical movement, to get to the point on the unit circle, so it would be sine over the cosine.

Jim then made the connection between the tangent as the slope of a line, and these vertical and horizontal “movements”. In stating that the cosine *is* the horizontal *movement*, his thinking was clearly metaphorical. This language seemed to come much more naturally to him than his hesitant attempts to define sine and cosine in terms of coordinates of points on the unit circle. In this interview, he also spoke of the trigonometric ratios in terms of distances, using a *journey* metaphor (cf. Lakoff & Johnson, 1980; Johnson, 1987; Lakoff & Núñez, 1997). The metaphor of a journey was also present in a striking way in his language in the last interview (May 9), in which he referred to the x value of the cosine as “the amount you go over”. The journey metaphor was also found in language used by Raj and Brian in interviews. In the third interview (February 13), Brian spoke of “distances” from the x and y axes when referring to the sine and cosine ratios respectively. In the first interview (January 30), I asked Brian if there were any aspects that had helped him make the transition from triangle definitions to trigonometry in the coordinate plane.

Brian: I think the thing that made the transition easiest, was that, when we worked on the positive [that is, in the first quadrant], it was just like working with a normal triangle. Your sine was still the opposite over the hypotenuse, your cosine was still the adjacent over the hypotenuse. And that made it easy to see the **translation** that, sine was y, which was still the same as the opposite, over the r; cosine was still the adjacent, over the r.

Brian’s characterization of a “translation” in this context could be a metaphoric reference to the trigonometric definitions in terms of movement in the plane. However, the metaphor might also have its source in a linguistic context: one translates between two languages in a way comparable to the translation between the registers of the triangle definitions and the designation of points in the plane.

The most dynamic instantiations of the journey metaphor were those used by Jim, who was obviously at home in the visual environment of a journey. But Jim’s metaphorical thinking in terms of spatial orientation sometimes led him astray, as illustrated next. In the final interview of the trigonometry project (May 9), each of the four students was asked to identify the “big ideas” of the trigonometry part of the course they had completed, and to explain to the interviewer how these big ideas are “all connected together, if they are”. The wording was deliberately broad, in order to elicit those ideas that each student considered significant, and possible connections amongst them. About one third of the way through the 20 minute interview, Jim had identified the trigonometric ratios in terms of distances again, in the coordinate plane with a unit circle. Then he sketched the graphs of sine, cosine, and tangent of an angle.

Interviewer: Can you link the graphs with the circles?

Jim: Um, for this one [sine] if you had the origin or zero for the y at um, zero, here, um ... And this would be one on the unit circle and negative one here. The same thing for all of them. Then you have for the cosine, reaches,

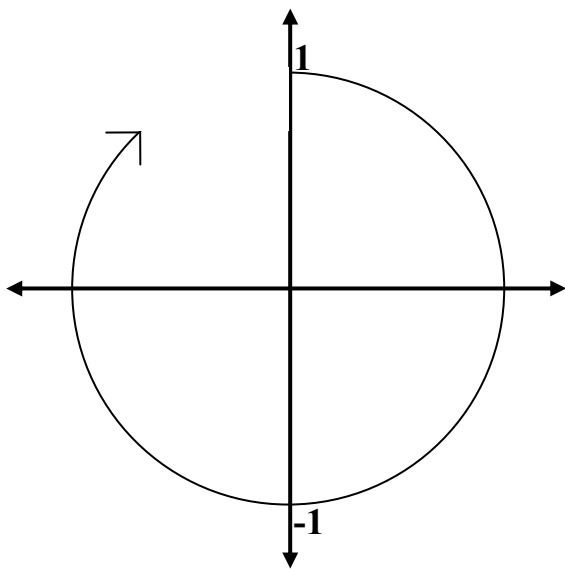
you'd start up here [at one on the y axis] for the cosine graph and then, see it goes down as you go around. So, until it goes down to the bottom, and then it goes back up.

Interviewer [based on Jim's gestures]: So you're going clockwise, from the top, in the cosine?

Jim: Yes!

Interviewer: Is that how it works? Are you thinking it's one here at the top, on the y axis?

Jim: Yeah. And then negative one at the bottom.



Jim's gestures and words suggested that his image of the changing values of the cosine, based on the cosine graph and its sinusoid shape, was as in figure 2.

Figure 2. Jim's attempt to link the graph of $y = \cos \theta$ with the unit circle.

There was a certain logical quality in Jim's gestured inscription, because the sinusoid shape of the cosine graph does "start up here" at one on the y axis, move to negative one after half a revolution of the angle, and then return to positive one after completing the revolution, and the negative part of the cosine wave happens in the "right" places. But what makes this conversion a cognitively complex task, and what Jim was overlooking, was that he had formerly defined cosine θ as the "horizontal movement" or the "x value" of points on the unit circle. His journey metaphor was not helpful in this instance, because he converted the y "distance" quite literally from the graph of $y = \cos \theta$, in which the axes represented θ and y , to a unit circle with axes x and y respectively, on which cosine should have been represented by x coordinates rather than y . The discourse continued as follows.

Interviewer: What is cosine again?

Jim: Um ... cosine is the x coordinate over the radius.

Interviewer: Yeah. So what you're telling me here [at the start, on the y axis] is that the x coordinate is one?

Jim: Oh, yeah! So ... starting here then [positive x axis], that would be the top, the highest cosine value of one, then work backwards this way [counter-clockwise].

Then Jim completed the comparison correctly. It is interesting that Jim's imagined inscription conforms to the way that angles are measured in large-scale space, from due north clockwise, in surveying and navigation.

There was one further spontaneous metaphor, of an anthropomorphic nature, in the first interview with Brian. I had asked him whether the triangle trigonometric definitions were still viable in the coordinate plane.

Brian: Um, in a sense, yes, but these [the triangle definitions] all have to account for positive integers, and **your calculator thinks the same way**. Looking at the coordinate plane, you have to deal with negative x now, so it could be negative like, and negative y. So in both senses you could work with a negative adjacent for your x. Because the right angle, when we've been doing these rotations it's always on the x axis. This change, this angle is here. So basically, the opposite is the y.

Brian drew a perpendicular from a point in the third quadrant to the x axis, and completed the right triangle. He did not refer to the bow tie metaphor, which had not yet been introduced in class. He did not need it. His metaphoric reference to a calculator "who" can think captures the principle that the calculator has to work economically with trigonometric ratios and thus deals only with positive values, as in the right triangle definitions. "Basically" the adjacent is the x coordinate and the opposite is the y now, but the thinking calculator can deal only with positive values, so it is up to the student to supply any relevant negatives in the four quadrants.

Spontaneous metaphors are especially powerful in illuminating the unique ways that individual students attempt to make sense of mathematical principles. However, in interpreting students' words and inscriptions, it is necessary to bear in mind that every metaphor has a *tension* as well as a *ground*, and that the tension may result in some difficulties unless it is taken into account, as illustrated in Jim's case.

SIGNIFICANCE OF STUDENTS' USE OF METAPHORS

Particularly when the research focus is ways that students construct connections amongst various mathematical registers (as is the case in the projects outlined in this paper), awareness of the role of metaphors in these connections is a useful research tool. Lakoff and Núñez (1997) presented a thought-provoking foundation for the position that mathematical cognition is structured by metaphors of various kinds. However, these authors wrote in terms of the canonical structures of mathematics itself (as accepted by professional mathematicians), and in my view did not pay sufficient attention to the idiosyncratic, spontaneous metaphors that individuals construct in their attempts to give meaning to mathematical ideas.

In the teaching and learning of mathematics, either by serendipity or design, it sometimes happens that a metaphor introduced by the teacher becomes part and parcel of the fabric of classroom discourse, and thus helps many students in their building of connections for mathematical concepts. In the current research projects the metaphor that *a good construction is unbreakable* (serendipitously introduced by the instructor, Jeff) served the double role of connecting the target dynamic geometric computer constructions with the real life source context of objects that can break or shatter or be pulled apart, and of connecting—by usage—the world of dynamic geometry software with the contrasting world of static Euclidean constructions. As in the bow tie metaphor, one source domain (breakable objects) was mapped on to two target domains as the students used the metaphor in both worlds, thus moving between the registers of these worlds in working with the same geometric ideas. Of course, *ground* and *tension* were present in both modes, but the students who picked up and used this metaphor seemed to have no difficulty in distinguishing between elements that were common in the structures being compared, and those that were not. The same could be said of Susan's deliberately introduced metaphors of the bow tie and the boom crane in connecting registers of trigonometry. Both of these metaphors were of evident value to students as reflected in their words and actions in interviews designed to elicit their ways of making connections.

More subtle, but not on that account less valuable, were the idiosyncratic metaphors introduced spontaneously by the students themselves. That they were constructing metaphoric connections with real life contexts was revealed in some cases by the kind of language they used. Many terms related to journey and position metaphors: *you go over; you go up*. In one case the metaphor was anthropomorphic: *your calculator thinks the same way*. The metaphor of a *translation* in connecting triangles and the coordinate plane had overtones both of *movement* and of *translating one language into another*. In all cases the implicit comparisons helped the students to find contexts that were personally meaningful in making sense of the mathematics that they were learning. Mnemonic advantages were also apparent: Alison did not forget the mathematical principle that was the target of her *water level* source, and nor did Susan's students who made the *bow tie* their source for the same target principle. Despite the perils introduced by the tension present in every metaphor, the power of metaphor in students' constructing of connected knowledge of mathematics is clear.

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METAPHORS AND IMAGE SCHEMATA IN CONCEPT FORMATION AND REASONING

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Metaphor theory and cognitive science provide viable models of the functioning of the brain and our way of thinking that is highly relevant for mathematics education. In this paper I discuss how the metaphor analysis of Lakoff & Núñez (1997, 2000) and the concept of image schemata proposed by Johnson (1987) and Lakoff (1987) can be used to improve the learning of mathematical ideas and reasoning. The theoretical tools of prototypes and protocols from Dörfler (1991, 2000) are central for the application of image schemata and metaphors to this field. A central focus of the paper is how interplay between the formal and cognitive parts of mathematics is necessary in order to achieve learning with understanding.

INTRODUCTION

Lakoff and Núñez think that metaphor theory and cognitive science will change both mathematics itself (1997) and mathematics education (2000). My focus is how the field introduced by Lakoff, Johnson and Núñez can be used to change mathematics education by actively promoting new ideas and pointing to weaknesses in contemporary practice. Changes closely related to mathematical content will be discussed. Knowledge of embodied cognition gives the hope of better and more systematic learning of both mathematical ideas and mathematical thinking. In order to achieve this, the formal parts of mathematics should not be forgotten. Dörfler (1991) gives some advantages of the formal approach to mathematics. For instance, it gives high security and accuracy. Also, formal reasoning can in principle be mechanized and automated. Lakoff and Núñez (2000) mention much of the same, but also that algorithms minimize cognitive activity. My claim is that interplay between the cognitive and the formal is fruitful and necessary. Improvements are possible in both areas, and often a change on one side has consequences for the other.

REIFICATION, DISCOURSE AND METAPHORS

Traditional textbooks and classroom practice emphasize the formal part of mathematics. Techniques are taught systematically, but usually not ideas. Lakoff and Núñez state that mathematics should be taught in terms of the latter.

To overstress either techniques of formal proof or techniques of calculation is to shortchange students. Students have a right to understand mathematics in terms of its ideas – and especially when the ideas are controversial and conflict with one another. Since a great many mathematical ideas are metaphorical, teaching mathematics

necessarily requires teaching the metaphorical structure of mathematics (Lakoff & Núñez, 1997, p. 85).

To achieve this, a link is needed to connect the analysis of ideas with learning. Sfard has pointed out that efforts of introducing or bringing about the appropriate metaphors are often rewarded with limited success (Sfard, 1994). In several papers Sfard (1991, 1994 e.g.) discusses her theory of reification. According to Sfard, “reification is the birth of the metaphor of an ontological object” (1994, p. 53). The sudden appearance of reification is at the same time the moment of real understanding. The way of thinking is drastically changed from operational to structural. Sfard states, “Reification – a transition from an operational to a structural mode of thinking – is a basic phenomenon in the formation of a mathematical concept” (p. 54). Reification is normally preceded by considerable effort and work. The dominant part of this work takes place in what Sfard calls the condensation phase (1991, p. 19). The approach in this phase is predominantly operational (p. 14), but the learner becomes more and more capable of thinking about the process as a whole (p. 19). The phase is a struggle for understanding, often including a search for structural metaphors. Then reification can also mean that a system of structural metaphors for the new concept is established.

Sfard gives emphasis to the potential role of names, symbols, graphs and other representations in condensation and reification (p. 21). In the case of negative numbers, the student initially sees only operations like subtractions and movements to the left on the number line. However, the symbols for these processes resembling those for the natural numbers, suggest that they behave like the latter. Involved is the metaphor that “negative numbers are numbers”. The use of the well known signs ‘+’ and ‘-’ in the new setting has a potential of communicating the metaphor.

The emphasis of Dörfler (2000) on the Wittgensteinian (2001) idea of language game is an important and consistent supplement to Sfard’s theory of reification. The former implies that the learner has to jump into the discourse of the subject in order to learn. You cannot learn the meaning of the piece called knight in chess without playing the game and taking part in the discourse among the players. In the theory of Sfard, symbols and operations have to be used before they are understood. Dörfler can be seen to describe phases similar to condensation and reification. The grasp of the as-if attitude of Dörfler (p. 122) is comparable with reification. His formulation “to be inducted into the mathematical discourse about a concept” (p. 111-112), seems to indicate the same phase. This is preceded by a period of action and communication. Dörfler does not often mention metaphors, but they are implicitly present, both as prototypes (pp. 102) and in the concept of discourse. The number line is a possible prototype of the whole numbers Z (p. 103) and its introduction may be viewed as a final trigger for the reification of negative numbers (Sfard, 1991, p. 21). Lakoff and Núñez (1997) regard the number line as a linking metaphor when we metaphorically

understand numbers as points on a line (p. 34). In this case we are linking arithmetic and geometry.

IMAGE SCHEMATA AND PROTOCOLS OF ACTIONS

Image schemata or embodied schemata were introduced by Lakoff (1987) and Johnson (1987). The two notions are used interchangeably by Johnson (p. 28). Image schemata are associated with patterns and order in actions, perceptions and conceptions. According to Johnson (1987, p. 29), “A schema is a recurrent pattern, shape, and regularity in, or of, these ongoing ordering activities.” Image schemata explain how metaphors are connected to bodily experience.

Human ideas are, to a large extent, grounded in sensory-motor experience. Abstract human ideas make use of precisely formulatable cognitive mechanisms such as conceptual metaphors that import modes of reasoning from sensory-motor experience (Lakoff & Núñez, 2000, p. XII).

Dörfler (1991, 2000) in his work underlines that many mathematical concepts are related to systems of actions and their products. When such systems are available, the formation of image schemata and conceptions can be supported by focusing attention on the relevant aspects of the actions.

With respect to image schemata for mathematical concepts I propose a special mechanism which is hypothesized to lead to the conscious construction of image schemata by the individual (1991, p. 28).

This mechanism is called protocols of actions. Protocols consist of records, notations or descriptions of mental or physical manipulations and interactions with object-like models. According to Dörfler, a protocol is a cognitive process (p. 28). However, the internal mental processes and the external signs take part in close interplay. Guidance from teachers or other competent persons is necessary for successful use of protocols.

As an example consider the learning of isometries in the plane. Rotations are more easily comprehended when physical objects are rotated than abstract mathematical points. A protocol of actions for the former situation can be some kind of record of the rotation angles and the changing positions of the rotated objects. This helps the student to be aware of the patterns in compositions of rotations and to link the activity to the discourse of mathematics. Reification means that an image schema for rotations is established. The kind of situation and devices used for the rotations may then be turned into a prototype for mathematical rotations. When this is achieved, the physical system can be seen as the source domain of a metaphor for mathematical rotations. An example of further metaphorical use of mathematical rotations is to give meaning to orthonormal matrices in linear algebra.

IDEAS AND THE FORMAL IN MATHEMATICS

Sfard mentions several dichotomies of mathematics and mathematical understanding (1991, pp. 7). For instance, mathematics can be divided into abstract and algorithmic. The main such distinction in her paper is the duality between operational and structural mathematical understanding. The word duality is used in another sense than dichotomy. A duality refers to inseparable, though dramatically different, facets of the same thing (p. 9). Beside the operational/structural duality of mathematical understanding, I will also put emphasis to the distinction between ideas and the formal in mathematics. Conceptions, metaphors and image schemata are classified on the cognitive or idea side of mathematics. The formal consists of algorithms, formulas, symbols, definitions and axioms. This is the part of mathematics that in principle can be written in a formal language and be represented on a computer. The distinction between ideas and the formal is not the same as that between the structural and operational. Certainly, symbols and rules are needed in working with an operational approach, but the phase of condensation is more than that. Ideas are important in the struggle for meaning. Also, structural thinking is not at all an escape from the formal part of mathematics.

Concepts are often considered to be on the formal side, given by definitions. One origin is Aristotle and his theory of essences. Following Sfard I use the word “conception” as the cognitive version of concepts (p. 3). In objectivist thinking conceptions are the internal representations of concepts. However, from the perspective of metaphor theory or constructivism concepts cannot be considered independently of conceptions. The meaning of a concept is not inherent in formal definitions or operations. The theories of Sfard and Dörfler show how meaning develops in interplay between operations, ideas and formal aspects like symbols and rules. Neither of these factors can give meaning in mathematics independently.

Certainly ideas and the formal are very different aspects of mathematics. In order for the distinction to be a duality, the two have to be inseparable and facets of the same thing. At least they seem to be inseparable. Ideas in advanced mathematics heavily involve or relate to the formal aspect. Also, no formal concept is pursued without accompanying ideas. It is more interesting however, to look for a deeper connection. I suggest that some definitions and use of formal language have more potential of being related to clear ideas than others. For instance our positional system of numerals is better suited for thinking about numbers than the roman numerals. Something similar is suggested by Sfard.

Although such property as structurality lies in the eyes of the beholder rather than in the symbols themselves, some representations appear to be more susceptible of structural interpretation than others (Sfard 1991, p. 5).

Further research is needed to achieve a good understanding of the phenomena. Bergsten (1999) gives a contribution by studying the figurative and spatial characteristics of formulas. He refers to Dörfler (1991) who thinks that formulas in

mathematics can play the role of a carrier for an appropriate schema (p. 28). A carrier is the mental, drawn or physical “objects” whose manipulation leads to the schema (p. 21). Also Dörfler’s concept of protocols is relevant to the question. The formulas or other formal records associated with protocols are related to the involved image schemata.

THE CONCEPT OF INFINITY

If protocols of actions are the link between ideas and the formal in mathematics, some parts of contemporary mathematics are not included in this. According to Dörfler, we cannot devise a protocol whose structure reflects infinite processes such as limits, derivatives and integrals. In his words “it is difficult to base the limit concept on experiences as they are provided by prototypes and protocols” (Dörfler 2000, p. 121). Tall (1989) rejects the classical introduction to calculus through limits, secants and a formal definition of the derivative. He claims cognitive obstacles to follow that kind of introduction and gives as an example “that many students encapsulate the *process* of getting smaller as an *object* that *is* arbitrarily small - a cognitive infinitesimal” (Cornu in Tall, 1989). The cognitive process behind this is described by Lakoff and Núñez (2000, pp. 268). Tall puts forward an alternative based on local straightness and tangents, concepts close to physical experience.

Lakoff and Núñez state that “Mathematics is ultimately grounded in the human body, the human brain, and in everyday human experience” (1997, p. 84). Grounding metaphors allow us to project image schemata structure from everyday domains to the domain of mathematics (Lakoff & Núñez, 1997, p. 34). It is relevant to ask if the ideas and ways of thought closest to physical experience are easiest for students to grasp. Finite line segments may for instance be easier to comprehend than infinite lines. The former was the first to be introduced historically. Only finite lines and potential infinity is used by Euclid in his Elements. Line segments are well suited manipulative objects associated with protocols of actions and image schemata. After all, only finite lines can be drawn on paper or a screen. The absence of infinite lines leads to awkward formulations in theorems, but such statements can wait for the reification of the line concept.

Lakoff and Núñez describe the creation of the conception of actual infinity by their basic metaphor of infinity, BMI (2000, pp. 158). However, they do not claim the infinite to exist as physical reality. This view resembles that of Hilbert and his ideal elements.

The role that remains for the infinite to play is solely that of an idea – if one means by an idea, in Kant’s terminology, a concept of reason which transcends all experience and which completes the concrete as a totality - ... (Hilbert, 1983, p. 201).

Lakoff and Núñez describe currently used ideas of infinity in mathematics. Some of these ideas pose avoidable difficulties for the learners of the subject. The example of transfinite numbers (Sfard 1994, p. 53) shows clearly what kinds of problems that are

posed by actual infinity. For example, the concept of transfinite numbers violates the fundamental, experientially established principle “the part is less than the whole” (p. 53).

POSITIVE AND NEGATIVE DEFINITIONS

The concept of prime number is sufficiently advanced to see a link between ideas and formal definitions. Usually prime numbers are defined as the natural numbers larger than one, not divisible by any other numbers than one and itself. The innocent word ‘not’ means that this is a negative definition. Prime numbers are defined by the lack of a property. This does not directly give the learner obvious objects and actions to carry out. Moreover, the word ‘any’ suggests that an infinite number of possible divisors have to be checked. We know that there are only finitely many candidates for dividing a natural number, but the use of language is distracting. Fortunately, the primes can also be defined positively. They are the natural numbers with exactly two divisors. The latter definition relates to the action of finding divisors for numbers from one and onwards. An associated protocol can be a table of divisors with a marking for those numbers with exactly two divisors. This supports the formation of an image schema giving a rudimentary understanding of primes. Of course, a continued process of learning is needed to gain a mature understanding.

HYPOTHETICAL REASONING

Hypothetical reasoning is an imaginative way of thought involving objects not known or not existing at all. In the equation $x^2 + 3x = 10$, it turns out that x denotes two possible integers, 2 and -5. Similarly, if we ask who the pilot of a plane is, there may be two pilots in the cockpit. The equation $x^2 = 2$ has no integral solutions. This corresponds to the frightening situation of no pilots. The x and the pilot are metonymies which usually have one referent, but sometimes several or none. Presmeg (1992, 1997) are actual references for the use of the concept of metonymy in mathematics education research.

Solving equations by inspection is a direct and positive approach. This is not immediately applicable if we ask for rational solutions of the equation $x^2 = 2$. Then we are looking for natural numbers a and b such that $a^2 = 2b^2$. The standard proof of the non-existence of such a pair of numbers uses formal reasoning with divisibility and the fundamental theorem of arithmetic. We are asked to suppose the existence of two numbers solving the equation, letting those be referred to metonymically by letters a and b . A positive alternative is to view the equation $a^2 = 2b^2$ as a sequence of equations

$$a^2 = 2 \cdot 2^2, a^2 = 2 \cdot 3^2, a^2 = 2 \cdot 4^2, a^2 = 2 \cdot 5^2, \dots$$

Cases like $b = 4$ and $b = 5$ are prototypical. Any factorization of a gives an even multiplicity of the factor 2 on the left side of a prototype. Trying to separate the factors on the right side in two equal parts, gives a remaining factor of 2. This gives

an odd multiplicity of the factor 2 on that side. Given a belief in the uniqueness of factorization, this is a contradiction meaning that no number a can be found. The involved actions and appropriate protocols have potential of developing an image schema for elementary understanding of the impossibility of finding a rational square root of two.

At advanced level a proof of the fundamental theorem using algebraic technique is needed. The reason is that uniqueness of factorization is not at all easily seen for large numbers. Experience with computer assisted factorization of large numbers can be a help to see that factorization is inherently difficult and time consuming. Numerical software and calculators are situated on the formal side of mathematics. This gives another example that the formal can take part in interplay with understanding and ideas.

THE PRINCIPLE OF GENERALIZATION

Generality through the special was used by the Babylonians long time before the birth of symbolic algebra. The dominance of formal mathematics lowered the status of that kind of presentation and reasoning. Metaphor theory can change this. The empiricist philosophers Berkeley and Hume claimed the general to be particular in the mind's conception of them, (Hume, 1739. Part I, sect. vii). Dörfler's concepts of prototypes and protocols of actions let image schemata be used to rehabilitate general reasoning through the special. However, an image schema is something more than a collection of particulars. One aspect of this is elaborated by Johnson (1987, pp. 24) taking Kant as a starting point.

In the sequence of equations

$$a^2 = 2 \cdot 2^2, a^2 = 2 \cdot 3^2, a^2 = 2 \cdot 4^2, a^2 = 2 \cdot 5^2, \dots$$

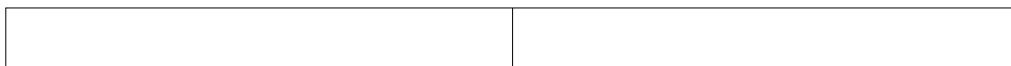
the factorization operations on both sides are actions with a clear pattern. The recognition of this pattern makes the inference that this kind of action can be done for all the equations in the sequence. I call this an application of *the principle of generalization*. Another example is the formula for the sum of the geometric progression

$$1 + 2^1 + 2^2 + \dots + 2^n = 2^{n+1} - 1.$$

A prototype for this identity is

$$1 + 2^1 + 2^2 + 2^3 = 2^4 - 1.$$

A power of two, like 2^3 , can be regarded the three times iteration of doubling, starting with the unit one. The prototype identity above can be proved starting with a rod of length 2^4 . This rod can be divided into two rods consisting of length 2^3 .



The left part of these 2^3 rods can be divided into two 2^2 rods. The process continues until termination is achieved by splitting 2^1 into $1 + 1$.



The entire 2^4 rod now consists of the sum $1 + 1 + 2^1 + 2^2 + 2^3$. Even if literally only one value of n is investigated, the proof is general. The same pattern of actions and reasoning can be applied for every n . A help for the learner to be aware of the generality of the actions with the rod, is to make a protocol. This can be a figure for each splitting of rods with accompanying symbolic formulations. The rods are operative prototypes and the prototypical equation is relational and symbolic, using concepts of Dörfler (2000). In the language of metaphors, the rods are based on the metaphor “numbers are measuring sticks” (Lakoff & Núñez, 2000, pp. 68).

It is more likely that the metaphors for numbers are activated when the student sees a prototype than the version with variables and three dots. If facing the more abstract identity first, the student mentally has to make a translation into prototypes in order to understand. First when a value is chosen for n , it is possible to make or draw a rod. These carriers open the possibility of well grounded ideas. My point is not to argue that the dot version should totally disappear. The latter makes totally clear that generality is involved. This may be overlooked when reading something intended to be a prototype. Formulations with variables also open the way for the forceful techniques of algebra.

THE ROLE OF THE TEACHER

The theories of Lakoff, Johnson and Núñez have strengthened the idea that mathematical ideas can be learnt. To achieve such learning both the teacher, the curriculum and the school system are very important. This is emphasized by several researchers, for instance Tall (1989). Learning takes place in individuals, as underlined by constructivism. Individual concept construction involves the formation of image schemata and metaphors. Both Presmeg (1992) and Dörfler (1991) point to the personal and idiosyncratic in this process. However, shared meanings are possible to a sufficiently high degree. Dörfler (1991) explains this by social communication and the supply of socially normed and standardized carriers (p. 22). According to Dörfler (2000) and Tall (1989), the teacher has to introduce the student to cognitive tools and learning environments. The students are not expected to invent such tools themselves. They need the guidance of a teacher who points to the crucial and typical characteristics for the intended image schemata, extends or delimits the attention etc. (Dörfler, 1991, p. 24). The concrete carrier by itself is no guarantee that the student constructs the intended image schemata (p. 27). My conclusion is that highly competent teachers are needed. They must have both general knowledge about the formation of image schemata and metaphors and content specific knowledge of the

ideas to be learnt by the students. Realistically, the teachers also need support. Examples are good software, material tools, written resources, collaboration and updating courses.

QUESTIONS FOR RESEARCH

The metaphorical analysis of mathematical concepts is far from complete. For instance, geometry is sparsely covered by Lakoff and Núñez. Also, they do not include metaphors developed for the purpose of learning. The concept of cognitive root from Tall, McGowen and DeMarois (2000) involves these kinds of metaphors. The metaphor of the function machine from that paper is of the type Lakoff and Núñez call an extraneous metaphor (2000, p. 53). However, the idea of local straightness, also introduced as a cognitive root, is embodied and natural (Tall, 1989). A research question is to develop a fruitful classification of metaphors for the support of finding cognitive roots. Inspired by Freudenthal (1983) I suggest the name phenomenological metaphor for the local straightness kind of metaphor. A comparison of the use and effects of cognitive roots based on different kinds of metaphors seems interesting. Another question worth pursuing is the connection between cognitive roots and the operational approach of protocols of actions. Finally, the relation between image schemata introduced by Lakoff and Johnson and the already existing literature about schemata should be clarified. This is elaborated to a certain extent in Presmeg (1992).

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TEACHING SPECIAL RELATIVITY

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Abstract. The recent convergence of ideas in neuroscience and cognitive linguistics has revealed that the unconscious mind is surprisingly powerful and extensive. This convergence has resulted in the emergence of cognitive frameworks that seem able to generate novel perspectives in the domain of conceptual change. We are investigating the application of conceptual blending, visual metaphor and embodied cognition in teaching the extension of Newtonian mechanics into special relativity. The pivotal role of visualisation and visual metaphor is discussed. This theoretical paper represents the first step towards a teaching experiment.

1. EMBODIED KNOWLEDGE OF MECHANICS

If we believe that mental processes can only exist by virtue of neural processes in the brain, then human ideas about such things as number, force, space and time must have their origins in bodily perceptions, and are not disembodied abstractions. They are mental constructions forged out of human experience over an evolutionary time scale. The idea of *embodied cognition* is supported by the convergence of cognitive science, neuroscience, cognitive linguistics and evolutionary anthropology. (Dehaene, 1997; Edelman, 2004; Fauconnier and Turner, 2002; Lakoff and Nunez, 2000).

Lakoff and Nunez (2000) argue that basic human mental processes are largely metaphorical, and that the conceptual system is metaphorically structured. They present linguistic evidence that reveals the metaphorical structure of mathematics. However Schiralli and Sinclair (2003) argue that metaphor itself is constructed from more basic mechanisms, which cannot be ignored, of identification, discrimination, generalisation and synthesis. These processes are heavily dependent on the ability to categorise and to make analogy. Metaphorical extension provides the mind with the ability to construct complex ideas on the embodied foundation, as a layered network of metaphor built on metaphor. As this structure develops, concepts become progressively more disembodied and less intuitive. But certain basic concepts are difficult to reduce further because they are semantic embodied knowledge, laid down by genetics and evolution within the brains of our ancestors and developed by experience of the external world. When this idea is applied to understanding mechanics, neural motor control circuits, muscle contractions and limb movements *provide the basis of an embodied force and motion schema.*

Motor activities are processed *unconsciously* by dedicated brain modules, and conscious thought is supported by a large raft of unconscious processing, Edelman, (2004). This basic mechanism provides the semantic structure for our understanding of dynamic and static forces. Dynamic force is associated with body movements, such as lifting or of throwing a rock, which are produced by unbalanced muscle contractions. The idea of static force is associated with opposing balanced muscle contractions, made in response to decisions to keep the body rigid and still, for instance when stalking prey. Unconscious motor routines are a basic component of our intuitive force and motion schemas which permit an individual to participate in exacting activities, such as ice dancing or taking a running catch that demand an extensive intuitive knowledge of mechanics. Real knowledge of mechanics must exist in the brains of all individuals to enable the execution of everyday activities. Embodied force and motion schemas are *metaphorically extended* to structure understanding of the force and motion schema perceived in the external world. This mechanism also extends knowledge of real world forces and motions into the abstract force and motion schemas of formal mechanics. From the point of view of embodied cognition, the vector force arrow is representing an unconscious push of the hand because we have *no other way* of subjectively knowing the nature of force. The arrow is a visual representation of the force where an unseen hand is pushing the arrow.

Spatial and temporal awareness are also functionally embodied. The central nervous system is characterised by intrinsic *spatial awareness*. The body's sensory system transmits signals resulting from skin surface stimulation along parallel links to the primary somatosensory cortex, where the skin surface activation is represented *topographically*. The body is intrinsically aware of the spatial organisation of its surface even though the sensation itself is generated in the brain. The visual system is characterised by a precise spatial awareness. A function of the visual system is to provide a prompt mental representation of spatial aspects of the objects situated in the visual field. The visual signals progress along the optic nerve to the visual cortex, while maintaining the spatial organisation of the retina. The visual cortex has numerous layers of cells (V1, V2, V3 and many others) through which *form, colour, orientation and motion* are processed in separate areas, Zeki (1991). The retina and the visual cortex generate precise responses to orientations; for instance, column cells in V1 respond strongly to orientation and direction, Bonheffer and Grinvald (1991). These *innate* neural structures enable the visual system to extract spatial information from the visual stream and create awareness of the structure of external space.

The temporal organisation in the brain makes extensive use of synaptic delays between neurons. Pulvermuller (2002) describes *synfire chain* mechanisms which are able to generate the temporal sequencing necessary for speech, music and motor actions. The mind has a strong sense of temporal order. We are able to imagine a sequence running in mentally generated time which matches accurately the timing of external sequences of real events. The mind also has subjective time awareness. Linguistic studies have revealed that we consistently think of time as a flow of events from the future to the past, passing us at now, Lakoff and Johnson, (1980). Anticipated future events are coming towards us and the experience of waiting for the arrival generates awareness of time.

Innate schemas of the real world do not allow space and time discontinuity and demand invariance of the size and shape of solid objects. Even very young children recognise the violations of these schemas, which are the basis of cartoon humour. We believe that an object we have placed in another room will still be there in the same place when we return to the room; if the object has vanished, we believe that there must be a cause for its removal. However the *means of construction* of this knowledge, innate and developed during early life, is hidden from consciousness and is beyond introspection. It is upon this foundation that the constructions of intuitive mechanics are built.

2. FUNCTIONAL WEBS AND THE CORRELATION LEARNING PRINCIPLE.

All neural connections are synaptic, and the strength of connection is variable. Dormant synaptic connections between neurons in the cortex are normally weak and neural signals are attenuated as they pass through. However, connections become much stronger if they are used repeatedly. This strengthening mechanism was proposed by Hebb (1949, p.70) and quoted by Pulvermuller (2002, p.19). When pre and postsynaptic signals activate a synapse simultaneously the synapse strengthens and the mutual influence of the two connected neurons becomes stronger. They therefore become *associated*. The mechanism allows the formation of durable long distance reciprocal links between groups of sensory and motor neurons. Each neuron is massively connected and may send information to, and receive information from thousands of others, enabling the extensive mixing of information. Pulvermuller (2002, p.21) refers to this mechanism as the *correlation learning principle*. The correlation learning principle implies that frequently co-occurring patterns of activity can be interconnected. This mechanism allows the creation of durable networks, termed functional webs by Pulvermuller, which establish associations between frequently

coactivated areas in the cortex. It is a mechanism that enables the association of two concepts simply by the action of their repeated and simultaneous activation. *The correlation learning principle is therefore the basis of a critical cognitive process.*

Many thousands of webs exist in the cortex. Although associated areas of the web are permanently connected by strengthened synapses, these networks are normally inactive. A strongly connected neural web can be stimulated into activity by external input, even if only a fraction of the neurons in the web are stimulated. The activation spreads to the rest of the web, leading to full activation. This process is known as *ignition*, and is described by Pulvermuller (2002, p.29), and has been simulated in neural network models by Palm (1981,1982). If the web is a memory representation of an object and each neuron is representing some attribute of the object, then ignition represents the activation of all attributes of the object, triggered by input to at least one. Web ignition can be stimulated either by external perceptual input or by internal input from other cortical neurons, outside the web. This activation of the web is the basis of short-term memory, whereas long-term memory merely requires the web to be in a passive state.

An object is represented in the brain by a web of interconnected sensory and motor areas, which binds together the object's stored attributes. In addition, webs in the specialised language areas represent words. An object web, and the word web representing the object, become highly associated by repeated coactivation. Perception of the object will automatically evoke the word and the word will automatically evoke a mental representation of the object. The extended functional web provides the essential link between the word and the attributes of the object, which allows the word to have *meaning* when the web is activated.

Metabolic imaging studies have revealed that although the specialised language areas are always active when words are processed, there are also contributions from other cortical areas. Therefore *the type of entity which a word represents should be reflected in the topography of the functional web activated by the word*, Pulvermuller (2002, p.56). Investigations using metabolic brain imaging techniques have revealed that animal names, tool names and action verbs produce the expected brain responses both in core language areas and in motor and sensory areas. The specific meanings of words are reflected by variations in the topography of cortical responses. Animal names are represented by activation in core language areas and visual areas because many animals have mainly visual representation in the brain. Tool names are represented by activation of core language areas and pre-motor areas. Thus it appears that spatial and action concepts are strongly

associated with premotor preparations for real actions, even though the actions are not necessarily realised.

Action words and concrete object words are learned by association with motor action programmes and representations of experienced objects, which are *grounded representations*, that is they are representations in the primary cortical areas. These representations provide a cache of embodied *semantic referents*. The process of searching for the meaning of a word entails the activation of the functional web containing the basic mapping of the word and the associated semantic referent. *The implication of this is that a word only has meaning when it is part of a functional web which is able to connect the word to primary sensory and motor areas.* It seems that the meanings of words which are learned in the absence of grounded referents can only be understood if they are grounded indirectly by association with grounded functional webs. This principle must apply to the vocabulary and ideas we expect students to acquire and use in physics and mathematics lessons.

Fauconnier and Turner (2002) have identified the powerfully creative role of conceptual blending. The essence of *double scope blending* is that two conceptual structures may be brought together so that they are within conscious attention simultaneously. The emergent structure has characteristics of both of the source structures but it also has novel features that were not evident before blending occurred. Double scope blending appears to be the essential mechanism that permits mental acts of imagination and creativity. It allows previously disassociated entities to be brought together, with the potential that the new blend might provide resolution for some existing enquiry or need. All human creativity thrives on the discovery of these blends, and physics and mathematics abound with blended structures. For instance, the intuitive concepts of *force* and *area* are blended, creating the non-intuitive concept *pressure*, which has new characteristics. Inertia, momentum and moment of a force provide further examples.

3. TEACHING SPECIAL RELATIVITY.

Special relativity is available as an option as part of the Advanced level G.C.E. physics syllabus of four examination boards but the take-up is generally quite low; for example the proportion for the southern area Advanced level physics students is only 30%. Despite the well-established technological position of relativity, students still regard it with scepticism. Relativity is regarded as ‘add-on’ rather than part of mainstream mechanics. Students and teachers informally report that:

- Relativity is poorly understood by teachers.

- Teachers have insufficient time to sort out their ideas and poor teaching methods are used.
- There is a necessity to demonstrate cognitive need for relativity before introducing the theory.

These difficulties are compounded by the current shortage of physics teachers. Another developing problem is a consequence of the decline in algebra skills of school pupils. (Royal Society, 1995) An indication of this decline is provided by the content of textbooks from the 1960s such as Marder (1968) which has an introduction to relativity for the 16-19 age group. This treatment would now be considered quite unsuitable by many teachers because of the heavy reliance upon the algebraic derivation of results. An exemplary modern text, described as a 'tour de force' by the School Science Review, is 'Physics' (Dobson, Grace, Lovett, 2002). In chapter 23 the authors make use of a diagram-based thought experiment to derive the time dilation formula but unfortunately simply quote other important results, which therefore appear as unsubstantiated conjecture. The concepts of special relativity deserve to be developed more robustly so that the apparent distortions of nature are seen as inevitable. The examination syllabuses require a discussion of time dilation, length contraction and mass increase. Usually only about five hours of teaching time are available to do this. We suggest that these results should be derived using procedures, which where possible, *allow the students to construct the reasoning for themselves, using a visual approach*. According to Mathewson (1999), visual-spatial thinking is an aspect of science overlooked by educators. Although vision and imagery are fundamental cognitive processes using specialised pathways in the brain, we cannot take the visual spatial aspects of cognition for granted. Sfard, (1997) and Presmeg, (1997a) have argued that there is a need to nurture visual representation and metaphor in school mathematics curricula. We may create a diagram in order to describe spatial organisation. The diagram then provides the focus for constructive elaboration and it enables the elements of the situation to be considered holistically. The representation allows further mental abstraction to take place, which might not have otherwise occurred.

Case study investigations of students' learning will be carried out which will address the following questions.

- Is the visual-spatial teaching producing the anticipated learning outcome?
- Is there evidence of retention of knowledge?
- Do the students have the ability to extend the ideas?
- Is their new knowledge meaningful or merely remembered?

- What are the students' affective reactions; are they comfortable with what they have learned?

4. SUBJECT CONTENT

The fact that light does not propagate instantaneously across space was demonstrated by the well-known experiment of Romer in 1676, using observations of the inner satellite of Jupiter, Jenkins and White (1957, p.382). The investigation of stellar aberrations by Bradley in 1728, showed that when the telescope which is being used to observe starlight from zenith stars is moving perpendicularly to the light with velocity v , the telescope must be rotated through a small angle θ , Jenkins and White (1957, p.384). The rotation is necessary to accommodate a rotation of the light path between the star and the telescope. Bradley's stellar aberration experiment is highly significant because it is a direct observation of dynamic light path rotation.

In 1905 Einstein *postulated* that the speed of light is invariant for all observers. It is important to mention that the idea of the invariance of the speed of light is not *empirically* based. We consider the profound modifications to Newtonian absolute time that this proposed restriction generates. Light rays cannot be observed sideways, and the time intervals for light trajectories may be extremely small; light paths must be inferred and visualised concretely by means of a line diagram. We first visualise the path of a beam of light as it travels from source S to mirror M then back to S (fig.1a). For an observer in this static frame who is stationary with the apparatus, the light takes time $2t$ and travels distance $2ct$.

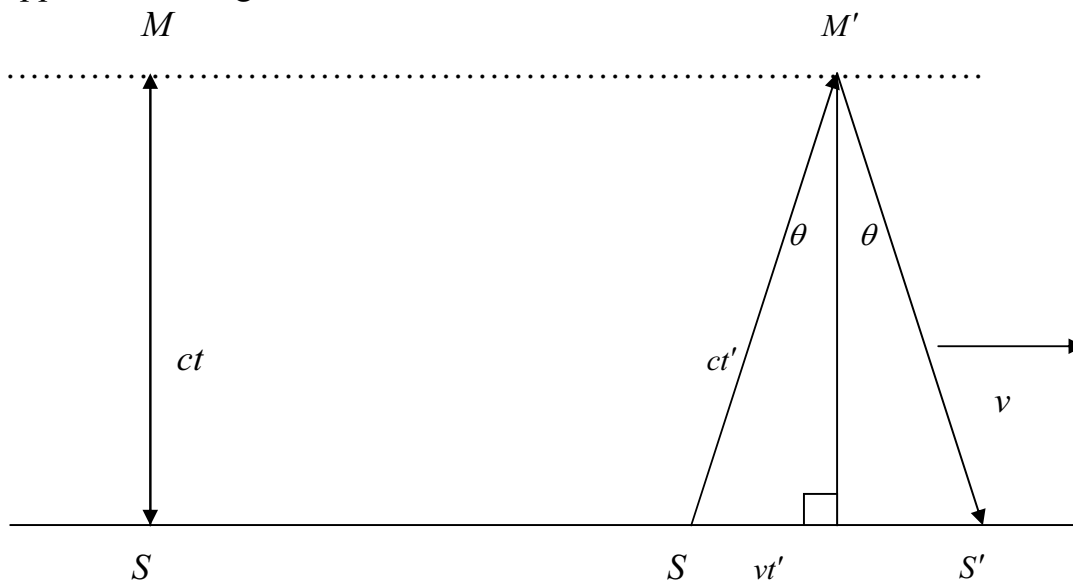


fig.1a. Light path in the static frame

fig.1b Light path in the moving frame for an external observer

We now consider the effect on the path of the beam of light of uniform transverse motion v . According to the observer moving with the apparatus, the light path is still as in fig.1a. but according to a stationary external observer watching the apparatus pass by, the light path is as in fig.1b. The resulting light path in the moving frame is a blend of the light path in the static frame and the uniform motion. This blend is grounded by perceptual experience, such as the appearance of wind driven rain. For the external static observer, the path travelled by the light in the moving frame is now rotated through angle θ and the path length has *increased from SMS to SM'S'*. What is the time now taken by the light to travel the increased distance from S to M' ? Were we to use the ideas of Newtonian mechanics the speed of the light must increase, keeping the time for the static and moving light paths the same for both observers. But when adopting Einstein's postulate, the simple addition of the vector velocities v and c is ruled out. Instead we must blend the moving light path with the idea of the invariance of the speed of light. The adoption of the Einsteinian option immediately leads to the idea that the time interval in the moving frame is not the same as time interval in the static frame. The effect of the motion on the time intervals can be calculated by evaluating ct'/ct . From fig.1b, in the right angled triangle $\cos\theta = t/t'$, $\sin\theta = v/c$ and so, using $\sin^2\theta + \cos^2\theta = 1$ we obtain the result

$$\frac{t'}{t} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma. \text{ The factor } \gamma \text{ is ubiquitous in}$$

special relativity. For real values of γ , $\gamma \geq 1$ and $t' \geq t$, and this effect is known as *time dilation*. Time intervals in the moving frame according to an external static observer are always increased and depend on the speed of the moving frame relative to the static observer. Thus in special relativity, time intervals are relative, and are not observer invariant. An important point is that all timings are *proper*, that is times are measured by the static external observer. Time intervals in the moving frame are longer by the factor γ , and so the rate of flow of time appears to be slower in the moving frame. This result is highly counter-intuitive; it violates intuitive knowledge of the familiar environment.

The range of values of γ as the relative speed is varied can be investigated by students, and the range of possible time dilations inferred. Students can investigate relative time for the cases when $v=0$; $v < c$; $v=c$; $v > c$? At what speed do relativistic effects ($t' \neq t$) start to become significant? Do we abandon Newton's laws? What happens to time dilation

as the speed of light is approached? What would be the consequences of exceeding the speed of light? The calculations are much less laborious if they are done using the trigonometrical expressions $\sin \theta = v/c$ and $1/\cos \theta = \gamma$ rather than the derived square root expression. Time dilation can also be explored quantitatively using scale diagrams in which the angle θ is varied. The idea that time flows at different rates in different frames of reference is highly counterintuitive but the reason for this conclusion emerges clearly from the blending of the light path diagram and Einstein's postulate.

Abundant empirical evidence for the reality of time dilation is available from the measured lifetimes of energetic elementary particles, both in the environment and in particle accelerators, Ohanian (1985, p.430). Such evidence supports Einstein's speed of light postulate, and the idea that Newtonian mechanics is incomplete, but not incorrect.

Summary.

The existence of functional webs provides an understanding of *meaning*. Abstract ideas can only have meaning if they are associated by functional webs with grounded concepts. Awareness of the process of conceptual blending is important; the emerging concept has characteristics which differ from those of the initial concepts. Violations of intuitive knowledge of mechanics provide understanding of the basis of counter-intuition in abstract mechanics. A cognitively efficient visualisation can be used to represent a situation with unobservable or abstract elements which allows an exploration of the situation. Teachers and students may well benefit from more precise knowledge of cognitive mechanisms.

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METAPHORS AND COGNITIVE MODES IN THE TEACHING- LEARNING OF MATHEMATICS

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The didactic role of metaphors and cognitive modes as well as their interplay is discussed, based on examples. Transition from one cognitive mode to another is illustrated, in case studies with students and in-service teachers. Its relevance to the learning process is appraised.

INTRODUCTION

In this paper we intend to continue the research undertaken in Soto-Andrade (2006), presenting further examples of didactical uses of metaphors and cognitive modes, as well as exploring the interplay between them, as they emerge in the didactical praxis.

This exploration requires a first-person approach, in the sense of Varela & Shear (1999). Indeed, metaphors having a deeper cognitive thrust are usually those that entail a switch in the cognitive mode of the subject perceiving them. For instance, when you approach solving linear equations with the help of the “scales metaphor”, you switch from a verbal cognitive mode to a non verbal one: Instead of checking an equality by an arithmetic or algebraic calculation, you put and take out weights on both pans of a scale, trying to preserve balance.

After setting up our tentative theoretical framework, we set down our main research hypotheses, related to our teaching experiments, and proceed to report on some specific examples of metaphors and cognitive modes in action, that give preliminary experimental evidence to support our hypotheses and suggest further research along these lines, that we discuss in the final section.

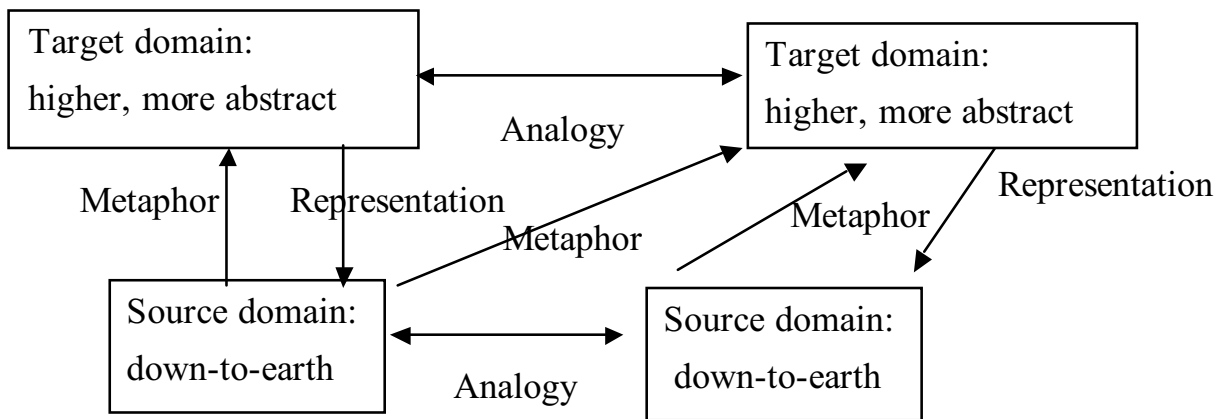
THEORETICAL FRAMEWORK

Nature and Role of Metaphors

It has been progressively recognized during the last decade (Araya, 2000; Bills, 2003; Edward, 2005; English, 1997; Ferrara, 2003; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Parzysz et al., 2003; Pouilloux, 2004; Presmeg, 1997; Seitz, 2001; Sfard, 1994, 1997, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way: “metaphors we calculate by” (Bills, 2003). We meet conceptual metaphors (Lakoff & Núñez, 2002), that appear as mappings from a “source domain” into a “target domain”, carrying the inferential structure of the first domain into the one of the second, and enable us to

understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent.

The term “metaphor” is often nowadays taken in a loose sense, as a synonym of “representation”, “analogy”, “model”, “image”, etc. (Parzysz et al., 2003). We intend nevertheless to be more precise: the following diagram may be helpful to clarify our viewpoint on the difference among metaphors, representations and analogies.



So, as indicated, in operational terms, conceptual metaphors “go up”, representations “go down” and analogies, “go horizontally” both ways. Notice that we take analogy in a rather narrow sense, kin to a “simile” (that draws an explicit comparison between two different things), symmetric in nature, and not as an “umbrella” concept embracing metaphors, representations, similes, etc. So our viewpoint is closer to Sfard’s (1997) than to Presmeg’s (1997). We may have, moreover, metaphors going up from different source domains to the same target domain and also from the same source domain to different target domains.

Notice however that this scheme doesn’t impair the subjective aspects of the difference between metaphor and representation. For instance, if probability is a new concept or a concept under construction for us, then “probabilities are masses or weights” is clearly a metaphor for us, that helps us to grasp the concept of probability, or better, to build it. On the other hand, if we are to some extent already familiar, albeit not quite comfortable, with probabilities, we may realize that probabilities may be represented by masses, we feel more at home with.

Others might want to say that there is an analogy between probabilities and masses, because they see analogy as a symmetrical relationship and they see probabilities and masses on the same footing.

Anyway, be metaphor, representation or analogy, we gather that to solve probabilistic problems we may just solve mass or weight problems, where we can take advantage of our physical intuition, in static or dynamic settings.

Example: Metaphors for multiplication

Both addition and multiplication of numbers are commutative. It is however an interesting metaphorical insight of Lakoff & Núñez (2000) that while addition is a priori commutative, multiplication is only a posteriori commutative, we might say. For instance, addition of vectors, complex numbers, quaternions or matrices is as commutative as the addition of real numbers. However, multiplication of quaternions or matrices is not in general commutative. So, in a “good” metaphor for addition commutativity should be built-in, but not so in a good one for multiplication.

This may be compared with Soto-Andrade (2006), where the “product is area” metaphor and the “multiplication is concatenated branching” metaphor are presented, as two ways of “seeing” that $2 \times 3 = 3 \times 2$, as illustrated below:



In the first one, commutativity of multiplication is perceived as invariance of area under rotation in one fourth of a turn. So you “see” that $2 \times 3 = 3 \times 2$, without even knowing that it is 6, like the Amazonian Indians in Dehaene (2004). In the second one, commutativity of multiplication is less obvious: it is perceived as the fact that the order in which we concatenate branchings is irrelevant for the final harvest. If one has played around a lot with tree diagrams, this metaphor may become a “metbefore” in the sense of Tall (2005). But otherwise one would rather count... This suggests indeed that multiplication is not a priori commutative.

Cognitive styles and modes

The concept of cognitive styles emerged from work by Neisser (1967), Luria (1973) and de La Garanderie (1989) and was further developed by Flessas (1997) and Flessas & Lussier (2005), who pointed out to their impact on the teaching-learning process.

A cognitive style is defined nowadays as one’s preferred way to think, perceive and recall, in short, to cognize. It reveals itself, for instance, in problem solving.

The term “cognitive mode” is often used as a synonym to “cognitive style”. It suggests however a way of cognizing that is usually more transient and not as stable, or even rigid, as a cognitive style is. Since one of our theses is that the ability to switch from one way of cognizing to another is trainable, we will rather say “cognitive mode” instead of “cognitive style” in this paper, so that Flessas and Lussier’s “styles cognitifs” will become “cognitive modes” from now on.

To generate what they call the 4 basic cognitive modes, Flessas and Lussier (2005) combine 2 dichotomies: verbal – non verbal and sequential – non sequential (or simultaneous), closely related to the left – right hemisphere and frontal – parietal dichotomies in the brain (Luria 1973) This affords the following table:

The 4 cognitive modes	VERBAL	NON - VERBAL
SEQUENTIAL	S-V	S-NV
NON - SEQUENTIAL	NS-V	NS-NV

Example: Solving a problem through different cognitive modes.

How can you check that you have the same number of fingers in your hands?

You can approach this problem with different cognitive modes.

You can count the fingers in your left hand first: one, two, three, four, five. Then, you do the same with the fingers in your right hand and you discover that you have the same number of fingers in both hands, indeed. This a typical *verbal and sequential* cognitive mode approach.

But you can also, in a single gesture, put into a one to one correspondence the fingers in both your hands, in a natural way. This is a typical *non verbal and non sequential* approach. You don't name the numbers, you don't count, you don't write any formulae. Moreover your checking is simultaneous, non sequential, because you can make your homologous fingers touch in just one simultaneous gesture.

If you make each finger of your right hand touch one finger of your left hand, one by one, you would be using a *non-verbal, sequential* cognitive mode.

Flessas and Lussier emphasize that effective teaching of a group of students, who may display a high degree of cognitive diversity, needs teachers supple enough to be able to tune easily to the different cognitive modes of the students. This necessary competence has a neurological correlate that can be imaged and monitored in contemporary neuroscience (Dehaene, 1997, 2004; Varela & Shear, 1999).

In what follows we adhere mainly to the framework laid by Lakoff & Núñez (2000) for metaphors and Flessas & Lussier (2005) for cognitive modes.

PROBLEMATICS

We claim that most in-service teachers are not familiar with either metaphors or cognitive modes, in relation with their teaching of mathematics. More precisely:

Most teachers are frozen in just one cognitive mode, unaware of it to begin with, and so unable to switch to another one. They are also unaware that their teaching is shaped by unconscious and misleading metaphors, like the container-filling metaphor or the gastronomic metaphor (called “*métaphore alimentaire*” in Soto-Andrade (2005)).

Moreover, their metaphor quiver is poorly furnished: they would rarely have more than one metaphor for each mathematical concept or process and they have trouble creating “unlocking metaphors” for their students.

Students, on the other hand, are not stimulated to work in more than one cognitive mode and they have very often the feeling that tackling a problem in a different cognitive mode that the one it came wrapped up, is definitely bad manners.

RESEARCH HYPOTHESES

Our main research hypothesis is that metaphors and cognitive modes are key ingredients in a meaningful teaching-learning process. Moreover the deepest impact on this process is usually attained by metaphors that involve a switch in the previous cognitive mode of the subject.

We also claim that competences regarding multi-mode cognition and use and creation of metaphors and representations are trainable and that measurable progress can be achieved in a one semester course or even in a one week workshop. This, in spite of the fact that most teachers report that their initial training included no metaphors and privileged just one cognitive mode: the usually dominant verbal-sequential one.

We conjecture that on the average primary school teachers will be more successful in learning to switch cognitive modes and evoking and using metaphors than secondary school teachers.

Regarding students, our working hypothesis is that they would significantly improve their learning if they were able to approach problems with more than one cognitive mode and to draw from a suitable spectrum of metaphors.

We present below some examples, tested in teaching mathematics courses to various audiences of students, to illustrate the didactical use of different cognitive modes for approaching mathematical objects and their interplay with the use of metaphors.

RESEARCH BACKGROUND AND METHODOLOGY

The background for our experimental research consisted in several courses, to wit:

- **Mathematics 0:** A one semester, general mathematics course, given to first year students of the Bachelor in Humanities and Social Sciences Program at the University of Chile. Its formal aim is to teach the students “all the mathematics” they will need during their university studies, besides statistics. Its real aim is to introduce them to the mathematical way of thinking and to the cognitive attitudes of mathematics. Classes have 35 students, lessons are 3 hrs a week. Experiments are carried out during the lessons (90 minutes each) and during exams (2 hours each, 4 in a semester). Lessons are interactive, questions and activities are suggested and students propose ways to tackle them. Students engage often in “horizontal” discussions but not so much in group work. Gleaned knowledge is periodically “harvested” and recorded in more formal language.

- **Random walks in “Metaphorland” (Paseos al azar en el país de las metáforas):** A one semester optional mathematics course, addressed to all students of the University of Chile. The aim of the course is to introduce them to the power of

metaphorical thinking in mathematics, while performing a “random walk” through several key topics, like randomness, symmetry, infinity and the systemic approach. The class had 70 students in 2006. Lessons consist mainly of group work, in small groups of 4 to 5 students. Activities and problematic situations are proposed, to be tackled by the students, each group working on its own first, then putting together their findings.

- **Numbers:** One yearly module (220 hrs approx.), for 2 classes of 30 primary school teachers, in the post-graduate program of the University of Chile, for in-service teachers who didn't major in mathematics in their initial formation. The aim of this module is to review the mathematics as well as the didactics of numbers, specially fractions, ratios, decimal and binary description of numbers. The teachers usually work in interactive sessions, forming small groups of 3 to 4 teachers.

The methodology consisted in observing the students and teachers, as they carried out various activities, as in the examples described below, that were proposed during lessons, group work sessions and as a part of exams and diagnostics. Records of this observation comprised the written and drawn production of the students and some transcriptions.

EXPERIMENTAL ACTIVITIES AND PRELIMINARY RESULTS:

Example 1: Who has more marbles?

John and Mary have a bag of marbles each, all of the same size. How can they decide who has more marbles?

They could take the marbles out of each bag, one by one, count them and compare the resulting numbers. This is a verbal-sequential approach, the most frequent one.

Working with two separate classes of 30 in-service primary teachers, organized in small groups (3 to 4 each), we invited them to figure out other approaches, which would involve non-verbal or non-sequential modes. They were also asked how they would work this problem with their students. In a few minutes, they came out with:

- John pulls out his marbles and Mary hers, one by one, and without counting them, they put them side by side, in pairs, sequentially, until one of them, or both simultaneously, runs out of marbles. They recognized this as non verbal – sequential.
- To weigh the bags in one's hands to assess which is heavier. If it is hard to tell, weigh then in a scale, one bag in each pane (non verbal – non sequential approach).
- After some 15 min. discussion on the non sequential – verbal approach, 3 teachers got the idea of weighing the bags simultaneously in 2 digital scales and compare the readings.

However, roughly 80% of the teachers reported that they had never before tried to employ more than one cognitive mode to solve this type of problem.

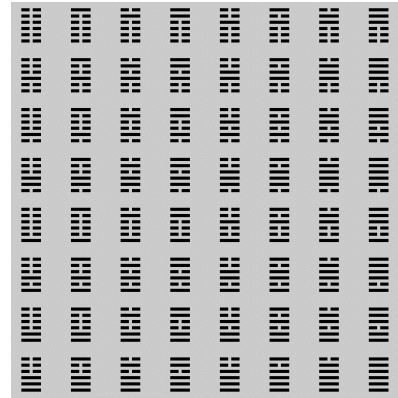
Example 2: The number sequence, otherwise...

Is it possible to represent the numerical sequence 0, 1, 2, 3, up to 63, let us say, in a non verbal and non sequential way?

We presented this challenge to the courses described above, suggesting to try first representing the sequence in non verbal - sequential way. As a preliminary, we proposed to the students in the classroom to try to get the binary description of their number without counting themselves first, as in Soto-Andrade (2006).

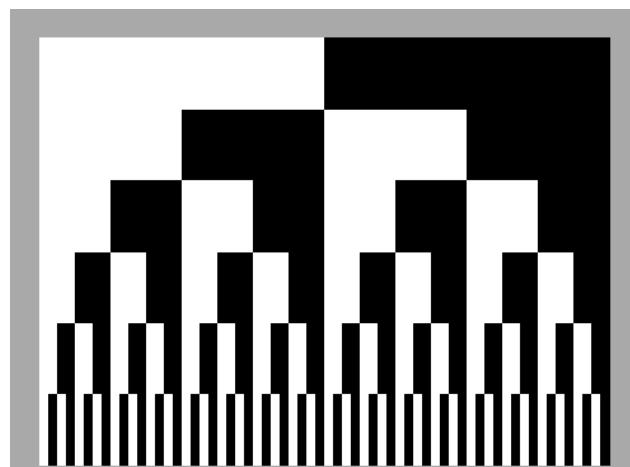
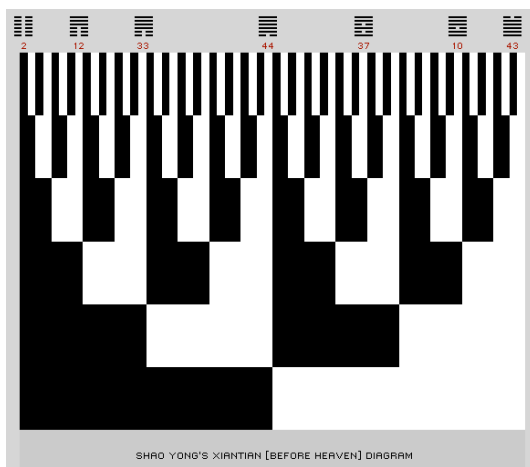
To do this, they just stand up, trying to match up in pairs, checking whether there was one “odd man out”. Then the pairs did the same, and so on. When the pairing game was over, we asked: Is there an unmatched person? An unmatched pair? An unmatched quadruplet? and so on. They answered: YES–NO–NO–YES–NO–YES.

When prompted to codify this in a non verbal way, they eventually rediscovered the I Ching (Yi Jing) codification: a broken line for NO, a continuous line for YES, or something equivalent. Reading hexagrams from top to bottom, they got the one in the 2nd column, 6th row, in the figure to the right (i. e. number 41). Notice that this square arrangement due to Chinese philosopher and mathematician Shao Yong (1011-1077), displays the binary sequence of numbers 0 to 63, in their natural order. Students in all 3 courses successfully completed this activity and reported later having understood for the first time the binary description of a number or quantity. After this non-verbal approach to the binary description of numbers, we suggested the harder challenge:



Approach the binary hexagram sequence in a non verbal and simultaneous way, by encapsulating it in a single image that can be reconstructed from just a glimpse of it.

In a first class of 30 primary school teachers, after 30 min. work in small groups, 5 of them came up with diagrams equivalent to famous Shao Yong’s Xiantian (“Before Heaven”) diagram or its inverted form (Marshall, 2006), illustrated below.

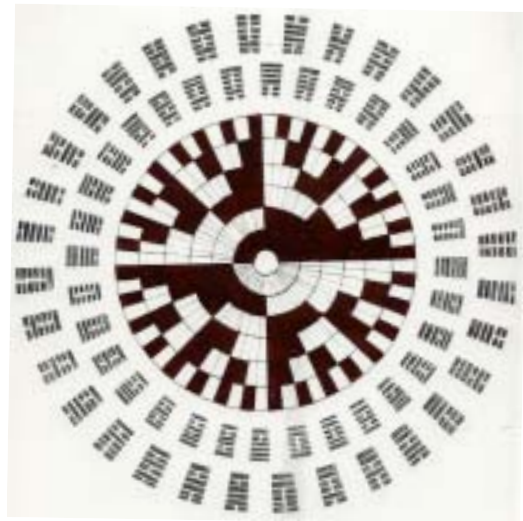


Working Group 1

Notice the underlying binary tree! In a second group of 30 teachers, 6 rediscovered Xiantian and, most remarkably, one of them, Ofelia, draw all by herself a circular version of the Xiantian diagram, that is a rotation of the classical one, unrolling counterclockwise. See below the classical version (left) and Ofelia's version (right).

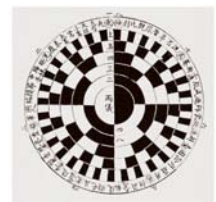
Yijing Dao - A 'compass rose' star in the circular Xiantian diagram

<http://www.biroco.com/yijing/xiantianstar.htm>



They also noticed, as pointed out by S. J. Marshall (2006), that this circular diagram reveals a compass rose when looked at from a distance. When interviewed, they recurrently reported:

- I had just learned by heart a recipe to transform usual numbers to binary form, but now (after playing the pairing-off game) I understand it for the first time!
- I would have never thought of this way of approaching binary numbers!
- This is really new to me! I have some trouble in getting used to it, because I am too structured and used to seeing things always from the same viewpoint.



Approx. 5% of the students in the Random Walks course, when exposed to Xiantian (without the 7 examples above the square), quickly realized how to recover the whole sequence from this image. The circular Xiantian was tested in Maths. 0 exams, as an optional question: out of 40 students, 50% did choose this question and 78% of them reconstructed the circular diagram after two 2 second glimpses of it. Out of the latter, 40% explained correctly how to recover the binary sequence from circular Xiantian.

DISCUSSION

We have shown, through several activities carried out in the classroom, how classical mathematical objects and problematic situations may be unexpectedly approached with cognitive modes different from the usually dominant verbal-sequential one.

We have seen how to facilitate the activation of these less usual cognitive modes, even for in-service teachers who never had this sort of experience before. After some prompting, a high percentage of students and teachers were able to switch from their dominant verbal-sequential cognitive mode to a non verbal or non sequential one. In this way, some of them rediscovered representations of - or metaphors for - familiar mathematical objects, developed in other cultures (like the ancient Chinese, for instance), that favoured more than ours non verbal and non sequential cognitive modes: “One image is worth 1000 words...” they said.

According to their reports, taking advantage of more than one cognitive mode fostered their understanding of important mathematical objects and processes, like the binary description of numbers.

Our observations show that the ability to approach the same object through various cognitive modes and transiting from one cognitive mode to other, is trainable, in students as well as in teachers. Experimentation suggests that this is facilitated by group work. First person reports by students and teachers bear witness of the impact and meaningfulness that this sort of cognitive experience had for them.

The experiences carried out reveal that often the activation of a different cognitive mode, when approaching a mathematical object, entails the emergence of a metaphor, or a representation, depending on the previous background of the subject. It remains to be checked that, vice versa, students and teachers looking for a useful metaphor to get hold of or to build a new concept, will learn to switch to another cognitive mode.

It would be interesting to test and measure the depth of learning that students may achieve when taught with the help of a broad spectrum of metaphors and various cognitive modes and to undertake the design of unblocking metaphors.

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