WORKING GROUP 13. Modelling and applications 2034

REPORT FROM THE WORKING GROUP MODELLING AND APPLICATIONS - DIFFERENTIATING PERSPECTIVES AND DELINEATING COMMONALTIES

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1. Introduction

The modelling and applications working group at CERME5 was again characterised by a heterogeneity of approaches to modelling research. There was a sense of continuity in the work of the group from CERME4 in Spain due to the presence of a core group of researchers representing the different approaches. Concerning growing clarity and common understanding of the different approaches progress has been achieved in the working group from the meetings at CERME4 to CERME5. One of the leading goals of the organisers was to ensure both a continuity for the present discussion as well as accumulate current perspectives coherently into the existing literature for use by modelling researchers.

The participants of the group represented a big variety of countries: Participants from 8 European countries (Cyprus, Denmark, France, Germany, Great Britain, Netherlands, Portugal, Spain) and 4 non-European countries (Brazil, Israel, Lebanon, USA) attended the working group.
In the working group 18 papers and 1 poster were presented. The papers were classified into 3 groups: papers

- with theoretical reflections,
- promoting research towards action,
- presenting empirical research.

Most papers belonged to the last group, which was structured along the age level of the cohort addressed in the studies, i.e. lower secondary level, upper secondary level, university level, in-service-teaching level.

2. Revisiting the classification of approaches

The discussion was structured using a classification of the variety of approaches developed by Kaiser & Sriraman (2006) on the basis of the discussion at CERME4. This classification was based on the goals of modelling and distinguished various perspectives within the discussion according to the central educational aims in connection with modelling. It describes briefly the backgrounds these perspectives are based on as well as their connection to the initial perspectives. Two issues of the Zentralblatt für Didaktik der Mathematik (see Kaiser, Blomhoj, and Sriraman 2006 and Sriraman, Kaiser, and Blomhoj, 2006) were devoted to papers representing the different perspectives. We refrain from reproducing the whole classification table and restrict ourselves to the main categories:

Name of the perspective	**Central aims**
Realistic or applied modelling	Pragmatic-utilitarian goals, i.e.: solving real world problems, understanding of the real world, promotion of modelling competencies
Contextual modelling	Subject-related and psychological goals, i.e. solving word problems
Educational modelling; differentiated in a) didactical modelling and b) conceptual modelling	Pedagogical and subject-related goals: a) Structuring of learning processes and its promotion b) Concept introduction and development
Socio-critical modelling	
Epistemological or theoretical modelling	Theory-oriented goals, i.e. promotion of theory development

The following approach can be described as a kind of **meta-perspective:**

Cognitive modelling	Psychological goals: a) analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes b) promotion of mathematical thinking processes by using models as mental images or even physical pictures or by emphasising modelling as mental process such as abstraction or generalisation

Figure 1: Shortened version of the original classification of current perspectives on modelling (from Kaiser and Sriraman, 2006, p. 304)

Several researchers agreed that the classification as given was useful and could aid understanding of the interrelations between the very different and complex approaches adopted by researchers and practitioners. As one of the main points of criticisms the need to separate out didactical approaches and research perspectives was pinpointed. Didactical approaches are characterised by a normative orientation concerning the overall aims of applications and modelling in mathematics education in contrast to research perspectives which guide studies on special aspects concerning applications and modelling. This missing distinction leads to difficulties, because individual researchers and practitioners usually operate across several perspectives both concurrently and consecutively. It was concluded that classifications of approaches allow a systematisation of the debate, but it has to be stated clearly that such a classification is only a working instrument in order to facilitate the understanding of the debate, based on idealisations and simplifications.

In the extensive discussions during the group sessions, several fundamental issues were raised. This discussion addressed the following issues:

- The need to devise and develop common notions and terms;
- The need to revise the classification of Kaiser & Sriraman;
- The usefulness of including concrete examples to illustrate the differences between perspectives;
- Prospects for future collaborative work of the group.

In order to solve a few of the aspects described above one of the sub-groups developed a task for each perspective in order to get a better understanding of the classification table, which is shown below.

	Task	Feature
Realistic Modelling	Create a price structure for a taxi driver.	This is an open task. You have to create a model and therefore you need the whole modelling circle
Contextual Modelling	A taxi driver has a fixed price of €2.00 and the price/km is €0.15. The age of the taxi driver is 43 and his taxi is 7 years old. How much costs a drive of 6 km?	This is more a word problem.
Educational modelling	The same task as in contextual modelling	A teacher can use a modelling task to explore linear functions. The teacher uses the understanding of the context to develop mathematical concepts. The question is how a task should be placed in the curriculum
Socio critical modelling	How should a taxi driver be paid?	You can think about different price structures, but the intention is to think also about the social question. You can argue that a taxi-driver should be paid for every hour he is working.
Epistemological modelling	How much money did the taxi driver earn at the end of a day?	In this question many concepts are hidden. You have to think about the price structure…. how many customers? How much gasoline was

		needed? The original cost of the car? A deep analysis is necessary considering various perspectives.

Figure 2: Original classification of perspectives on modelling exemplified using the problem of the cost of a taxi ride

In the discussion of this attempt it became clear, that it was difficult for some researchers to see the differences between educational and contextual modelling. The description of the socio-critical approaches is not unique: One could be critical about the model, the assumptions, the validity of the model, but one could also be critical about how modelling is used in society. The epistemological perspective was very difficult to understand and to exemplify.

3. Proposal for a revised classification system

As already mentioned one of the criticisms was the need to separate the different intentions which are underlying studies or publications, i.e. papers or studies might either be characterised as comprehensive didactical approaches or as approaches connected to distinct research intentions. Didactical approaches are normative theoretical approaches characterised by overall norms education shall follow and aims to be supported by applications and modelling, formulated by those belonging to this perspective. They characterise the teaching approaches connected with applications and modelling, which are favoured by the various perspectives and are strongly influenced by the theoretical background to which the perspectives refer. In contrast approaches connected to distinct research intentions guide empirical or theoretical studies concerning applications and modelling. They are of course not independent from the didactical approach, but they might either focus more on cognitive aspects such as concept development, development of modelling competencies or the affective domain of learners such as engagement or motivation, confidence, self-efficacy, and beliefs. Keeping that difference in mind, it is obvious that individual researchers and practitioners might operate across several perspectives both concurrently and consecutively.

Comprehensive didactical perspectives or normative theoretical approaches:

Name of the approach	**Central aims**	**Background**	**Authors of paper presented at CERME5**
Realistic or applied modelling	Pragmatic-utilitarian goals, i.e.: solving real world problems, understanding of the	Anglo-Saxon pragmatism and applied	Burkhardt; Schwarz, Kaiser;

	real world, promotion of modelling competencies	mathematics	Romo Vasquez
Contextual modelling	Subject-related and psychological goals, i.e. solving word problems	Problem solving debate and psychological laboratory experiments	
Model eliciting approach	Psychological goals, i.e. apply model elicited through solving the original problem to a new problem	Problem solving debate	Mousoulides, Sriraman, Pittalis, Christou
Educational modelling	Pedagogical and subject-related goals: a) Structuring of learning processes and its promotion b) Concept introduction and development c) Promotion of motivation and improvement of attitudes towards mathematics d) Promotion of critical understanding of modelling processes and models developed	Didactical theories and learning theories	Andresen; Berman, Verner, Aroshas; Blomhoj, Hoff Kjeldsen; Canavarro; Maaß
Socio-critical and socio-cultural modelling	Promotion of critical understanding of modelling processes and models developed as overall goal connected with recognition of cultural dependency of modelling examples and modelling approaches developed	Socio-critical approaches in political sociology, ethno-mathematics	Barbosa
Epistemological modelling	Promotion of connections between modelling activities and mathematical activities, re-conceptualization of mathematics and reorganisation of school mathematics from a modelling point of view	Anthropological Theory of Didactics	Barquero, Bosch, Gascón; Ruiz, Bosch, Gascón

Approaches connected to distinct research intentions:

Name of the approach	**Central aims**	**Background**	**Authors of paper presented at CERME5**
Cognitive approaches	a) Analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes b) Promotion of mathematical thinking processes by using models as mental images or even physical pictures or by emphasising modelling as mental process such as abstraction or generalisation	Cognitive psychology	Borromeo Ferri; Jurdak, BouJaoude; Roorda, Vos, Goedhart; Vos, Roorda
Affective approaches	Promotion of positive attitudes towards mathematics and mathematics teaching Promotion of adequate self-perception such as self-efficacy Influence of special aspects such as authenticity of the real world context	Related psychological approaches	Vorhoelter; Wake, Pampaka
Pragmatic, teaching-oriented approaches	Evaluation of the effectiveness of teaching proposals or the possibility to realise special examples in school, analysis of teaching strategies, intervention measures by teachers	General pedagogical research	
Theoretical approaches	Development of meta-analysis of models and modelling approaches		Peled

Figure 3: Revised classification of current perspectives on modelling

4. Future directions

The above described framework for the description of the modelling debate provides a basis for mutual understanding of the protagonists from different perspectives. The descriptions developed above give insights into the origin of the different perspectives and its relations to the underlying background philosophy. It makes clear, that the various approaches promoting applications and modelling in school or university teaching come from very different theoretical perspectives spanning the debate from ethno-mathematics to problem solving. They are characterised by different views on important aspects of applications and modelling such as their views on goals and intentions of applications and modelling, which vary from promotion of a better understanding of the real world to the promotion of learning mathematical theory. Accordingly, their views on the role of the context are highly differentiated ranging from the call to authentic real world examples to more or less artificial, mathematically oriented examples. In addition, their perception of the modelling process is also highly different demanding a modelling cycle starting from real world problems and coming back to them or modelling processes which start from a real world problem, but lead to mathematical reflections and the development of new mathematical theory.

Bearing in mind the different educational, philosophical, and cultural background of the various perspectives on applications and modelling developed internationally this overview will not only allow the identification of differences between the various perspectives, but as well the identification of commonalities. This will hopefully promote a mutual understanding within the debate and foster long-term intensive research collaborations between researchers from different perspectives.

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UNDERSTANDINGS OF 'MODELLING'

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***Abstract**: This paper meets the common critique of the teaching of non-authentic modelling in school mathematics. In the paper, non-authentic modelling is related to a change of view on the intentions of modelling from knowledge about applications of mathematical models to modelling for concept formation. Non-authentic modelling is also linked with the potentials of exploration of ready-made models as a forerunner for more authentic modelling processes. The discussion includes analysis of an episode of students' work in the classroom, which serves to illustrate how concept formation may be linked to explorations of a non-authentic model.*

Introduction: understandings of modelling in this paper

The title of this paper refers to the fact that the term 'modelling' is used with different meanings depending on the theoretical framework, the context of practice etc. In the paper, a distinction is made between: 1) modelling at functional level, which means expressive modelling, aiming at problem solving and involving certain applications of mathematical concepts, methods etc. This interpretation is in accordance with (Blum 1991 p 10). Modelling at functional level, thereby, requests modelling competence in the meaning described by Niss (2002), and 2) modelling at the level of concept formation, following the ideas of Realistic Mathematics Education (RME) like it is described in Gravemeijer (2000): the main design heuristics in RME are the horizontal and vertical mathematising, illustrated in fig.1 showing Gravemeijer's four-level-model of the development of mathematical ideas. In the four-level-model, 'horizontal mathematising' happens by changes from situational to referential level, by the creation of what is labelled 'emergent models'. Symbolising is a main issue for these changes. The modelling for concept formation continues during 'vertical mathematising' that happens by changes from referential to general level

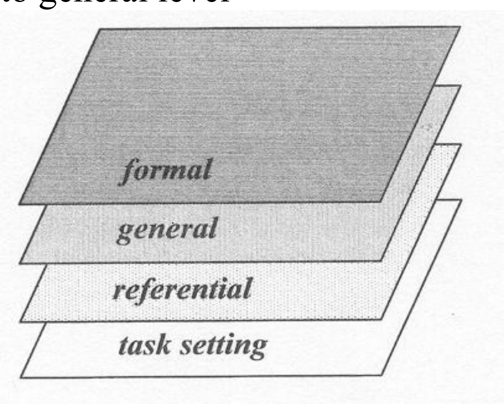

Fig 1. Levels of activity (Gravemeijer and Stephan 2002 p 159)

Background: modelling activities in school mathematics

During the last decades, there has been a growing interest for modelling activities in school mathematics. These activities are issues of (expressive) modelling at functional level, apparently regarding the larger part of curriculum, as mentioned for example by Blum (1991). The teaching of this modelling approach aims to let the students acquire the advantageous knowledge of authentic models' applications and their results. Its aim encompasses both technological knowledge on how to build and use models and democratic competence in relation to models, as mentioned in Blomhoej (1991 p 189). Following this point of view, students' expressive modelling processes take place when the student or a group of students start with a real world problem and build a mathematical model on their own, solve the problem mathematically and transfer the solution back to the real world situation. Blomhoej and Jensen illustrate the ideal process in Fig.2 showing the process divided into sub-processes. Important arguments, though, are brought forth by teachers against letting the students' do the full modelling process (Fig.2). The main arguments claim that the process is too complex and time-consuming. Further, it is hard for the teachers to control the open-ended process in order to ensure the desired result or learning outcome for all the students. These arguments exemplify what Blum calls C2: Counter-arguments from the learner's point of view and C3: Counter-arguments from the teacher's point of view (Blum 1991 p 17).

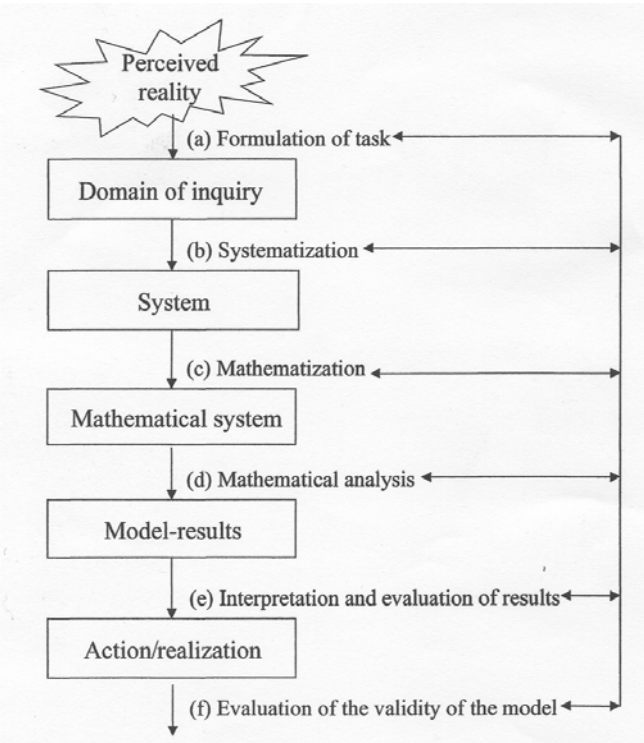

Fig.2. Blomhoej and Jensen. (2003) p 124-125

Change of view on the intentions of modelling

A change of view on the intentions of modelling, from knowledge about applications of mathematical models to modelling for concept formation, makes it possible for teachers (and researchers) to take these common teachers' experiences into account. The two understandings of modelling are naturally linked together in the sense that modelling at the functional level offers possibilities of modelling for concept formation. The potentials for concept formation, though, are not in general realised independently of the settings and the teacher's guidance.

Such a change of view on the intentions of modelling follows Blum's argument P4: psychological arguments for applications and modelling (Blum 1991 p17-18) and allows the teaching of sub-processes rather than full, expressive modelling processes. The great advantage of the change, however, is that it serves to realise the learning potentials for the students embedded in exploration of ready-made mathematical models. Later in this paper the idea of concept formation through exploration of a ready-made model is illustrated with an episode of two students' work with a task, picked out from (Andresen 2006). Based on the episode, the paper discusses the role of explorative work for concept formation as a forerunner for expressive modelling.

The use of non-authentic models

Further, modelling for concept formation allows the teaching to consider non-authentic mathematical models. This is in accordance with Blum's short remark, that besides really real, also artificial problems and traditional word problems may be used – the latter are sometimes 'better suited to educational purposes than are genuine real world applications' (Blum 1991 p 22). In line with this, Blomhoej also states that well-chosen, relevant pseudo-realistic models necessarily should be treated besides authentic models (Blomhoej 1991 p 191).

The authenticity of models and modeling is an issue, frequently put in front by students and teachers: the transition goes gradually from authentic models of real world problems into a sort of artificial "textbook-real world-problems" which tend to oversimplify the complexity of the problems they intend to deal with. The teaching of mathematics in school necessarily implies some restrictions: only certain parts of applied mathematics can be considered. School mathematics must be practiced in contrast to 'mathematising in practice' like it is described in (Hoyles, Noss and Pozzi 1999). According to Hoyles et al., mathematising in practice may involve a 'pay-off' for making mathematics visible, in terms of personal or professional empowerment. Focus in their study is on the process of computational modelling, showing how connections may be forged between implicit and explicit mathematical thinking, and the process is driven by the aforementioned pay-off. In school mathematics, in contrast, it is understood that the silent agreement between the students' and the teacher implies that the students follow the teacher's advises. Therefore, it is of special importance for the agreements between the teacher and the students to avoid that the students feel "cheated" by oversimplifying or misleading mathematical models. Negotiations between the teacher and the students, and a clearly statement of the assumptions, intend to overcome this problem.

Excerpt from a classroom episode

In the following, excerpts from an episode of teaching differential equations models serve to illustrate how two students' 'thinking loud' gives them mutual support to their concept formation. My interpretation of the process followed the idea of mathematising represented in (fig.1). The theme of this episode was the changes between the real world problem and the model during their exploration of the model. The two students, like the rest of the class, worked in pairs one lesson with the task (Hjersing et al. 2004 p 46) in the classroom. The task concerned an elementary, oversimplifying model of the learning process:

> If we let *L(t)* be the fraction of the list learned at time t, where $L = 0$ corresponds to knowing nothing and $L = 1$ corresponds to knowing the entire list, then we can form a simple model of this type of learning base don the assumption:
>
> • The rate *dL/dt* is proportional to the fraction of the list to be learned.
>
> Since $L = 1$ corresponds to knowing the entire list, the model is: $\frac{dL}{dt} = k(1-L)$
>
> where k is the constant of proportionality.
>
> For what value of L, $0 \leq L \leq 1$ does learning occur most rapidly?

Fig 3. Excerpt from the textbook (Hjersing et al. 2004 p 46)

The aim of the first task, apparently, was to make the students aware that the equation expressed the learning rate. The students explored the model by substituting one and zero:

S1: That is, if we let this one be 1, then it wil increase...oh no because otherwise it will...no, it has to be... it can not equal zero so it will never be zero.
S2: But it will not equal zero if you let L be zero
S1: Oh no it becomes one, then. One minus zero... then it is one... k then it is one..
S2: It is just k, isn't it?
S1: Yes but that does not make sense because if L is zero then you do not memorise anything
S2: Yes but then it has to be one, you know, because if...then it has to be zero
S1: Yes
S2: Because then..
S1: Yes that makes sense you know because...
S2: If you see a paper for the first time then there will always be something... there will always be someone who forgets...

This short dialogue reveals how these two students thought aloud in common. They related their calculations very closely to the real world problem without questioning the model. They changed fluently between the model and the real world problem. Then S1 grasped the idea and shared it with S2:

> S1: When it equals zero, the learning rate is fastest
> S2: In the very, very beginning...
> S1: From the very beginning
> S2: Just so – the less L the faster you learn

Now, both students had established the model. They read further in the text:

> Suppose that two students memorise lists according to the same model: $\frac{dL}{dt} = 2(1 - L)$
>
> (a)If one of the students knows one-half of the list at time t = 0 and the other knows none of the list, which student is learning most rapidly at this instant?
>
> (b) Will the student who starts out knowing none of the list ever catch up to the student who starts out knowing one-half of the list?

Fig 4. Excerpt from the textbook (Hjersing et al. 2004 p 46)

The students wondered, whether it was possible to know half part of the list without ever seeing it. They, obviously, enjoyed their conversation and the work with the model. Even if they, apparently, found the model a little crazy, they were definitely willing to work with it:

> S2: Not just this one...
> S1: It does not make sense that he memorises one-half at time zero, but it does not matter
> S2: Yes because then he just know...
> S1: Without seeing it
> S2: Yes

In the following dialogue, both tried to reason with their good sense:

> S1: But then it will nevertheless be the student who still does not memorise anything who is learning most rapidly at that time. Because he has the possibility of knowing 100%
> S2: Yes and the other has 70
> S1: Yes he had already learned something
> S2: He started before...
> S1: This student will memorise faster than this student
> S2: learn faster, eh?
> S1: Or the other
> S2: Yes because the less number of... something
> S1: This model

Then S2 changed to the model perspective, talking about the variable *t* and *substitute*, in contrast to the ealier words *from the beginning*, and the students continued in this perspective:

S2: You can just substitute... t is zero
S1: We can let this one equal...
S1: One of them learns 1 and the other learns 0.5, eh?
S1: No, the one lerns 0 and the other learns 0.5, so...
S1: When we substitute – you know, one of them learned nothing...
S2: Then it says 2
S1: And the other learned one-half, eh?
S2: Oh yes..
S1: Then you substitute 0.5 then it is 2 times a half, then it says 1. So this is the bigger and then the other learns faster..

The students continued their work with the next question in a similar way.

Explorative and expressive work

For the following discussion, a distinction is made between explorative work and expressive work at functional level: explorative work aims at inquiry of existing constructions or artefacts like mathematical models or statements or computer commands for standard routines. Expressive work aims at creation, for example creation of a solution or description of a mathematical problem. The modelling process in (Fig. 2) is expressive work since each of its six sub-processes requires creative non-routine activities. In general, students' works in school math tend to be explorative rather than expressive.

Exploration of models as a forerunner for expressive modelling

Since the teaching of applied models during expressive modelling is complex and time consuming, exploration of mathematical models as a forerunner of expressive modelling seems to be of interest. The excerpt from the classroom episode serves to illustrate the students' exploration of the model. During the whole teaching sequence that encompassed this episode, the students trained isolated parts of a full modelling process separately by exploring and revising ready-made mathematical models, as a forerunner of expressive modelling. The idea of the teaching sequence was to let the students:

- Study the 'mechanics' of the single terms in the sense of *symbolizing* and *creating relations* between the single terms
- Focus on the potential roles and the meaning of particular, mathematical conceptions like for example derivative, slope etc.
- Train their ability to *recognise* different types of mathematical models and be critical to their use in the actual context

Whereas the first point relates to concept formation, the third point concerns applications of mathematical models at the functional level. The second point, thus, acts as an in-between. These ideas, apparently, may apply to other topics than differential equations models. During the teaching sequence, the students' modelling activities appeared less time consuming and complex, and to a high degree the activities were under the control of the teacher. Though explorative, the experiments capitalised on the students' creativity and thereby resembled expressive activities.

Still, they allowed the exploration of conventional, powerful symbolisations. This is in accordance with Gravemeijer's description of RME's conceptualisation of modelling. In the description, modelling shares some commonalities with both the expressive and explorative approaches to design (Gravemeijer et al. 2000 p 240 ff). Thus, not only the expressive work but also the students' explorative work was designed to facilitate the students' own construction of mathematical conceptions. For example, the concept formation in the case of explorative work was facilitated by classroom- and group discussions of shared models and negotiations of symbolising.

Conclusion

The following conclusion is based not only on the excerpt from the classroom episode that serves to illustrate the students' work, rather than on the entire research project, from which the excerpt was picked out. Based on the project's qualitative analyses and discussions, the research project concluded that sequences of explorative work may serve to support the students' concept formation and at the same time prepare them for expressive modelling. The explorative work may focus on parts of an authentic mathematical model, which is subsequently treated as a whole in one or both of the aspects 'technological knowledge' and 'democratic competence', respectively. Though, the episode illustrated that appropriate, non-authentic models can be used as well.

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USING *RESEARCH AND STUDY COURSES* FOR TEACHING MATHEMATICAL MODELLING AT UNIVERSITY LEVEL

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This paper focuses on the teaching of mathematical modelling during the first year of experimental sciences university degrees. Within the frame of the Anthropological Theory of the Didactic (ATD), we propose the design of Research and Study Courses (RSC) as a new didactic device to teach mathematical modelling with a double purpose: to make students aware of the rationale of the mathematical contents they have to learn and to connect these contents through the study of open modelling questions. We also show to what extent these courses can "cover" the considered mathematical curricula giving a clear functionality to its different contents.

1. TEACHING MATHEMATICS AT THE UNIVERSITY LEVEL

The written description of different mathematical courses of Spanish first year university scientific programmes states that the teaching of mathematics follows a double objective: on the one hand, they strive to give students basic mathematical training; on the other hand, they try to introduce students to mathematical modelling. Mathematical contents are then organized in "topics", "themes" or "modules" centred on a main concept (limits, derivatives, integration, linear applications, diagonalisation, ordinary differential equations, etc.) each including a number of definitions, properties, theorems, proofs, various techniques and types of problems. At the end of the study process, problems tend to turn into "applications" such as giving a "rationale" to the contents and showing their functionality.

This traditional organization has prevailed historically for various reasons of didactic economy that we will not discuss in this paper. One of the drawbacks of such organization is that it hides the problematic questions which constitute the rationale of the taught notions, properties, theorems and techniques. The contents are reduced to a finite set of pre-existent "works" to study, but generally the questions that motivated the construction of these contents disappeared from the school' culture. This situation leads to the formulation of the following didactic problem, which constitutes the core of our research:

Given the classical description of contents in a university course, how can we design a didactic organisation that locates problematic questions at the starting point of the study process, that is, as the main generators of the mathematical contents that, because they appear as an answer to these questions, can be connected and acquire a clear functionality?

Here we consider the particular case of a first course of mathematics in experimental sciences degrees: biology, geology, chemistry and physics. The corresponding study programme is generally structured in three main areas: linear algebra, one-variable differential and integral calculus, and ordinary differential equations.

In order to approach the didactic problem stated above, we postulate that it is necessary to use mathematical modelling in an explicit and central way during the study process. In other words, we assume that doing mathematics consists essentially in the activity of producing, transforming, interpreting and arranging mathematical models. This modelling activity starts with the study of an extra-mathematical system, where an initial problematic question is considered, followed by the construction of a model of this system that, duly treated, allows providing an answer to the initial question and bringing up new problems to be studied. According to this, we can reformulate the didactic problem as follows:

Given the classical description of contents in a university course, how can we design a didactic organisation where mathematical modelling has an explicit and crucial role? What kind of status should we confer to mathematical modelling in order to transform it into a true instrument to study problematic question and to articulate the contents used during the study process?

2. MATHEMATICAL MODELLING IN THE ATD

Several works within the framework of the ATD have analyzed and described the modelling activity (Chevallard 1992; Chevallard, Bosch & Gascón 1997). Following the recent works of Javier García (2005, see also García, Ruiz, Gascón & Bosch 2006) we will consider that in a modelling activity both system and model have a praxeological structure and that the modelling activity is a process of reconstruction and articulation of mathematical praxeologies which become progressively broader and more complex. That process starts from the consideration of a (mathematical of extra-mathematical) problematic question that constitutes the rationale of the mathematical models that are being constructed and integrated. Our objective is to study, analyze and describe the conditions and constraints that would allow the development of study processes respecting these characteristics.

García (2005) and Bolea, Bosch & Gascón (2004) discuss the constraints that the teaching of modelling has to face in the case of secondary education. One could imagine that the institutional environment of sciences or engineering degrees would provide better conditions for the teaching of modelling. Indeed, the corresponding curricula incorporate mathematics courses since they constitute an essential tool for the understanding, use and development of these sciences. However, reality is quite disappointing. Always left to the end, the teaching of modelling problems is basically absent from Spanish “real” university curricula. Mathematics tends to be organized in a traditional way, where the teaching reproduces the logic of the axiomatic construction of concepts. In fact, as we have already pointed out, the ruling "ideology" in mathematics teaching at university could be called the "*application of pre-established knowledge*", leaving no place for the *process* of modelling. The modelling activity hence degenerates and is restricted to the algorithmic use of pre-existing models. On the way, any questioning related to the origin of the models and adequacy is eliminated.

In order to face this problem, we decided to use the notion of Research and Study Course (RSC) as a didactic device to facilitate the inclusion of mathematical modelling in educational systems, and, more importantly, to explicitly situate mathematical modelling problems in the centre of the teaching and learning process. That is, we will use RSC as a possible way for both teaching mathematical modelling at university level and covering, through it, a given mathematical curriculum. As we will see now, RSC will provide some adequate conditions to formulate questions concerning: (a) the origin of the models and their evolution; (b) the relations (adequacy) between models and modelled systems; (c) the effectiveness of the model to address initial questions; (d) the limitations of each model and the resulting need of broadening their reach (by remodelling the limited models, etc.).

3. TOWARDS A NEW ORGANIZATION OF MATHEMATICS CURRICULUM: THE RESEARCH AND STUDY COURSES

Chevallard (2004 & 2006) introduces the concept of *Research and Study Course* (RSC) as a general model for designing and analyzing study processes. Our teaching proposal will take into account the main characteristics of the RSC as proposed by Chevallard and, at the same time, the constraints that hinder mathematical modelling activities at university level (Barquero 2006).

A RSC must be generated by the study of a question Q, of real interest to the students (“alive”), and strong enough to generate many other questions. The study of Q and the subsequent questions it generates lead to the construction of a large body of knowledge that will outline a field of possible RSC and their limits. The sequence (or “tree”) of questions generated by an initial question Q is, in fact, a sequence of pairs questions/answers: (Q_i, R_i). The RSC thus permit retrieving the original relationship between questions and answers, or between problems and theories. This relationship is a key for the construction of scientific knowledge in general and for the activity of mathematical modelling in particular. Below, we highlight some of their characteristics:

- They let mathematical modelling play a central part in the study process. One must emphasize in each step that modelling has an essential role as a tool for the construction of new knowledge.

- To certain extent, they induce in some sense the transformation of the goals and evaluation methods. Because there is a problem to answer and not a concept to teach, it becomes necessary to evaluate the study process itself (the course followed to obtain the answer) and not only the final product.

- They allow making clear, crystallizing, institutionalizing and evaluating the very same process of modelling. This can be achieved if the study process has certain continuity over time, thus breaking the atomization of mathematical questions.

- They allow the questioning of the models that are being developed. This questioning is the driving force of the entire study process, and gives rise to the need of restructuring, correcting and interpreting current models into wider and more complex models.

Next, we present the design of a study process which places modelling at the heart of the mathematical work proposed to the students. It allows covering, through the RSC, a wide range of the mathematics taught in a first year university course.

We start from a generative question about the study of population dynamics. It constitutes the thread of the entire learning process. We propose to answer this question by constructing different mathematical models, which allow to frame the problems at first, and to broaden them progressively. Indeed, when studying the links between the system (the growth of a population) and the initial model, new questions will appear that can only be addressed through the construction of more comprehensive mathematical models that, in turn, will generate new questions, and so on. A sequence of successive enlargements of the considered mathematical models is thus generated. We postulate that this process helps connecting and making functional the different mathematical contents that are programmed in the course.

4. DESIGN AND TESTING OF THE RSC

4.1. A priori design: mathematical map of the RSC

We consider a system X (population), where a given value x_t (population size) changes over time t. The study of the population evolution, i.e., the *dynamics* of population X, gives rise to the following *initial question:*

$\boldsymbol{Q_0}$: Given the size of population X over some time period, can we predict its size after n periods? Is it always possible to predict the long term behaviour of the population size? What sort of assumptions on the population and its surroundings should be made? How can one create forecasts and test them?

Depending on whether we consider time in a discrete or continuous value, we develop two families of models: *discrete models* or *continuous models* in population dynamics.

If we assume, initially, that time t is measured in discrete units, and that x_t depends, among other factors, on past states $x_{t-1}, x_{t-2}, \ldots, x_{t-d}$ $(0 < d \leq t)$, studying question $\boldsymbol{Q_0}$ leads us to consider two main types of models:

- When x_t only depends on x_{t-1} (population with independent generations), we study models based on *recurrent sequences of order* 1: $x_{t+1} = f(x_t)$, where f is a real valued function of one variable. We can thus cover the field of one variable calculus and the study of sequences.

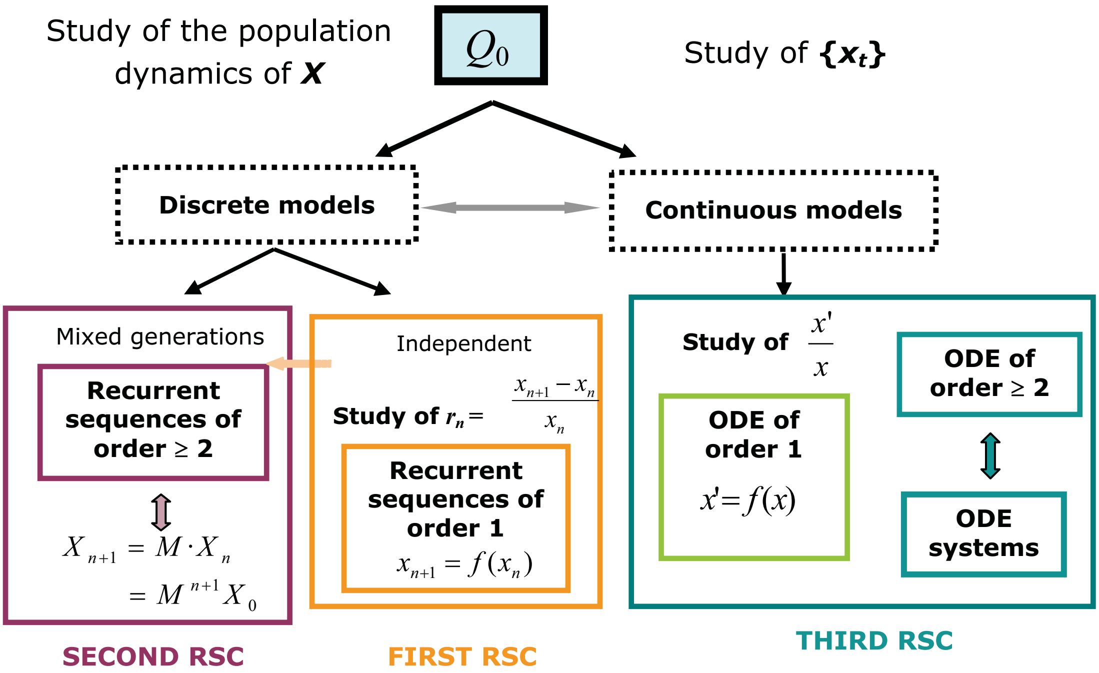
- When x_t depends on the $d > 1$ past generations $x_{t-1}, x_{t-2}, \ldots x_{t-d}$ (population with mixed generations), we study *recurrent sequences of order* $d > 1$, which can be expressed as *vector recurrent sequences* with $X_{n+1} = f(X_n)$ where $X_0 = (x_0, x_1, \ldots x_{d-1})$ is the vector of the d initial generation sizes and $X_i = (x_{id}, x_{id+1}, \ldots, x_{(i+1)d-1})$ is the i-th vector of d generations, $0 \leq i \leq n$. We thus cover the field of linear algebra.

If we assume that time t is measured as a continuous variable, we study the continuous evolution of the population, which has an analogous structure, in some sense parallel to the situations described above. This allows us to study mathematical models with *ordinary differential equations* (ODE) of order ≥ 1.

4.2. General conditions for testing and limits of the RSC

We tested the use of the suggested RSC during the academic year of 2005-2006, with first year students of a technical engineering degree (3 years program), in the Industrial Chemical Engineering department, at the Universitat Autònoma de Barcelona (UAB). The testing took place within the one-year course “Mathematical Foundations of Engineering”.

The test was performed in what we called a “Mathematical Modelling Workshop”, which was optional for the students, and would at the most provide a bonus of 1 point out of 10 in the final grade of the course. Since the RSC were relatively independent of the course, and since there were a number of institutional constraints on the curriculum, we divided the map of a possible modelling work into three modules that gave rise to the following three RSC.



4.3. The first RSC: the dynamics of a population with independent generations

The first RCS on the discrete study of the dynamics of a population with independent generations took place during 6 sessions of 2h each. The first session started with the study of data corresponding to the size of a population of gooses in an isolate island, for a period of 5 years. The students were asked to analyze the data and provide an initial answer to the generating question.

Students worked in teams of two or three. During the first session, the students took three approaches to the problem: a number of teams tried finding the best polynomial interpolation to the data; another group of teams tried an exponential fit on the data; the rest tried using recurring sequences to model the population dynamics. In the class discussion, the three approaches were presented by the teams, and the students, with the guidance of the instructor, decided that the discrete recurring sequences approach would be explored first, leaving the continuous approach for a later RSC.

In the second session, the instructor described the work done by the students that focused, in the first session, on recurring sequences. In particular, the class agreed on the notation to be used and some requirements, such as considering a population with independent generations. x_n was defined as the size of the n-th generation of population X and the study of the population evolution was thus characterized by the study of the sequence $(x_n)_{n\in\mathbb{N}}$. The assumption of independent generations leads to consider several indicators of the population growth. The instructor suggested to focus on: $r_n = \dfrac{\Delta x_n}{x_n} = \dfrac{x_{n+1} - x_n}{x_n}$, the *relative rate of growth* of X between consecutive generations, on which several assumptions were placed, leading to the construction of a variety of mathematical models of increasing complexity.

During all the sessions of the first RCS, students were working in the same teams. In the beginning of each session, the teams had to deliver a report of all the work done during previous sessions, and there was one team in charge of explaining and defending its report. It was a very good way to compare and discuss the work done during all the process, and particularly, a way for the study community to formalise all the questions treated and their successive partial answers. This allowed them to agree on how to continue with the study process. At the end of the first RSC, teams gave in a final report of the entire study. Below, we summarize the *assumptions on X* (H_i) that were considered in the course, along with the *main problematic questions to be addressed* (Q_i), the *mathematical models* (M_i) interpreted as tools to address the questions, and the successive temporary *answers* (R_i).

- ***From the malthusian model to the logistic model***

$\boldsymbol{H_1}$: The relative rate of growth is constant: $r_n = r$
$\boldsymbol{Q_1}$: What are the dynamics of a population with a constant rate of growth?

The situation can be modelled with the following model ($\boldsymbol{M_1}$), known as the *malthusian model*: $\frac{x_{n+1} - x_n}{x_n} = r$. Given the initial population size $x_0 = c > 0$, we can calculate x_n for any generation n: $x_n = (1 + r)^n x_0$ (1)

If $\alpha = 1 + r$, the following answer $\boldsymbol{R_1}$ to $\boldsymbol{Q_1}$ can be provided:

$\boldsymbol{R_1}$: If $\alpha < 1$, the population is wiped out; if $\alpha = 1$, the population size remains constant independently of the time elapsed; and, finally, if $\alpha > 1$, the population grows indefinitely.

This result raises a new question about the limitation of the model when $\alpha > 1$ and the considering of a new hypothesis $\boldsymbol{H_2}$:

$\boldsymbol{Q'_1}$ (limitation of M_1): The case of $\alpha > 1$ assumes the existence of infinite resources. How can we overcome this unrealistic fact?
$\boldsymbol{H_2}$: The size of the population cannot exceed a given maximum value K. Therefore, the rate of growth must decrease when the population size approaches this maximum value. For example, we can assume the simplest case of a rate of growth decreasing linearly with size.
$\boldsymbol{Q_2}$: What are the dynamics of a population under H_2 conditions?

One of the simplest models that satisfies $\boldsymbol{H_2}$ is summarized by the equation $x_{n+1} = \alpha\, x_n \left(1 - \frac{x_n}{K}\right)$, known as the *logistic equation* (discrete) or *Verhulst model*. In contrast with the malthusian model, this equation cannot be solved in closed-form, i.e., with a general formula $x_n = f(n)$.

$\boldsymbol{Q_{2.1.}}$: What does the convergence of (x_n) depend on, in the logistic model?

- ***A functional model that generalizes the malthusian and logistic models***

The work performed so far can be described by recurrent relationships $x_{n+1} = f(x_n)$ where f represents the functional relationship (linear or quadratic) between two consecutive generations of the population.

The teacher proposes then to study directly the more general model $x_{n+1} = f(x_n)$, of which M_1 and M_2 are specific instances. A **general functional model** ($\boldsymbol{M}_3$) is then considered and initial question is rephrased as:

$\boldsymbol{Q_3}$: What are the dynamics of the sequence of a sequence (x_n) generated by the relationship $x_{n+1} = f(x_n)$ where f can be any C^1 function?

New techniques are now necessary to analyze the behaviour of $x_{n+1} = f(x_n)$ for a general function f. This induces a huge *enlargement* of the *types of problem* that could be considered and marks the end of the RSC as was followed with the students.

5. CONCLUSIONS AND DISCUSSION

5.1. Feasibility and local compatibility of RSC with traditional teaching

As we proposed in the design phase, we verified that the sequence of questions arising from the generating question Q_0 cover most of the traditional curriculum of a first year mathematics university course, plus some additions. As the RSC were taking place, we observed that the issues arising from the RSC had more and more impact on the class lectures and problem sessions. As a matter of fact, this would be the ideal situation for a mathematics course that integrates the RSC as an essential didactic tool. The appearance of mathematical organisations in the course would then be subordinated to the needs appearing from the study of a small number of problematic questions. Anyway, we have to admit that the teaching system *ecology* has only been partially modified. The diffusion of such instructional devices to other first course degrees is clearly a more complicated problem that we would have to tackle in the future.

5.2. Didactic functions of the RSC and mathematical modelling survival

First, we should emphasize the importance, during the RSC, of the *exploratory* and *technological* moments. Indeed, we believe that promoting the "experience" of these moments allows the students to individually create hypotheses, formulate questions, compare experiment and reality and choose the relevant mathematical tools. These constitute the first phases of mathematical modelling. In addition, we would like to point out that the students were highly involved in the *evaluation* and *institutionalization* moments. Specifically, two didactic devices undoubtedly permitted changes in the ruling didactic contract: Team presentations, followed by delivery and defence of the reports in each session; Class materials without a priori fixed contents. These two moments helped conduct rather successfully the phases of production of knowledge and interpretation of the system, as well as the criticism and study of the limitations and links between each model constructed. These are crucial phases in the modelling process, which are not covered in traditional educational systems.

5.3. Changes towards a new didactic organization

The distribution of responsibilities during the management of the RSC was very different to the one of a traditional course. Since the beginning, the instructor acted as the director of the study process instead of lecturing the students. She gave as much autonomy as possible to the students and negotiated explicitly some aspects which are usually under the responsibility of the teacher: scheduling the study process, selecting mathematical contents, using computer and bibliographical resources and evaluating partial results. This *increasing autonomy* taken on by the students during the RSC is a necessary condition to carry out the activity of mathematical modelling. In this sense, we observed the importance of always keeping in mind the generating question, which we used as the thread of the entire study process. This allowed us not to consider the mathematical tools as a goal in itself, but rather to give an answer to a certain question. We thus obtained a functional teaching of mathematics as opposed to the traditional “monumentalism” (see Bosch & Gascón 2006).

5.4. The status of mathematical modelling within the RSC

Considering the work done by students in a particular model (the discrete logistic model, for instance) we can observe that students pass through the *different stages of the modelling process*. A didactic organisation starting from a *generative question* and structured by a sequence of questions and provisional answers not previously determined that helps keeping the initial question “alive”, seems appropriate to promote the mathematical modelling activity. However, if we analyse the study process that includes a whole RSC, or what is more the course structured by the three RSC, it becomes clear that one of the main didactic functions of this device consists in their capacity to produce a progressive enlargement of the models that are being built up which helps the connection between models and, finally, promotes the internal dynamics of mathematical modelling. One of the main didactic functions of the RSC comes from their capacity of broadening and articulating different mathematical models, and hence giving momentum to the internal dynamics of mathematical processes. This momentum can only be generated if, during the RSC, the modelling activity can progressively achieve the status of a (*mathematical*) *study object*, beyond the status of being a (*didactic*) *tool* to study a number of mathematical organisations. Only then, the students will be able to formulate all the questions related to mathematical and didactical techniques without constraints, and to take responsibilities traditionally assumed by the teacher.

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THE TEACHING CALCULUS WITH APPLICATIONS EXPERIMENT SUCCEEDED – WHY AND WHAT ELSE?

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Abstract

An education design experiment in which applications were integrated in the Multivariable Calculus course at the Technion was carried out in 2002-2005. The purpose of the study was to emphasize the connections between the mathematics course and the science and engineering disciplines, and examine the effect of applications on students' understanding and learning motivation. In the experiment we extended the conventional curriculum by optional applications motivated recitations. Data analysis indicated that the experiment succeeded to improve the understanding of calculus concepts, and had significant positive effect on students' achievements and motivation. In this paper we briefly describe the experiment, try to understand its result through and a learning processes analysis, and discuss further initiatives of teaching with applications.

Introduction

Applications and modeling, the subject of our CERME 5 Discussion Group 13, is a central theme in mathematics education research (Blum, 2002; Haines et al., 2006). In engineering education, the ability to apply mathematics has been recognized as one of the main learning outcomes required from graduates (Criteria for Accrediting, 2000, pp. 32-34). With the general agreement about the importance of this goal, there is a debate on the way of achieving it (Blum and Niss, 1991). Some educators propose teaching mathematics from the application perspective (Kumar and Jalkio, 2001) with focus on the mathematical skills required by the engineering disciplines.

This paper considers an experimental study of integrating applications in the Multivariable Calculus course at the Technion. Traditionally, the basic mathematics courses at the Technion are taught at a high level of theoretical abstraction and without applications. Our experience shows that a considerable part of the students encounters difficulties in their studies. This is in spite of the fact that in order to be accepted to the Technion, the students need to have high grades in advanced high school mathematics and in the scholastic aptitude test.

In our study integrating applications was suggested as a possible remedy to the students' difficulties, and our goal was that the experiment will influence the way the basic mathematics courses are taught. The research was done in the framework of the doctoral thesis of Dr. S. Aroshas written under the supervision of the first authors. Since our experiments of teaching calculus with applications were conducted within constraints of the given course, we had to change the learning processes and the research methods during the study. This motivated us to conduct the study in the form of education design experiment (Cobb et al, 2003). The experiment was conducted between 2002 and 2005. Details of the experiment were presented in (Aroshas et al., 2006). Motivated by the positive results we extend our study in theoretical and practical directions, as described in this paper.

The paper is organized as follows: we first present the experiment and its results. Then we discuss learning through solving applied problems in terms of mental representations, and try to use this model to explain the effects observed in the experiment. Finally we mention further initiatives of teaching through applications of other mathematics courses.

Study Framework

Conventionally, the Multivariable Calculus course at the Technion included four lecture hours and two hours of recitation. With the development of the Technion Mathematics Web tutoring system the recitations were reduced to one hour a week. In our study the course schedule was extended by supplementary applications tutorials which were given voluntarily in conjunction with teaching a conventional Multivariable Calculus tutorial. The study included three teaching experiments which examined different forms of supplementary applications tutorials.

In the first (pilot) experiment Supplementary Applications Tutorials (SATs) were given by Aroshas in the fall semester 2002-2003. About 75 students from different science and engineering faculties participated regularly in the optional classes. The SACs were coordinated with the main calculus class so that theoretical concepts and their use in real problems were studied in the same week.

The tutorials were followed up by a pilot study. In conjunction, we developed and tested applied problems and teaching methods in class. To characterize the applied problem solving skills we asked a number of experts from science and engineering faculties at the Technion for their opinion regarding the need for learning applications in the mathematics courses, and how important it was for their students. We also accepted their advice on how to present and analyze applied problems in the course.

The second (central) teaching experiment was conducted in the spring semester of 2003-2004 and designed in the following way. Two groups (experimental and control) of Multivariable Calculus students were created with the same number of students (N=33). Both groups attended the same Calculus lectures with no emphasis on applications, but their tutorials were different. Throughout the course the control group participated in two weekly hours of conventional tutorials (without applications), while the experimental group participated in one weekly hour of conventional recitation and one of SATs. There were no significant differences between the experimental and control groups with regard to mean grades in One Variable Calculus and in the pre-course test, interest in studying calculus with applications. Both groups represented the same engineering faculties. In this experiment the study focused on testing the effect of teaching calculus with applications through its comparison with the conventional teaching approach. The comparison related to learning achievements, understanding calculus concepts, and students motivation.

In the third (additional) teaching experiment two supplementary 2-hours SATs were given in the fall semester 2004-2005 and attended by more than 50 students from the multivariable calculus course. The goal of the tutorials was to introduce the calculus concepts of Lagrange multipliers and multiple integrals prior to their formal study in the lecture. In the sessions the concepts were recreated from the practical need and through the analysis of applied problems. In the follow-up we examined to what extend the SATs helped students to understand the concepts taught in the lectures.

Research Questions, Instruments, and Applied Problems

The research questions were as follows:

1. What is the effect of the applications-based mathematical instruction on learning achievements, understanding calculus concepts, and motivation in the course?
2. What are the possible ways of integrating applications in the Multivariable Calculus course while keeping its existing constraints and limitations?

In the first teaching experiment we used pre-course and post-course questionnaires. Four questions of the pre-course questionnaire were repeated in the post-course one. They tested student opinions on the following aspects:

1. Anticipated effect of integrating engineering and science problems on understanding the calculus concepts.
2. Interest in solving calculus problems from the area of specialization.
3 Viewing the calculus capabilities as a condition to succeed in the area of specialization.
4. Interest in attending the applications motivated course in addition to the conventional calculus class.

The post-course questionnaire also inquired student opinions about the contribution of the three teaching methods used in the course: demonstrating mathematical problems of science and technology, constructing and solving mathematical problems in context, visualization through computer simulations.

The central teaching experiment applied pre-course and post-course questionnaires and tests. The pre-course questionnaire examined students' attitudes and learning styles, and collected personal information. This information and a One Variable

Calculus applications test were used for creating experimental and control groups. The two post-course questionnaires were also conducted in both groups. The first one tested students' opinions about the value of tutoring sessions and their preferences in learning with applications. The second one was an understanding test which examined students' possible misconceptions of Multivariable Calculus concepts. The midterm exam and the final course exam grades of students in the control and experimental groups were also collected.

In the third teaching experiment we used an attitude questionnaire which was conducted after each of the supplementary applications classes and the following lecture. In the questionnaire the students evaluated the contribution of the classes to their understanding of the calculus concepts given in the lecture.

Results

First teaching experiment

The pre-test findings were as follows:

- The absolute majority of students pointed their high level of expectations from integrating applied problems on understanding the calculus and interest to solve problems from the area of specialization.
- A majority of the students (70.7%) recognized the connection between success rates in the first-year mathematics and the majoring subjects.

The post-test indicated:

- The students did not change their opinion about the effect of integrating applied problems, and continued to be interested in solving problems related to the area of their specialization (as given by the t-test). This shows that the course met the students' expectations.
- There was an insignificant increase in the average evaluation of the importance of the calculus capabilities for success in the area of specialization (as given by the t-test).
- All the students reported that they would recommend attending the applications course to their classmates.
- All three teaching methods significantly contributed to the understanding of calculus concepts (as revealed by F-test). The contribution of visualization through computer simulations was the highest.

Second teaching experiment

The pre-course questionnaire was used to divide calculus students into two "equal" groups: the experimental group and the control group. As a result, there were no significant differences between the control and experimental groups in One Variable Calculus grades, interest in learning mathematics with applications, results of the one-variable calculus applications test, in representation of different faculties and in learning styles. The comparison between the experimental and control groups at the end of the course indicated the following features:

1. The mean exam grades of the experimental group were 8.9 points higher than that of the control group in the midterm exam and 6.3 points higher in the final exam. The differences between the groups in both exams were found significant (t test).
2. The experimental group gave a significantly higher evaluation for the contribution of the course in relation to all the aspects mentioned in the table.
3. The two groups are of the same opinion about the need of addressing different learning styles and integrating applied problems in the course.

It is worth mentioning, that in the teacher evaluation survey of the Technion Centre for Promotion of Teaching, the mean grades given by the control and experimental groups to the calculus teacher were similar (4.85 and 4.86 out of 5). This fact supports the view that the advantage of the experimental group was a result of studying calculus with applications.

The calculus understanding test consisted of 14 theoretical and applied questions related to the following concepts: equipotential lines, directional derivative, gradient, tangential plain, Lagrange multipliers, and extremum. The test was validated by two experienced practicing lecturers of the Technion Multivariable Calculus course. Significant advantage of the experimental group over the control group in the percentage of correct answers to the test questions was indicated. For some of the questions correct answers were given by 70-80% of students in the experimental group vs. 25-35% in the control group. Typical student reflections to a question on the contribution of studying applications for understanding calculus concepts were as follows:

> *"Through applications I grasped the complex calculus concepts".*
> *"The impact of a one-hour application session is the same as of a regular (two-hour) tutorial".*
> *"It is a pity that applied problems were not given in the first Calculus course".*

Third teaching experiment

The attitude questionnaire conducted after each of the two supplementary applications classes asked to evaluate its contribution in the following aspects: understanding a lecture, following the lecturer's explanations, geometric interpretation of the concepts, linking new and formerly studied concepts, and development of problem solving skills. The answers were very positive. More than 90% of the students mentioned high positive contribution of the SATs in all the abovementioned aspects. In response to a question on preferred teaching methods, an absolute majority of the students supported teaching calculus with applications through practical examples, discussions of possible applications, and visualization of calculus concepts.

Learning Process Analysis

In order to understand how applied problem solving contributes to mathematical learning we need to analyze the underlying knowledge acquisition process. We do it by using the prevalent model of internal mental representations of knowledge perceived through cognitive processes (English and Halford, 1995). By this model concepts and ideas are internally represented by networks of knowledge units which include spatio-temporal images and predicate-argument structures (propositions). Links between knowledge units depend on associations rather than on logic. Different internal networks also can become connected by links and form organized bodies of knowledge (schemata) representing complex concepts and categories. It is accepted that this model of mental representations provides a psychologically realistic description of different aspects of cognition such as mathematical problem solving. Here we will use the model to explore some effects of applied problem solving on developing mathematical knowledge.

Explication of knowledge

Implicit knowledge is knowledge to perform a task, but without the ability to explain and modify the performance which characterizes explicit knowledge. Following Karmiloff-Smith (1990), explicit knowledge is created through constructing higher level representations of knowledge from that existing at an implicit level. This view is shared by diSessa (1998) who stresses the need for mediating between formalisms and experience in physics and constructing representations from theoretical and intuitive "knowledge pieces".

With regard to the multivariable calculus course considered in this paper, the students study it after and in parallel with basic science and engineering disciplines, such as physics. These disciplines strongly use calculus methods but present them as implicit procedures, while the underlying concepts are not yet internalized by the students from the mathematics course. As a result, students have difficulties in problem solving and understanding the disciplines. The value of solving applied problems from different disciplines in the calculus course was mentioned by many of our students. The students discovered meaning of the procedures that they used in the disciplines and their typical reflection was: "Now we understand the mathematical procedure used by the (science or engineering) teacher and acquired the mathematical method for solving a class of similar problems".

Enhancement of problem solving skills

The major factors that underlie human reasoning (English and Halford, 1995, pp. 29-30) include memory retrieval, strategies, and analogies. The role of applied problem solving practice in stimulating mathematical reasoning processes can be clearly observed. Indeed, the characteristic feature of memory functioning is that information which is communicated and used more frequently is better retrieved in memory. In our case, when studying calculus with applications the students create associations which link representations of mathematical concepts with representations of different real situations. Through these associations, the mathematical concepts recur to students' memory when they face the situations in their everyday practice.

Strategies are constructed on the basis of mental models which represent the structure of the relevant concepts and delimit the workspace of their use for problem solving (English and Halford, 1995, pp. 45-50). In our case study, calculus application experience facilitated the students to develop, test and revise their problem solving strategies in two ways. First, by dealing with new situations and non-standard problems the students extended their notion of possible application of calculus concepts. Second, by exploring the properties of a variety of calculus applications the student developed rules, algorithms, and heuristics that became components of their applied problem solving strategies.

An analogy is a mapping between two represented situations in which elements of the first representation (the source) are mapped into elements of the second representation (the target) in such a way that relations in the source and the target correspond (Holyoak et al., 2001). Del Re (2000) pointed that analogies embodied in physical and mental models have essential role in scientific research. He considered mathematical models as tools of argumentative analogical thinking. In our calculus course, we saw that solving applied problems affected students' analogical reasoning in two directions. First, when solving various real problems by common mathematical models the students discovered analogies between different situations. Second, this practice helped to understand abstract mathematical concepts by mapping them into real representations which were clear to the students from their experience.

Additional Work

Discussions and initiatives

The results of our experiment were discussed in a special committee that was formed at the Technion to suggest ways to improve the mathematics courses given to engineering students. The committee included professors of different engineering departments and from the department of mathematics. The committee was chaired by the first author but this was not the reason for the unanimous agreement of all the committee members that one of the ways to improve teaching of mathematics is to integrate applications and modeling and the ideas that were tested in our experiment can serve this purpose. The committee members felt that developing tutorials according to our suggestions is a good way to increase the students' motivation and interest in the course without compromising the abstract theoretical level of the course.

The committee recommended offering the engineering students an option of applied problem solving practice in the basic mathematics courses. Following this recommendation, our experiment was extended to the one-variable calculus course given to mechanical engineering students. In addition, the second author is developing a teacher training course on mathematical models in science and technology.

Another effort in this direction was made in teaching linear algebra. The teaching-with applications course was given by the first author. In the course the way of teaching applications differed from those tried in the experiment. Applied problems were an integral part of the lecture. Some of the problems were chosen from the major disciplines of the engineering students. Examples are: computing currents in electrical circuits (systems of linear inequalities); controlling chemical processes (eigenvalues); calculating air flow over wings. In the first two examples the knowledge acquired in the course enabled the students to perform the tasks. The solution of the third task was not presented in the course because it involves partial differential equations that the students had not studied yet. We decided to consider it in order to construct a bridge between the course and a central problem for aeronautical engineers. We hope that this connection can motivate students, especially aeronautical engineering majors, and help them to recall the material when facing this problem in future courses. We conclude the discussion of the Linear Algebra applied course, and the paper, with two additional examples.

An inquiry problem – the case of Google

Searching the Internet is an issue of general interest. We used this issue to develop an inquiry problem that motivates the study of eigenvectors. When the Web is searched for information about some concept, many sites may be offered. For this reason, it is of utmost importance to rank the usefulness of the sites. In the course we explained how the ranking is made by Google, using linear algebra (Berman, 2006). Accordingly, a site gets high rank if it is pointed to by high ranked sites. This is similar to the principle of tennis or chess player ranking. Let x_i denote the probability that a virtual surfer is in site i. Let $x = (x_i)$ be the probability vector which describes the surfer's location. Applying the ranking principle the students obtain a system of linear equations $Ax=x$, where A is a stochastic matrix. The vector x that satisfies the system gives the ranking of the sites. At this point we can only assure the curious students that there is a unique x without proving it, but we can use the obtained system to motivate the definition of eigenvectors. This is an example where the application preceeds the definition and perhaps makes the definition more natural.

Simple cryptography

This example is used to suggest an application of matrix inversion, extend the interest of the students in this operation, demonstrate the importance of the computing the inverse as a multiple of the adjoint and discuss the pitfall of round off errors. We start by choosing a 2×2 invertible matrix A and a message expressed in numbers that we want to send. We write the message as a matrix with 2 rows B and compute the product AB. The students, who know at this point of study the formula for inverting a

2×2 matrix, multiply $A^{-1}AB$ and read the message. Our experience is that a question follows: can the same be done with a 3×3 matrix or with bigger matrices so it is now natural to discuss the inversion of such matrices. Once this is done we show the effect of the computation accuracy on the results of the operations and conclude that it is recommended to choose the matrix A so that it and its inverse are integral matrices. We return to this point after the students learn the relation between the inverse of a matrix and its adjoint and choose A to be a product of a lower triangular matrix and an upper triangular matrix, both with all diagonal entries equal to 1. The advantage of this discussion is that the students appreciate the formula for inverting a matrix, that otherwise seems too theoretical.

Conclusions

The results of all three experiments were positive. Learning achievements of students who attended supplementary applications tutorials were significantly higher than those of other students. Through applications the students explore mathematical modeling cycles, and this practice resulted in better and easier understanding of calculus concepts, higher learning motivation and interest in the subject. The absolute majority of the students involved in the study supported integrating applications in the multivariable calculus course, recommended to continue teaching with applications in the future, and even extend this practice.

We used a representational model of perception in order to explain the effect of applications on mathematical understanding. We hope that our design experiment and the representational model can be used to continue the study of theoretical and practical aspects of teaching with applications.

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LEARNING THE INTEGRAL CONCEPT THROUGH MATHEMATICAL MODELLING

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We investigate how mathematical modelling activities can support students' learning of mathematical concepts. From experiences in developing, teaching, and observing students at work in a modelling course for first year university students, we argue that students' contextualised reflections in modelling situations are crucial in this respect. We illustrate how these types of reflections can be provoked and how they can enhance students' learning of mathematical concepts by analysing dialogues from a project, designed to support students' development of the integral concept.

INTRODUCTION

One of the main reasons given for teaching mathematical modelling is to support students' learning of mathematics (Niss, 1989), (Blomhøj, 2004). Through modelling, mathematics is used to describe, understand, predict, and prescribe the reality we live in, thus modelling can create connections between the students' extra-mathematical experiences and the mathematics involved in their modelling activities. Working through the modelling process students are challenged to use different aspects of their mathematical conceptions in various situations. Hence, it is of interest to design and evaluate modelling situations from the perspective of their potential to challenge the students' mathematical conceptions.

Accordingly, the objective for the (empirical) study presented in this paper is to answer the following general research question:

> How can mathematical modelling activities' potentials to support students' learning of mathematical concepts be realised through interactions with students during their modelling activities?

In what sense such learning potentials exist is discussed in a forthcoming paper. In this paper we briefly describe the mathematical modelling course that constitutes the empirical basis for our research. Then we discuss how our research has progressed through interplay between development of practice, theory, and pedagogical observations. To illustrate how our approach can provoke students to use the context in the modelling situation as a means for making sense of their mathematical conceptions we present a detailed analysis of a learning situation where groups of students used the concept of the integral to model the CO_2-balance in a lake. Finally, we provide some conclusions about our general research question.

THE EMPIRICAL BASIS – AN UNDERGRADUATE MODELLING COURSE

We have collected our data during eight years of co-operation developing and teaching a mathematical modelling course for first year university students at Roskilde University, Denmark.

The mathematical content of the course covers linear regression, calculus, numerical, analytical, and qualitative methods for solving and analysing differential equations (Blomhøj et al., 2004). All topics and concepts are discussed as tools for modelling. Besides demonstrating the relevance of mathematical modelling in the sciences, and contributing to the development of the students' mathematical modelling competency (Blomhøj & Jensen, 2003), the course is also designed to support students' learning of fundamental mathematical concepts and methods the last being our focus here.

The course is taught in classes of roughly 40 students with two weekly sessions of two and a half hours over two semesters. It is structured around six mini-projects, intended to make the students work on those parts of the mathematical modelling process that involves mathematisation, mathematical analysis, and interpretation and evaluation of results, see (Blomhøj & Kjeldsen, 2006). The students work on the mini-projects in groups of three for a period of three weeks, during which there will also be lectures, exercises and homework. For each mini-project the students can choose among a list of problems. The groups produce a written report. At the end of the term every student defends orally one of her/his mini-projects, randomly chosen.

INTERPLAY BETWEEN PRACTICE, THEORY, AND OBSERVATIONS

Our course differs from the Realistic Mathematics Education programme (RME) in that it is a university course and has the development of modelling competency as its main agenda (Treffers, 1987), (Gravemeijer & Doorman, 1999), (Gravemeijer, 1998). The recent work by Zbiek & Conner (2006) has a focus similar to ours but at a different educational level and context.

In general, the effects of modelling activities on students' conceptual learning are not that well researched. In our own work on how to carry out the learning potentials through interactions with students during their modelling activities we can identify three important issues related to the development of practice, theory, and pedagogical observations, respectively.

First, the mini-projects in the course are developed deliberately to create opportunities for challenging and enlarging students' understanding of the central mathematical concepts and methods in the course. However, the potentials for learning mathematics through the mini-projects do not realise themselves automatically. We need to pay careful attention to the learning potentials of each mini-project to notice when it can be pursued in the contemplations of a group of students, and to find ways to challenge students to engage in the relevant reflections. Therefore, the development of course material and teaching practise is an ongoing process that has benefited from our close collaboration and from having team-taught the course several times.

Secondly, the design of the course draws on theories about the teaching and learning of mathematical modelling (Niss, 1989), (Blum & Niss, 1991), (Blum & Leiß, 2006), concept formation (Sfard, 1991), concept images (Tall & Vinner, 1981), (Vinner & Dryfys, 1989), and the importance of the epistemological relations between symbol, concept, and object in learning of mathematics (Stienbring, 1989).

Thirdly, during dialogues with the groups we have recorded pedagogical observations that pointed out specific difficulties students had when they needed to draw on not well-developed images of their mathematical concepts. In such situations the students' focus changed from the task of modelling towards their own understanding of the involved mathematical concepts. Students experienced what Tall & Vinner (1989) refer to as a cognitive conflict in their evoked concept images. Besides creating frustrations, these situations contain opportunities for students to engage in certain types of reflections, situated in the students' modelling activity but directed towards their understanding of mathematical concepts.

In the following section we illustrate how students' modelling activities can create incidences of cognitive conflicts, where students can be challenged to engage in these types of reflections, which then eventually support their conceptual learning of mathematics. Moreover we show how these opportunities can be realised through dialogues between a group of students and a teacher that is well aware of the possible cognitive conflicts in the situation.

LEARNING THE INTEGRAL CONCEPT BY MODELLING ACTIVITIES

The mini-project "CO_2-balance of a lake" is designed to challenge students' understanding and images of central features of the concept of the definite integral.

Students' perception of the integral concept prior to the project

Prior to the project work we introduce the integral concept in the classroom with focus on its interpretation in different modelling situations and on numerical integration. To support students' understanding of the definite integral as a sum we use a considerable amount of time on numerical integration – by hand as well as by MatLab. In the "integration by hand"-exercises the students integrate by "counting rectangles". That this method challenges their concept image is clearly conveyed by the students' reactions towards this rough and "childish" method of integration. It makes them uncomfortable, and seems to go against their conception of calculus.

For many of the students their concept image is limited to the operation: find an anti-derivative, insert the endpoints of the integration interval in the anti-derivative, and subtract the two magnitudes. This limited concept image has the consequence that students will not be able to evaluate the definite integral of a function not given by an analytical expression or a function to which an analytical anti-derivative cannot be determined. Usually students have not experienced this consequence as a conflict, presumably due to their limited concept image of a function. It is our experience that in the beginning of the course students cannot imagine functions not given by an analytical expression. Nor can they imagine the existence of analytically given integrable functions for which analytical expressions for the anti-derivatives cannot be found. Also, most of the students are confused about the arbitrary constant *C* appearing in the general formula for the anti-derivative. When should it be taken into account and when can it be left out? Apparently, it is not included in the formula:

$$\int_{t_0}^{T} f(t)dt = [F(t)]_{t_0}^{T} = F(T) - F(t_0)$$

The mini-project of modelling the CO_2-balance of a lake is designed to challenge students' understanding of the concepts of the definite integral and the anti-derivative, the significance of the constant, and the interpretations of these concepts in different problem situations.

Modelling of the CO_2-balance of a lake

Biologists are interested in how the net rate of change of CO_2 in a lake changes over a 24 hours period, and on the basis of the data, partly displayed in table 1, students are asked to analyse the CO_2 balance in the lake.

Hours after dawn	CO_2 mmol/l/hour	Hours after dawn	CO_2 mmol/l/hour
0.666	-0.027	12.666	0.028
1.333	-0.048	13.333	0.058
…	…	…	…
12.000	0.000	24.000	0.000

Table 1: Data express the rate of change in CO_2 concentration (mmol/l/hour) over a 24 hour period. There were 2.600 mmol CO_2 in the lake at dawn.

Through a number of helping questions and by means of challenges given to the groups in dialogues with the teacher during their modelling work we establish a framework for the students' work. To give a first impression of the situation the students have already been prepared to produce a plot of the data (see figure 1).

In the first part of the mini-project the help-questions are designed to make the students interpret the plot and realise what information they can read off directly. That the sign of the rate of change is negative during the day and positive during the night is straight forward, but what that says about the CO_2 content in the lake is not as clear. The students' problems come from two sources: (1) they are not completely sure about the mathematical relationship between the CO_2 rate of change and the CO_2 content, and (2) they do not really understand what the CO_2 rate of change tells them about the system.

The CO_2 rate of change in mmol/l/time

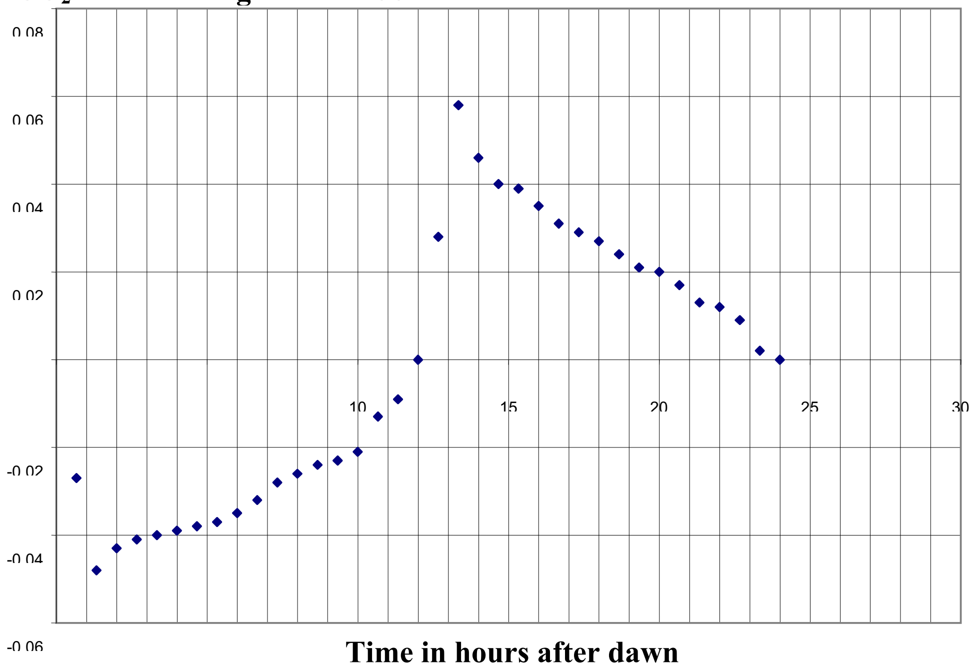

Figure 1: Shows a plot of the data indicated in table 1.

We often see them confuse the graphical picture of the CO_2 rate of change displayed in figure 1 with the CO_2 content. To deduce facts about a function from a graphical representation of its derivative is not strongly represented in the students' conceptual framework. At this stage the students do not think of the CO_2 content as an anti-derivative to the CO_2 rate of change. Instead they think of the CO_2 rate of change as the derivative of the CO_2 content. Their concept images of these two different ways of looking at the relation between the CO_2 rate of change and the CO_2 content is not integrated at this point. One of the purposes of this modelling project is to create situations where students get opportunities to develop links between these relations.

In the second part of the project students are asked questions such as: When will the CO_2 content in the water be at its lowest, and how much CO_2 will be in the water when that happens? Is the lake in equilibrium with regard to the CO_2 content? How can this question be decided graphically? How much CO_2 was released to the water during the 12 hours of night and how much was removed during the 12 hours daytime? These questions all require the students to regard the CO_2 content as an anti-derivative of the CO_2 rate of change. They force the students to shift focus and view their plot of data as a plot of a function whose anti-derivative they are searching information about. At this point, only a few students have a concept image so well developed that it does not matter to them whether they use one way of considering the data plot or the other. For most students however, there is a considerable difference in the two points of view. Since the students only have a discrete set of data they experience the numerical integration method of "counting rectangles" as a valuable and sensible tool that can answer relevant and significant questions.

The use of pedagogical observations

We have made extensive use of pedagogical observations as a tool to get a deeper insight into students' learning processes and problems of learning various mathematical concepts. Modelling projects are an excellent context for this tool because they effectively create situations where the observer can get insights into the challenges facing students as they try to understand and learn. The modelling context stimulates students to talk about the mathematics, formulate problems and questions of a mathematical character, and test their understanding in group discussions with each other. Below we will show in detail how we have used pedagogical observations to get insights into and challenge students' understanding of the integral concept.

The object in the third part of the project is to address both students' perception of the definite integral as an accumulated sum and the problem they have with understanding the role of the arbitrary constant C. Students are asked to produce a graph of the CO_2 content as a function of time. For most students this either creates a conflict with their concept image or reveals a huge gap in their understanding. They realise that they need to integrate the function (the CO_2 rate of change) represented by their plot of data, but that is followed by their confusion caused by vague or missing connections between the concept definition and their concept image of the definite integral, as well as their problems with the distinction between the definite integral and anti-derivative functions.

The following dialogues are reconstructed from our experiences and observations over time with groups of students working on the project:

The teacher (T) has been summoned by the group. The students (S) are stumped.

T: What are you being asked to calculate?
S: The CO_2 content of the lake.
T: Correct, how can you determine that?
S: Well, we know the rate of change …
S: That means we know f' …
S: Is it f we have to find? (Everyone looks at the teacher)
T: I don't know. It is the CO_2 content you are asked to find.
S: We know the velocity so it is f we have to find.
S: We must integrate.
T: Yes, what is it you need to integrate?
S: Our function (points at the plot of data).

Then the teacher leaves. After a while the teacher is called back in:

T: How is it going? Have you integrated your function?
S: No. We can't.
S: We don't have an expression for the function.
T: OK – but aren't there other ways to integrate?

S: Yes, numerically, and we did that, but we got -0.025 which cannot be true. The CO_2 content cannot be negative.
T: How did you arrive at this number?
S: We integrated the function by counting rectangles. The ones below the ordinate axe are negative.

Here the students clearly have problems understanding how the integral concept is connected with the problem of determining an anti-derivative.

T: Let's try to focus on what it is you have to find.
S: We have to find the integral.
T: Yes, but why is it that you want to integrate?
S: … ??? …
T: What does it say in the text?
S: Oh yes, we have to find the CO_2 content.
T: Yes, when?
S: … ??? … What do you mean?
T: Is it at midnight … or … ?
S: Yes … no … it is at dawn.
T: Where does it say that?
S: Ehh .. it doesn't.
T: No, what does it say?

Rereading the text they realise they must find the concentration as function of time.

S: But then we have to calculate many numbers!
T: Yes.
S: But the integral is just one number … right? How can it then become a whole function?
T: How will you estimate the CO_2 content at time 0.666?
S: Integrate.
T: Yes, but what?
S: The function – the rate of change.
T: Yes, but where from and where to?
S: Ohhh – from 0 to 0.666.
T: Yes.

The teacher leaves again – and is called back in after a while.

S: We still don't get it. Do we need to count rectangles all over?
T: What do you mean by "all over"?
S: We have done it for the first data point from 0 to 0.666. And we also did it for the second data point from 0.666 to 1.333. Is that then the CO_2 content at time 1.333?
T: That is a good question.

After some discussion the group realises that they must add the first integral from 0 to 0.666 to the second to get the CO_2 content at time 1.333. First at this point, do the students really understand that they can tabulate an anti-derivative (the CO_2 content) by successively adding "the next column" of rectangles. They realise why there is no conflict between the definite integral being one number and the possibility of tabulating an entire anti-derivative function by integration.

S: That is a huge calculation …

T: Yes, but the function to be integrated is tabulated in the data set, so maybe you can use one of your MatLab programs.

S: Oh yes, numerical integration.

After such discussions most of the groups are able to take over their modelling work to produce the requested graph, to use it, and to reflect on it in relation to biological questions about the lake.

One particular group decided to use Excel to integrate instead of one of their MatLab programs, because they felt that in Excel they were able to keep track of what was added when and to what. It is noteworthy that this group realised by themselves that they had to add the initial amount of CO_2 in the lake, and that this amount was the constant *C*. Some of the other groups at first hand overlooked that, and found negative amounts of CO_2 in the water, which gave rise to comments about the feasibility of these results in relation to the context.

DOCUMENTATION AND CONCLUSION – WHAT DID THEY LEARN?

In this mini-project students' learning of mathematics is supported through the connections the students are challenged to explore between the concepts of the derivative function, anti-derivative functions, and the definite integral. Through this project students develop abilities to apply these concepts in particular contexts. The project enables them to juggle with the concepts and to interpret them in relation to each other and in relation to a perceived reality.

The project demystifies the arbitrary constant *C* in the formula for the anti-derivative. Through area-observations students are able to answer many questions regarding amounts of CO_2 being removed or added to the lake. In such questions the constant *C* plays no role. But when it comes to determine the actual content of CO_2 in the lake at a given time the constant and its interpretation play a crucial role. The constant determines the actual level of CO_2 and that idea is encountered in the third part of the mini-project, where students realise that they cannot solve the problem without taking the initial level of CO_2 in the lake into account. This experience enabled them to read new meaning into the general formula for anti-derivatives interpreting the constant *C* as the value of the function $F(t)$ at some chosen time $t = t_0$. It is within the modelling context that this becomes clear for students. They found that without the constant they calculated only increasing and decreasing amounts of CO_2, and because they calculated with signs they ended up finding negative amount of CO_2 in the lake. This contra-intuitive result was what triggered their reflections.

Students' understanding of the concept of the integral got connected to the definition of the concept, and they experienced the usefulness of integrating by counting rectangles. This is clearly demonstrated by the group of students who used Excel to tabulate the function of the CO_2 content. This group of students understood the integral concept at a deeper level than before the mini-project, and at the end they were able to transfer the definition of the definite integral to new modelling situations. Students' reflections are contextualised, and therefore one should not presuppose transfer of students' conceptual understanding of the integral concept developed in relation to a particular mini-project to other modelling situations. The question of transfer has to be investigated separately. However, we have some soft evidence showing that students who have worked with the CO_2 mini-project afterwards are able to use the integral concept in other modelling projects.

Mathematical modelling provides opportunities for students to learn mathematics in different ways. Here we have focused on situations where the students reflect on their mathematical understanding in activities related to mathematisation, analysis of the mathematical system, and the interpretation of the results. These types of reflections are situated in students' particular modelling activity and the learning potentials of such reflections are triggered by students activating different aspects of their conceptual knowledge, experiencing possible conflicts in their understandings, and enlarging their concepts images by connecting the concepts to their modelling activities.

In course planning and developing of modelling projects, one can look for such learning potentials. It is our experience that the projects have to be genuine modelling projects in order for the students to engage in reflections about their mathematical understandings. However, we recognise that the student-teacher interaction during the modelling activity is absolutely crucial.

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PERSONAL EXPERIENCES AND EXTRA-MATHEMATICAL KNOWLEDGE AS AN INFLUENCE FACTOR ON MODELLING ROUTES OF PUPILS

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Abstract: *In this paper one aspect of my current study (COM²-project) will be presented. The phenomenon or rather the term of "individual modelling routes" will be clarified and exemplified by means of two different modelling routes of pupils. The focus lies further on the analysis why these routes are different. Two main reasons could be reconstructed, which is firstly experiences and secondly mathematical thinking styles of the pupils. Concerning these aspects it will be shown how these modelling routes look like, that means which phases between reality and mathematics were preferred while solving the modelling task.*

Introduction

The frame of the presented results in this paper is the COM²-project (**Co**gnitive-psychological analysis of **M**odelling processes in **M**athematics lessons)[1]. This study tries to close a part of the lack regarding research on modelling under a cognitive perspective. In the COM²-project contextual mathematics lessons are analysed from a cognitive-psychological perspective and issues from the Ph-D thesis on mathematical thinking styles are taken up (Borromeo Ferri 2004) in which learners' different individual mathematical thinking styles were reconstructed.

In this paper I will show one aspect, which became more and more interesting during my data analysis. At this point I have to make clear that it was not a goal in my study to analyse pupils' experiences in connection with their modelling process. This was also not a research question. After reconstructing and characterizing the so-called "modelling routes" of pupils, I compared these modelling routes. During these analysis I could reconstruct, that experience/extra-mathematical knowledge of pupils' is beneath mathematical thinking styles an influence factor, why modelling routes can differ.

Because of that, it is important to know, what is meant by the term modelling route and to get an overview about the state of the art of "modelling under a cognitive perspective.

1 Theoretical framework

1.1 Short overview about "modelling under a cognitive perspective"

[1] Study was supported by the German Research Foundation as a Post-doc-project within the Graduate School "Learner Development and Domain Specific Educational Experience" (University of Hamburg).

Within the didactic literature on modelling there exist only a few studies, which have their focus on cognitive processes of pupils while modelling. Treilibs, Burghardt and Low (1980) focused on the aspect, how an individual builds a model (the so called "formulation phase"). Consequently they did not examine the complete modelling cycle. But this was the first step going more on a micro level analysing individuals modelling processes. Also Matos' and Carreira's (1995, 1997) research for example puts a special emphasis on 10th grade learners' cognitive processes. They analysed transition processes from reality to mathematics and backwards. But they did not reconstruct the transitions between the single phases, for example between real situation and real model or mathematics results and real results. This is one claim of the COM²-project.

Galbraith' & Stillmans (2006) new study about identifying blockages during transition processes of pupils has also a cognitive approach. They turned their attention on the kinds of mental activity that the individuals engaged in as they moved around the modelling cycle. A similar interest has the DISUM-project (Blum/Messner/Pekrun) (see Blum/Leiss 2003). They try to reconstruct cognitive barriers of individuals while modelling next to other aspects.

Nevertheless and to conclude briefly: the so called "cognitive modelling", seen as a meta-perspective within classification of current perspectives on modelling (Kaiser & Sriraman 2006), is neglected in the modelling discussion. To resume this short overview a characterisation of "modelling analysed under a cognitive perspective" will be given, which shall make the main goal of these studies more transparent:

If modelling is considered under a cognitive perspective the focus lies on the (individual) thinking processes which are expressed more or less through actions while modelling process.

1.2 Modelling cycle under a cognitive perspective

The modelling cycle under a cognitive perspective (Borromeo Ferri 2006, Borromeo Ferri, in press) which is shown below is a central instrument for analysing the modelling processes of pupils. I adapted this cycle from Blum/Leiss (2005) for the purposes of my study.

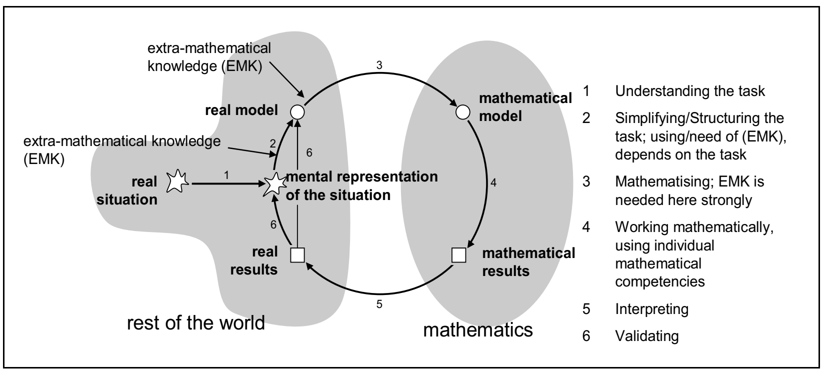

fig. 1: Modelling cycle under a cognitive perspective (Borromeo Ferri 2006)

Blum/Leiss (2005) used the term situation model, which is a well known term in connection with word problems (Kintsch & Greeno 1984, Reusser 1997, etc). But I replaced it for the term *mental representation of the situation* (MRS), because in my sense it better describes the kinds of internal prozesses an individual can have while reading and understanding a modelling task. Also MRS has fewer connotations to "word problems". Concerning these phases of the modelling cycle I wanted to analyse pupils' modelling processes on a micro-level. According to that claim my *main* research questions are:

1.3 Research questions

1 What influences do the mathematical thinking styles of the learners' and teachers' have on modelling processes in contextual mathematics lessons?

2 Can differences between MRS, real model, mathematical model and the other phases (as described in the didactic literature on modelling) be reconstructed from the learners' way of proceeding?

2 Methodological framework

In this chapter I will only briefly describe the methodology and design of the whole study and I will shortly explain how modelling routes were reconstructed.

COM²-project is a qualitative study which combines classroom research and analysis of single individuals respectively groups of pupils within these classes. For the investigations three 10th grade classes from different Gymnasien (German Grammar Schools) were chosen. The sample includes 86 pupils (65 pupils in the first phase and additional 26 pupils in the second phase of data collection) and three teachers (two female, one male).

The used modelling tasks are of central importance as they delineate the field of analysis and were taken from the DISUM-project. They were analysed under subject matter aspects and from a cognitive viewpoint.

All lessons including group works were videotaped and transcribed. Firstly, statements of the pupils were analysed concerning the aspect, in which phase of the modelling cycle they worked on. For every single pupil the statements were coded (Strauss & Corbin 1990), so that an individual modelling route could be reconstructed. Comparing these modelling routes one coded statements of each individual in which they were talking about their own experiences and extra-mathematical knowledge in connection of the given situation of the task. I did not interviewed them about their experiences to the task, I analysed, if pupils' were talking about this intuively.

3 Selected results of the study

Here I will not show all results of the study with regards to the focus of this paper. So one aspect will be presented and described in detail.

3.1 Individual modelling routes

As I said before, beginning with data analysis it was not a goal to reconstruct "individual modelling routes" less was the term in my mind. During the process of analysing different phases in the modelling process of pupils it became much clearer that they switched more or less between phases and that some of the modelling process were similar in some way, but some not. So I made the decision not to speak from modelling processes but from individual modelling routes, which is characterised as follows (see Borromeo Ferri 2006):

"Modelling route is the individual modelling process on an internal and external level. The individual starts this process during a certain phase, according to their preferences, and then goes through different phases several times or only once, focussing on a certain phase or ignoring others.

To be precise from a cognitive viewpoint, one has to speak of visible modelling routes, as one can only refer to verbal utterances or external representations for the reconstruction of the starting-point and the modelling route."

After reconstructing 20 modelling routes of learners I set up the hypothesis that different modelling routes are a result of different mathematical thinking styles of the pupils. In sense of theoretical saturation (Strauss & Corbin 1990) after analysing the remaining 15 modelling processes of the learners[2] I am able to formulate a result, which is more on a solid base and includes further aspects:

- Phases of the modelling process (which are described on a normative way in the didactic literature on modelling) can be distinguished and described empirically (for more details see Borromeo Ferri 2006).
- Modelling routes of pupils are different, because of influence factors on several levels. This is not a hierarchy, further more modelling routes result from a combination of these levels: (1) Level of Mathematical thinking styles, (2) Level of mathematical competencies and (3) Level of extra-mathematical knowledge and own experiences.

In this paper I will only describe (1) and (3) in more detail, which is the basis for understanding the statements of the pupils in the example later on.

(1) Main influence factors are mathematical thinking styles (visual, analytic, integrated) of pupils. Visual thinkers for example differ in their modelling routes compared with analytic thinkers. Visuals are more reality-based. They need the mental picture of the real situation or real model. That's why they often switch back, when they are between mathematical model and mathematics results. Their

[2] Per class and per lesson a group of five pupils were videotaped, so that 35 pupil altogether were in a special focus. Modelling routes of each individual were reconstructed with respect of the influence of the group. Methodological basis for this way of analysis is Personality and Social Psychology; see especially Slavin (1995) and Johnson/Johnson (1999).

argumentations are also very visual although they working on the mathematical model.

(3) It could be reconstructed, that strong extra-mathematical knowledge and experiences of pupils have influence on their modelling route in the following way: apart from affects, which are mostly connected with experiences, extra-mathematical knowledge contributes to determine the result of the given problem very exactly. This is a conclusion of the knowledge one has about the situation more or less in detail.

3.2 Daniel, Andreas and the level of extra-mathematical knowledge

In this chapter I will describe more in detail and with an example of two pupils, Daniel and Andreas, what is meant with the influence of extra-mathematical knowledge pointed out in (3). Of course, as said before, this level is a combination of at least one level more.

I selected Daniel and Andreas for this illustration, because both have different mathematical thinking styles (Daniel: Analytic thinker, Andreas: integrated thinker) and both have different experiences and knowledge about the given situation in the task. Andreas' experiences concerning the situation in the task were on a deeper emotional level, because this was a part of his life. Daniel and the other group members had not such experiences and extra-mathematical knowledge.

Both worked in a group with three other pupils on the following modelling task:

At the end of the summer one can see a lot of straw bails. Straw bails on the picture above are piled up in this way that in the bottom line are five, in the next four, then three, then two and on the top one ball.
Try to find out exactly, how high this mountain of straw bails is.

For a better understanding of pupils' modelling routes a short normative analysis of the solution process regarding different phases is given:

The real situation of this task becomes very visible through this picture, so a mental representation of the situation is perhaps very close to it. The individuals try to understand, that they must determine the height of the straw bail mountain. Simplifying and structuring this mental picture more on, one can think of circles instead of straw balls and perhaps draw them down for a real model. In this phase also the woman sitting on the straw balls can take into account for solving the problem. The level of extra-mathematical knowledge in this task is not very high, how it is in other problems, in which it is for example implicit asked after the diameter of the earth. Anyway one can suppose that a lot of pupils (living in the city) have seen these straw bails from a distance while driving in the car, but sitting on it or touching them is surely not the rule. Perhaps it is easier to know the height of a woman as the height of a straw bail. Both aspects can be used as a step to build a mathematical model: Woman's height of approximately 1.7 m can be piled and then added up to get the height of the straw bail mountain. Another way is using Pythagoras Theorem by estimating the height of one straw bail of approximately 1.5 m. Taking into consideration those straw bails in lines below will be pushed because of the weight of the other bails; one can build a model with a radius of only 3/4 of a straw ball. Inner-mathematical competencies such as Pythagoras Theorem or fractional arithmetic are needed to get mathematical results of about 6.7 m height. These results must be interpreted concerning the given problem to get real results. Validating real results means then to compare these with the mental representation and real model one had at the beginning.

Daniel and Andreas were very active within the group and made a lot of annotations although they worked not often on modelling problems in math lessons. That is why Daniel said at the beginning: " *It is not possible, because we have no numbers to calculate!"*

In the following I will describe Daniels and Andreas modelling route together with focus on their extra-mathematical knowledge/experiences and mathematical thinking styles[3]. So the first level was to analyse in which phase of the modelling process the individuals had been. It makes sense to know that concerning the aspect, at which point for example Andrea is telling or explaining about his experience or extra-mathematical knowledge.

Short after Daniel remarked, that numbers are needed to solve this task he had a decisive idea, which he formulated as the first of the group*:" You have to think about the height of the woman."* In this statement one can reconstruct a mental representation he has of the situation. He did not simplify the problem so far. Then he said to a girl in the group*:"The woman is perhaps as tall as you."* Here one can speak of a real model, because he structured the problem strongly. The interesting thing after this was, that Daniel repeated*:"Yes, but we have not a*

[3] Quotes made by pupils during the videotaped modelling process can only be an exemplary illustration of a change of modelling phase. The modelling processes are too long and complex to give an account of all the utterances in detail.

single number!" As an analytic thinker he focused on facts and numbers, which were not given in the problem.

This was not a problem for Andreas in the first minutes working on the task. As an integrated thinker he combines elements of visual and analytic thinking styles. So short time after Daniel repeated the statement Andreas said to the group members: *"Say, can you imagine that the woman is now standing up? Yes?"* This is really a wonderful example for MRS and Andreas continued*:" You have to think about the woman, that she stands up now, that means, she must be as big as the straw bail."* It is interesting how Andreas made his mental representation so visual for all group members.

Directly after that, Daniel came to a mathematical result, which he estimated*:" I would estimate that the height is 10m altogether, because a straw ball has a diameter of 2m."* So he built a mathematical model on an implicit level and used addition as an inner-mathematical competency. But most of the group members were of the opinion, that it must be less than 2m. So the group validated Daniels result in a certain way and he made again a comparison: *"How tall are you Julia?"* and started a new modelling process beginning with a real model. Julia said that she is 1.65m tall. Daniel concludes that the woman must be 10cm taller than Julia and tried to convince the others, that one straw bail is higher than the woman.

Andreas was not sharing that opinion and answered: *"No!"* This "No" was the beginning of Andreas' verbalised knowledge about straw balls, which he firstly did not express. He then began to measure the straw ball with his lineal. Although Andreas was in the phase of mathematical model his arguments were on a visual level:" *If one image that it is 10cm, then it is 1.50m."* Daniel answered with a good idea and switched back to real situation: *"You have to think, that these straw bails sink down! They don't really lie on top of each other, they glide in the gaps."* Of course this idea was from Daniel and this was again an impulse for Andreas to tell more about his extra-mathematical knowledge.

Andreas said*:" Yes, air must come through the straw balls and they are not stiff! If you cut the straw ball here that will be a quarter."* On the basis of this statement Andreas and Daniel and some of the other group members calculated and discussed about rounding up their results. But Andreas wanted to determine the height more exactly and tried to convince the others that is must be less because of the fact that the straw balls sank down. His knowledge was on an implicit level up to now. Following conversation makes clear that his experiences have influences on his modelling route and therefore on his transition processes between phases of reality and mathematics:

Julia: "I'm not sure, which effects it really has, if these straw bails sink down."

Susi: "I don't think that straw bails sink down so heavily."

Daniel: "Have you ever been on top of one straw bail?"

Julia: "In grade 5 we made an excursion and I climbed on top of such a straw bail."

Of course Julia wanted to make clear, that she has experiences. But for Andreas was that not enough and he answered: *"I grew up on a farm, don't tell me anything!"* After that, Daniel came again to a result of 6m, which was not interesting for Andreas. He wanted to talk about what happens, if these straw balls will become wet. Further on he explained other group members the difference between hay and straw. Through this knowledge and experiences it became clear, why he wanted to determine the height of the straw ball in such an exact way. For him these were real results, far more as for the others.

To summarize *all* the statements and actions of Andreas and Daniel while modelling, they will present as individual modelling routes within the modelling cycle below. There changes in the phases will become visible, but not how long pupils are in one phase.

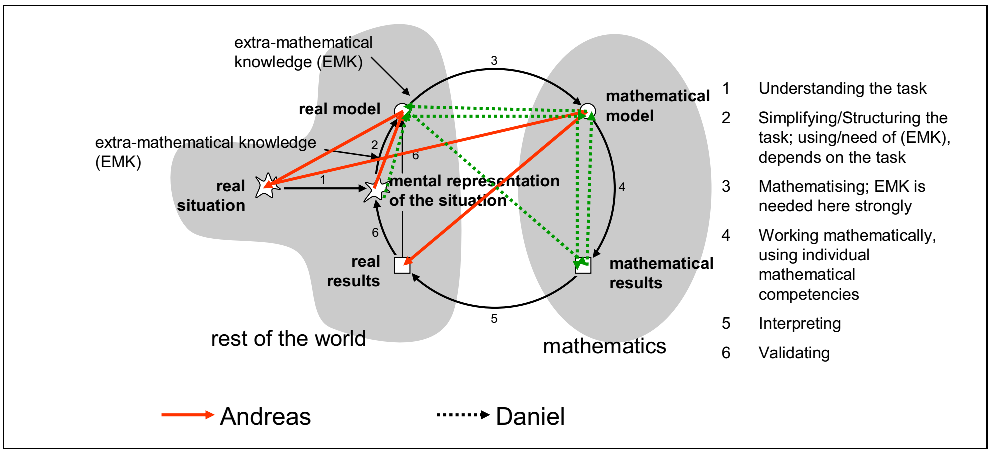

fig.2: Individual modelling routes of pupils (RS: real situation; MRS: mental representation of the situation; RM: real model; MM: mathematical model; MR: mathematics results; RR: real results)

I will shortly describe the modelling routes: Andreas spent a long time in reality at the beginning by switching from MRS to RM and again to RS before he went into mathematics. Daniel switched fast from RM to MM and even to MR, as described before. Then he needed time back in reality to modify his RM for creating in new MM and finally new MM, which he interpreted.

4 Summary and Discussion

The example showed that there are factors, which can influence modelling routes of pupils. The influence of extra-mathematical knowledge/experiences is such a factor. Andreas as an integrated thinker switched often back to reality, because he had his experiences in addition. His clear imaginations of the real situation let him determine the result very exactly. Daniel as an analytic thinker had more the idea to estimate his result. He had fewer experiences with straw balls and so not a clear picture of a straw ball like Andreas has.

I want to emphasise extra-mathematical knowledge as a special influence factor next to the others, because it came up more and more while analysing my data, although it was not a goal of the whole study to figure that out in detail. We know from several studies (e.g. Busse 2005) that context of a task influences the modelling process of pupils. But it is not my goal to point this out again. Further more I see experiences of individuals as an enrichment of their modelling route in various ways and as a strong factor compared with mathematical thinking styles. To make it more concrete: It could be reconstructed that analytic thinker switch very fast into mathematics and stay there. Visual thinkers switched more often back and forward between reality and mathematics. But if now an analytic thinker has a lot of experiences/extra-mathematical knowledge concerning a special task, does his modelling route changes? Is he still working more in mathematics or more in realty or does he also takes his knowledge to determine the results on a higher mathematical level? These questions came up to myself, as I analysed my data and recognized that extra-mathematical knowledge must be a factor and that I can’t let it outside. So I reconstructed that analytic thinkers who have strong extra-mathematical knowledge in connection with a task used really more imaginations of reality than in a task, in which he could not involve experiences and extra-mathematical knowledge. There is another aspect, which came up to me: In chapter 3.1 (3) I pointed out that experiences are connected with affects. Of course extra-mathematical knowledge can be a result of experiences, but not absolutely. So (personal) experiences can be seen closer to affects and emotions. If I take this in consideration I am leaving a little bit the area of cognition, which I not want to do.

Besides all these reflections I will finish with a more general remark: I think that the phenomenon of individual modelling route and the reason why they can differ is a possible diagnostic instrument for the teacher in two ways: (1) for a better understanding of pupils behaviour while modelling and (2) for himself while dealing with modelling problems in classroom.

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MAKING MATHEMATICAL LITERACY A REALITY IN CLASSROOMS

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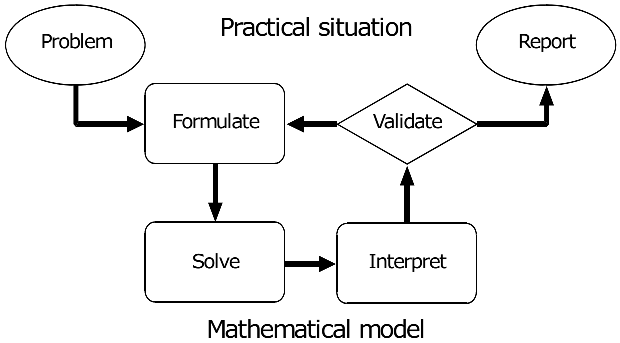
Abstract

Modelling of new problems is at the heart of mathematical literacy, because many situations that arise in adult life and work cannot be predicted, let alone taught at school. There are now plenty of examples of the successful teaching of modelling at all levels – yet it is to be found in few classrooms. How can every mathematics teacher be brought to teaching modelling reasonably effectively? This paper discusses how progress may be made, illustrating it with examples of „thinking with mathematics" about everyday life problems of concern to most citizens. It discusses the role that curriculum materials, professional development and various kinds of assessment may play, together with the challenges at system level. There are some reasons to be optimistic.

1. Background: the story so far

In a recent paper (Burkhardt, with Pollak 2006), we reviewed the history of the teaching of modelling in school mathematics curricula, focusing on developments in the UK and the US. The early explorations in the 1960s were followed by twenty years of more systematic development, so that by about 1990 there were proof-of-concept courses and course components of various kinds across the age range 10-21. These demonstrated that typical teachers can teach modelling skills if they have well-engineered teaching materials and some, relatively modest, professional development support. Students in these courses demonstrate a power over practical problems, from real-life or'fantasy' worlds, in which their mathematical toolkits play an important role in the analysis and reporting. They handle, for non-routine problems of appropriate complexity, the various phases of modelling shown in the diagram, and not only the *solve* phase on which school mathematics is normally focused.



Because these are *switch-on effects*, where students are showing kinds of performance that are new to school mathematics, evidence of progress does not require tightly structured research studies. Further, the social value of the skills involved is obvious, and rarely questioned[1]. The change in student motivation when working on real-life problems is equally dramatic.

The importance of these clear qualitative gains have kept the focus of work so far on development rather than insight-focussed research in depth, on the *engineering* rather than the *science* of the teaching and learning of modelling. There have been a few studies in greater depth with some interesting results, such as Vern Treilibs detailed study of formulation processes[2] (Treilibs et al. 1980). Among other things, it documented the „few year gap“ between the mathematics students can do in imitative exercises and those that they choose and use when modelling (some recent studies suggest that this gap is narrowed by teaching modelling). These examples underline the need and opportunities for research to provide further insights into the processes of modelling, how students learn the skills involved, and how teachers can help them. I hope that some studies will focus on design research that can help the field move practice forward, rather than simply academic studies (see Burkhardt 2006).

In summary, we know how to teach modelling, have shown how to develop the support necessary to enable typical teachers to handle it, and it is happenng in many classrooms around the world. The bad news? „Many“ is compared with one; the *proportion* of classrooms where modelling happens is close to zero.

Why is this, and how can the situation be transformed so that modelling is a feature of the mathematics curriculum for every student – the prerequisite for mathematical literacy? I shall first look at what we mean by mathematical literacy, its importance as a life skill, and its role in making mathematics itself meaningful and useful to most people. Then I shall list *barriers* that obstruct the large-scale implementation of modelling and, indeed, other curriculum improvements, linking these to various *levers* that promise progress.

From a societal perspective, the school mathematics curriculum is worse than regrettable; it is scandalous. Currently *most people in their adult lives use none of the mathematics they are first taught after age 11*. Further, study after study has shown that school mathematics gives them none of the aesthetic satisfactions that people get from, say, music or literature. Modelling is the missing ingredient.

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[1] Though some pure mathematicians argue that modelling should be deferred until more mathematics has been learnt – indeed, to a stage that most students never reach.

[2] 120 students age 17 of high ability in mathematics, but untutored in modelling, were tested on real world problems. Despite 5 years successful experience in algebra, none used it for modelling; though it seemed the obvious tool, they chose to rely on more elementary methods: numbers, tables and, sometimes, graphs.

2. What is mathematical literacy?

Many different terms are used in various places and circumstances: *mathematical literacy* (ML) is the most widespread, *quantitative literacy* is favoured in the US, *functional mathematics* is now fashionable in the UK, while *numeracy* was originally defined as „the mathematical equivalent of literacy". Distinctions between these terms are not widely agreed; for our purposes, they are unimportant.

PISA (Programme for International Student Assessment, OECD 2003) defines ML:

> Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.

However, such verbal descriptions on their own are ambiguous, particularly across countries and cultures – they are easy to re-interpret in terms of one's own experience. It is useful to complement them with examples – in education, of the kinds of task that represent learning goals. The following illustrate what I (and many others) mean by mathematical literacy. I begin with a PISA task.

ROCK CONCERT *M552Q01*

For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

A. 2 000

B. 5 000

C. 20 000

D. 50 000

E. 100 000

While the length and multiple choice format are limiting, this kind of ‚back of the envelope' estimation is central to ML. So are the following types of task.

MAKING A CASE

The spreadsheet contains 2 sets of reaction times – 100 each for Joe and Maria.

Using this data, construct and justify two arguments:

A: that Joe is quicker than Maria, **and**

B: that Maria is quicker than Joe

SUDDEN INFANT DEATHS

In the general population, about 1 baby in 8,000 dies in an unexplained "cot death". The cause or causes are at present unknown. Three babies in one family have died. The mother is on trial for murder.

An expert witness says:

"One cot death is a family tragedy; two is deeply suspicious; three is murder. The odds of even two deaths in one family are 64 million to 1"

Discuss the reasoning behind the expert witness' statement, noting any errors, and write an improved version to present to the jury.

PRIMARY TEACHERS

In a country with 60 million people, about how many primary school teachers will be needed? Try to estimate a sensible answer using your own everyday knowledge about the world. Write an explanation of your answer, stating any assumptions you make.

HOW RISKY IS LIFE?

"My parents won't let me go out on my own. They think I'll be mugged, or run over."

"My sixty year old granny is terrified by the stories she reads in the newspapers. One day she is afraid of being assaulted, the next she is frightened of terrorists."

What do you think? Collect and use data on different causes of death to estimate the chances of people becoming a victim of these and other events. Compare the likelihoods of these events with each other, with other risks, and with the 'base' risk – the probability that people of different ages will die in the next year.

It is clear from the above that mathematical literacy involves complex reasoning, linking models of the situation to data. Lynn Steen (2002) describes it as „The sophisticated use of elementary mathematics“, in contrast to school mathematics.

3. Barriers to large-scale improvement

Here I shall list some of the key implementation challenges we face. These are discussed in more detail in (Burkhardt with Pollak 2006) and Burkhardt (2006)

- **System inertia**: The limited large-scale implementation of modelling is not unique; it has proved difficult in many countries to establish *any* profound innovation in the mainstream mathematics curriculum. This should not surprise us. Teaching modelling requires changes in the well-grooved practices of teachers, their teaching skills, and their beliefs about the nature of mathematics – and those of parents and politicians. To become part of the mainstream curriculum, it is not enough to be "good" and "important".
- **The real world** is an unwelcome complication in many mathematics classrooms. The "purity" of the subject is something that attracts people to teach mathematics; for them, using mathematics to tackle real world problems is not their job. (First language teachers welcome the motivation it provides)

- **Limited professional development** In many countries teachers are expected to deliver a curriculum on the basis of the skills they acquired in their pre-service education, consolidated in early years in the classroom. In a changing world, continuing professional development is essential but in most countries is not yet an integral part of most teachers' week-by-week work.
- **The role and nature of research and development** in education, as compared with other applied fields, is not well organised for turning research into practice. Burkhardt and Schoenfeld (2003) looked at how this process can be improved, learning from research-based improvement in medicine, engineering and other fields. The growing role of ‚design research' in education is a move in this direction but more is needed if policy makers with problems are to turn to the research community to solve them.

The research and development agenda that these barriers imply is huge and work on it is at an early stage. Here I can only sketch some of the key ingredients that are likely to be important in establishing modelling. They are all worth working on.

4. The importance of communication

The story of modelling in school mathematics is one of mutual incomprehension between leaders in mathematical education and those they seek to serve. The public and most politicians see mathematics as "What I learned at school". The mathematical limitations of many students, which they regularly deplore, are seen as a failure to make every child mathematically ‚like them'. The changes in the mathematical skills that society needs are acknowledged, but their implications are not understood. This needs greatly improved communication[3].

Contributions to the media are the first area that needs attention.

- These need to explain and illustrate the changes. The mathematics curriculum is still focused on developing reliable technical skills in well-practised procedures; everywhere except in schools, these are now performed by technology. In this more technical world, where computers do the routine things that clerks used to do, people need a broader range of higher level skills so as to be flexible problem solvers who can handle change.
- This is not an easy communication challenge – people don't want to read about mathematics, so media are reluctant to publish such pieces. Skilled writers of ‚popular science' can provide help.
- Assessment tasks can be useful tools – they communicate new goals in a vivid and compact form, bringing to life verbal explanations; otherwise these are interpreted within each reader's experience.

[3] It is a salutary exercise to try to trace the paths by which your (excellent) research might influence practice in typical classrooms, who is responsible for each step, and how likely it is to get through. What changes in research might make its influence more direct? (see Burkhardt and Schoenfeld 2003)

Meetings with policy makers, both politicians and their senior civil servants, are crucial to improving the communication process. In addition to the above kinds of input, they will respond well to:

- Suggestions that are aligned with their existing policies – look hard for elements of declared policy to which you can attach the initiative you want, and adapt your proposals to maximise the alignment;
- Evidence on the learning outcomes from curriculum components of a similar kind that have been tried elsewhere, from evaluations and/or independent research studies;
- More comprehensive and detailed descriptions of the proposed changes, preferably with examples of assessment tasks, lesson materials and the professional development needs and methods;
- Estimates of the likely costs of development, and of implementation – it pays to offer alternative models, with varying scales and pace of change, including some that start inexpensively;
- Evidence of some public support for the changes proposed – policy makers are pressured to provide support for many things; they are more likely to respond to ideas that have public support.

Mathematicians are a key group that may need particular attention. In the US in particular, a small well-organised group („Mathematically Correct“) with conservative political support have led a highly effective opposition to reform. Most research mathematicians have little understanding of the complex dynamics of learning and teaching mathematics. Ignorant of the associative nature of learning, they tend to assume that the logical structure of mathematics provides the best learning sequence[4]. Further, pure mathematicians work in a field in which logical consistency is the sole criterion, so are often naive about empirical evidence, downplaying its decisive role. Perhaps most important of all, their unspoken priority is the education of students of high-ability like themselves. They emphasise particularly fluency and accuracy in manipulating algebra, the key language of specialist mathematics that only a few will ever use in their adult lives.

Thus the mathematical literacy of the many is sacrificed to the very-real specialist needs of the few who will work in engineering, science or economics. This important group can be catered for by additional options in specialist mathematics; the priority of the core mathematics curriculum should be high quality mathematical literacy.

The following can help to get support:

[4] Except in the UK, this plausible assumption was the basis of many mathematician-driven „new math“ movements in the 1960s. In one project that followed the Bourbaki reconception of pure mathematics, itself motivated by mathematics education issues, the curriculum began with a set-theory course for 5-year olds. The empirical evidence on these experiments was largely negative, but this is often forgotten.

- Local support from sensible university mathematicians who: are willing to spend some time learning about how school students learn; recognise that their limitations in this area; accept that mathematics education is an empirical field in which evidence is a better guide than pure reasoning.
- Formal and informal involvement of the representative societies of research mathematicians and scientists, negotiating with their leadership to ensure that those they nominate satisfy these same criteria. Formal approval by these societies of the processes by which reforms are developed and evaluated is an important asset.
- Professional associations of those involved in the teaching of mathematics, science and social studies form a key constituency. I mention them last because those who drive mathematics reform are often in close contact with some of these. However, for mathematical literacy the mathematics-focussed associations are not enough; those in science and social science can be powerful allies, or foes. Science teaching associations will be concerned that the mathematics needed for physics may suffer. Many social studies teachers will downplay the need for mathematics, often reflecting their own insecurity with it. Good ongoing communication, with reassuring evidence, is important.

5. The roles of assessment

In trying to reform curriculum, assessment is often an afterthought – important for evaluating progress and, perhaps, for holding schools to account but not a core part of planning and development. This attitude leads to a tragically missed opportunity. Why? There are two key reasons, one already noted:

- Assessment tasks provide a clear and vivid statement of the learning and performance goals of the change. Teachers, students, politicians and the public can understand them. In contrast, lesson materials are too bulky to be easy to comprehend – or for policy makers to read – while „standards" alone, focussing on separate ingredients of mathematics, do not specify performance.
- In systems with strong ‚accountability' pressures on schools, most teachers „teach to the tests". (WYTIWYG) Many people deplore this but the tests, whatever their limitations, are the main target that society sets for successful learning. *Thus the tests effectively define the implemented curriculum.* ‚Authority' is often reluctant to accept this, perhaps because it implies a responsibility for designing high-stakes assessment that reflects all the performance goals of the curriculum in a balanced way – this costs more.

However, viewed positively, this influence offers unmatched leverage for effecting changes in the implemented curriculum. Because everyone likes simple tests, this leverage usually impoverishes the curriculum, narrowing the range of classroom learning activities. Multiple-choice tests, dominant in the US, assess very short chains of reasoning, and favour elimination tactics focused on the wrong answers – performances that are only indirectly connected to curriculum goals. It is argued that these correlate with better measures but that is rarely proved by research and, crucially, ignores WYTIWYG. England mostly uses short items with constructed responses – better, but again with short chains of reasoning, totally different from those needed for modelling or most other *thinking with mathematics*.

What of systems that do not have high-stakes tests? They depend on teachers' professionalism, and are protected against the negative effects of WYTIWYG. However, it would be unfortunate if they lose the benefits that high-quality assessment has to offer. Professionals, generally speaking, are good at sustaining established practice; the introduction of improvements is more problematic, especially when these require new teaching skills. The modified professional practice that is needed to encourage greater student autonomy in non-imitative tasks needs explicit *support*. Assessment tasks are a key part of that support. Change also requires *pressure*. The anglophone countries tend to rely on pressure alone, with negative consequences; it remains to be seen if support alone can establish modelling or other curriculum reforms.

6. Tools for teachers

Here I shall be brief since this area is relatively familiar. People in all fields are much more effective when they have well-engineered tools. What are they here?

- **Classroom teaching materials** are part of the professional practice of most teachers, even in familiar areas. For new curriculum elements that need extended teaching styles, classroom materials are even more important – as is the design and development challenge they present.

  To develop such materials requires the ‚engineering research' approach that is used to develop effective tools in other fields, from consumer electronics to new medicines (Burkhardt 2006). What does this approach involve? Input from prior research and from other designs with similar goals. The design skills to turn these into draft materials that match the goals. Rich and detailed feedback from a sequence of trials that informs each revision, until the outcomes with users, representative of the target populations, match the goals. Gathering this feedback needs the methods of insight-focussed research. Finally, ongoing feedback ‚from the field' informs subsequent improvements.

  This methodology is more elaborate than the ‚author' model, more usual in education; however, it pays off – no-one would fly in an airplane or take a medicine that had been developed by the craft methods still used in education.

- **Professional development** (PD) support has an important role to play. Methodology is important here. Most PD is delivered ‚live' or on-line and designed by those who give it. It is usually evaluated by questionnaire, asking participants whether they found the experience valuable; feedback is powerful so the response is usually positive. However, *there is no feedback on whether the PD leads to any changes in teacher's classroom behaviour* – surely the main objective. The few studies that have used classroom observation before, during and after PD found no significant changes in teaching style.

It is not always like this. When observational feedback is part of the development, it leads to a different style of the PD – less concerned with teaching general principles and more with specific experiences, in the course sessions and in the classroom between them. Sharing these experiences among participants leads to general discussion on mathematics, learning and teaching, from which the principles emerge. It is constructivist learning for teachers. Well-engineered materials are important here, too.

The issue of *transfer* needs more research. How much of this kind of PD experience do teachers at various levels of sophistication need before they adopt the same broader teaching style in other teaching – of concepts and skills, for example. While these *can* be taught by ‚direct instruction', this is ineffective for resolving mistakes and misconceptions. The investigative, discussion-based methods that are effective (see, e.g. Swan 2005) have much in common with those needed for teaching modelling.

7. Models for systemic change

These components of successful change will only be effective if integrated. Piecemeal changes of the right kind have often been tried: new textbooks, but with the same tests; more professional development, but on an occasional basis; changes in policy involving new ‚standards'; and so on. Such attempts have proved inadequate, so that mathematics classrooms today are much like those our grandparents were taught in. What are key characteristics of a model that *is* likely to prove effective? Experience in other domains suggests:

- **Coherence** Policy, curriculum specification, classroom materials, assessment and professional development support all need to be closely aligned, developed together, and clearly communicated.
- **Sensible pace of change** Politicians, and many in education, like ‚Big Bang' solutions that will ‚fix the problem' once and for all. However, there is much to be said for gradual change. It gives the many groups, particularly teachers, who have to absorb profound changes time to absorb them. It also offers year-by-year gains that reconcile the few-year timescale of elections that drives politicians with the decade timescale of significant improvement in education.

  This model has proved effective. The Shell Centre (1984-86) worked with a leading English examination board to introduce specific profound changes to the mathematics examination at age 16, providing assessment, curriculum and professional development materials. These units were popular with teachers.
- **Realistic costing** In government initiatives the challenges are usually underestimated and the money provided for development is grossly inadequate. This guarantees failure. It is better to scale down or spread out the goals so that realistic costing can be reconciled with spending limits.

Success is never, of course, guaranteed but this kind of *sensible planning avoids guaranteed failure*. The need for further research and development is clear; the above analysis is a contribution to specifying such a program.

8. Scenarios for the future: optimistic and otherwise

History should make us cautious. The most likely scenario is little or no change. Most of those involved will be happy to avoid extra challenges in their already busy lives. However, there are some things that allow us to be more optimistic.

PISA is now the prime international comparison between countries' performance in mathematics, and it is designed to assess mathematical literacy. Politicians care about the results. Some countries are making policy moves to bring modelling into mathematics. Following the high-level Tomlinson (2004) and Smith (2004) enquiries, the British Government has made „Functional Mathematics" a central goal for English schools. Time will tell whether the government will make the moves needed to make functional mathematics a reality (Shell Centre 2005).

The problem of establishing modelling as a regular part of school mathematics remains work in progress – but progress there is.

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MODELLING BUNGY JUMPING: WHY IS IT SO DIFICULT?

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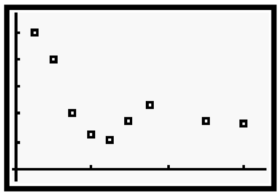
Mathematics curriculum orientations of many countries recognise the importance of developing students' capacity to use mathematical knowledge to better understand reality (Niss, 1996). But mathematical modelling is not a simple activity for students — neither for teachers. To model situations of reality we all need to develop competencies that were not present in maths classroom for many years. It involves new conceptions of mathematics classroom as a powerful knowledge that really applies to reality; the capacity of looking for a mathematical model that really explains the situation to model; the capacity of working critically with technology.

THE TASK PROPOSED AND THE ANSWERS OF THE STUDENTS

Bungy jumping is a radical sport. We can obtain data from websites that allow us to determine points of the trajectory of the jumper (time of the jump in seconds, distance from ground). A class of 30 upper level students was given a table with data from a jump and was asked to model it and describe it in several aspects. Students used graphic calculators to visualise the scatter plot and to experiment different possibilities of functions for models. But only one student obtained a good function, from the family of y = a + (sin bx +c) / x. The others used polynomial functions grade 3 or 4, or simple trigonometric functions, despite many of them were aware that this kind of model does not describe the jump in real life (Canavarro, 2004).



SOME QUESTIONS TO REFLECT ON

While working on that task, students seemed to be highly worried about obtaining a graph of a function that exactly fits the scatter plot of the real points given. The majority of students proposed as a model a function directly provided by graphic calculator, no matter if the function is a reasonable one to explain the jump. So, what are students' conceptions of mathematical modelling? Do they really value the power of mathematics to better understand and explain reality? Do they recognise modelling activities as important mathematical activities of the classroom?

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MATHEMATICAL MODELLING AND PARALLEL DISCUSSIONS

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In this paper, I put attention on those students' discussions which don't play a role in building of the mathematical models. Using qualitative data from a situation in the classroom, I propose the notion of parallel discussions to term them and analyze its nature. Arguments are underlined for the potentiality of these discussions to produce mathematical investigations or discussions about aspects of life in the society.

MATHEMATICAL MODELLING AND STUDENTS´ PRACTICE

Introduction

The debate about Mathematical Modelling and Applications has increased in Mathematics Education in many countries, what has generated an important agenda of research as that mentioned in the ICMI Study Document (Blum, 2002)

One of the current focus is the students' practice when they do mathematical modelling (to avoid repetitions, from this point on, I am only going to use the name modelling). Before developing the focus of this paper, I will discuss the theoretical perspective adopted, so that the reader can understand how this influences the other parts of the study.

A modelling perspective

According to what is discussed in Barbosa (2003), I define modelling as a learning milieu in which the students are invited to question or investigate situations with reference to the reality through mathematics. Any mathematical representation of these situations is called mathematical model. For an activity to be defined as modelling it is necessary to be a problem for the students, that´s to say, they do not have previous schemata, and have reference in the reality.

Modelling in school may have different features according to the objectives of the activity. Recently, Kaiser and Sriraman (2006) have characterized many perspectives in the field. Although it is not my objective to discuss them in this paper, it is important to point out that different perspectives generate different research agendas.

One of the observed perspectives by Kaiser and Sriraman (2006) is the socio-critical one, characterized in Barbosa (2003, 2006a), which emphasizes on the Modelling as an opportunity for the students to discuss the role of mathematics in the society and the nature of the mathematical models. The development of the competences or the learning of concepts and mathematical procedures, seen in other perspectives as primary proposals, is considered in the social-critical perspective as a "means" to enable the discussion on the roles that mathematical models may play in society.

The argumentation in socio-critical perspective is based on the recognition that the mathematical models are used in the society as power instruments (Borba & Skovsmose, 1997; Keitel, 1993; Skovsmose,1994). This requires that the students "read" the mathematical models critically, noticing that the mathematical results depend on the "place" where they are produced and how they can be used.

A framework for the analyses of the students´ practice

Many of the studies about the practice of students in the modelling milieu have focused on the analyses of competences and skills (Haines & Couch, 2005; Henning & Keune, 2005), putting some emphasis on the enculturation of the students in the practice of professional modellers. Other studies, as in Zbiek and Conner (2006), have analysed students´ practice in terms of the opportunities for the conceptual and procedural development in mathematics.

In this paper, I adopt another point of view, trying to extract implications from the socio-critical perspective to analyse students´ practice. However, it is not enough to generate theoretical understandings about this object, but a framework about students' practice is necessary too.

I will not expand the discussion about the notion of practice. In this moment, I will only define it as actions developed by individuals, which have its senses in the contexts in which they are produced. The fundamental hypothesis adopted here is that the human actions are linked to the cultural, institutional and historic contexts (Lerman, 2001; Wertsch, 1993, 1998). According to what Wertsch (1993) explains, "a social cultural approach to mind begins with the assumption that action is mediated and that it cannot be separated from the milieu in which it is carried out" (p.18).

Therefore, the comprehension of the actions in the specific milieu of modelling is not in the relations that students establish with the object (in this case, the problem-situation), but in the external conditions. It's not possible to dissociate the actions from the forms of mediation used, like the instruments and the language. According to Wertsch (1993, p.12),

> the relationship between action and mediational means is so fundamental that it is more appropriate, when referring to the agent involved, to speak of "indivudal(s)-acting-with-mediational-means" than to speak simply of individual(s).

The author argues that the human action employs means of mediation, in a way that the separation between the individual and the way of mediation is just analytical. Based on Vygotsky and Bakhtin, Wertsch (1993, 1998) puts emphasis on the communicative practices, understanding them as primary source to the human action. In Mathematics Education, Lerman (2001), among others, have made several studies using this assumption. This author elected the discourse category as the focus of his program of research. Discourse refers to all kinds of language, including gestures, signs, artefacts, mimics, and so on (Lerman, 2001). For him, analysing discursive practices is central, because the meanings precede us and we are constituted from the language and practices associated in several contexts in which we participate. So the understanding of the actions of the students in the modelling milieu may be searched in its discursive practices.

Students´discussions

As a result of the social cultural perspective, I understand the students´ practice in modelling as discursive. Previously, when focusing on the verbal interactions in the modelling milieu, I proposed a notion of interaction spaces as a unity of analyses (Barbosa, 2007). A space of interaction happens when students get together or students and teacher get together with the purpose of discussing the modelling activities (or even other kinds).

Borromeo Ferri (2006) has proposed the notion of modelling routes to denote the individual modelling on an internal and external level. This notion underlines the possibility of students giving different directions to the activities of Modelling.

Since I am realizing the discourse as object, the interest falls about the verbal interactions between students or between them and the teacher. Therefore, I will consider the modelling routes as a discursive instance, thus referring to the external level, differently from Borromeo Ferri (idem).

What constitutes the modelling routes? This question generates a large agenda of research. In a previous study, inspired in Skovsmose (1990), I have proposed the notions of mathematical, technological and reflexive discussions as parts of the modelling routes (Barbosa, 2006a). Defining them, we have: the mathematical discussions refer to the pure mathematical procedures and concepts; the technological discussions refer to the translation of the elected phenomenon to study in terms of mathematics; and the reflexive discussions refer to the nature of mathematical models and the influence of the criteria used in the results.

According to the purpose of the teacher, it is possible that one of these discussions be stimulated. In the specific case of the socio-critical perspective, it is interesting that the students are not restricted to the mathematical and technological discussions, but develop reflexive ones, because they constitute an opportunity to reflect upon nature and the role of mathematical models in the society.

The consideration of the mathematical, technological and reflexive discussions does not exhaust the practice of students in the Modelling milieu, since other discussions might appear and may not fit, exactly, in none of these cases. Thus, in this present paper, I take the example of the classroom in order to analyse the nature of these other discussions of the students, that´s to say, those ones that are not mathematical, technological or reflexive. With this, I hope to theorize the practice as well as have more elements for the teachers to follow students´ actions.

METHODOLOGY AND CONTEXT

This study takes part in the tradition termed in Guba and Lincoln (1994) as constructivist, which hopes that the reality happens in the multiplicity of perspectives, implying in the return to the subjective activities to build an understanding of the social world. The study has a qualitative nature. It is about giving sense to interpret the phenomena in terms of the meanings that the people bring to them, taking the words as data (Guba & Lincoln, 1994).Here the word also has the status of object as the focus is the discourse.

The data were collected through observation, that consists in collecting impressions of the world around through the relevant human faculties (Agrosino & Pérez, 2000). A group of students doing modelling activities was recorded by one of the members of our research group at State University of Feira de Santana (Brazil).

The context was an in-service education program to maths teachers, in May 2006, in the city of Feira de Santana, Brasil, taught by Professor Andréia Maria Oliveira, who is also a member of our research group. The students are experienced teachers, but with no academic titles, who returned to the university to take certification in a governmental project.

In that moment, the local city mayor had authorized an increase in the bus far from R$ 1,40 to R$1,50 (in Brazil, the currency is Brazilian Real, quoted as R$), which provoked big anger in the city. Taking advantage of the public debate about the subject, the teacher asked them to evaluate the impact of the increase in the price of the bus far in the monthly family budget.

At first, the teacher distributed a copy of a newspaper article that talked about the increase in the price of the bus far, which was read and discussed with everybody. In the sequence, they were organized in groups, having as a task to find a solution to the problem. In this present paper, I will consider the recording of a group composed by Lila, Selma and Maria.

The analyses of data was inspired in the grounded theory (Charmaz, 2006), using mainly some guidelines for coding. The recording was transcribed and coded line by line, trying to identify the extracts that referred to the mathematical, technological and reflexive discussions and those that did not fit in one of them. Next, the latter ones received codes, which were grouped in bigger descriptive categories. Then, the results were confronted with the literature, generating theoretical understandings for the purpose of this paper. The interpretations were discussed with the research group, which allowed later improvements.

DATA PRESENTATION

The group of students formed by Lila, Selma and Maria, immediately decided to search for a solution to the problem proposed by the teacher. Due to the limitation of space, I will just present two representative extracts of the data. It's possible to identify utterances in them that can be placed under the definition of mathematical discussions [M] and technological discussions [T]. However, there are other utterances that do not refer to these ones, which I will indicate as [?].

[T] Lila: What do we really want to know? We still have to figure out the monthly expenses. Four and a half weeks.

[T] Selma: A family that earns a minimal wage or that participates, at least, with three or four salaries.

[?] Maria: Gosh, the reality is really hard.

[?] Lila: That´s why children have to study at schools nearby. It´s not possible. They go on foot.

Maria: Say something Selma,

[M] Selma: We figure out 1,50 times six. The result is 9,00

[T] Maria: The week has five days: Monday, Tuesday, Wednesday, Thursday and Friday.

[T] Selma: In a month there are about 18 or 20 school days

The teacher approaches the group.

[T] Teacher: Are you counting full price or half price of the bus far? [Students pay half of the bus far in Brazil]

[T] Selma: We considered half price.

[M] Maria: Half and 18 times 4.

[?] Selma: What happens is that a worker pays 6%. The bus far law is like this: the employee pays 6% and the employer pays 94%[In Brazil, there is a federal programme informally called "transportation-ticket", in which the companies pay the amount spent on employees´ public transportation; on the other hand, the employees have 6% taken off their salaries].

[?] Maria: It is a transportation help.

[?] Lila: It is not possible, it´s very expensive.

[T] Selma: 85,5? It is wrong!

In this episode, it's possible to notice that the enunciations indicated by [?] refer to the students´ perception about the largest context of the problem. They talk about the high price of public transportation for the poor people that receive a minimal wage. They remembered the transportation-ticket programme, which the employees receive tickets for the transportation.

The students do not seem to consider these information useful for the solution of the problem, as they do not interfere in the group strategy. The extract suggests that the discussions defined as [?] go parallel to those called [M] and [T], since the first ones are not part of the building of the mathematical model. Nevertheless, they work as a theme of the reality outside the school, life in society, and specifically, the possibility to access public transportation by the population. They do not fit into the definition of reflexive discussions, because they do not work as the relation between the mathematical model and the criteria used in its building.

Nonetheless, I think that, from the socio-critical point of view, these discussions which are coded in the episode as [?] represent some reflection about the social reality outside the school. They bring to the mathematics class a debate about critical problems in the society.

Later on when the teacher visited the group, another kind of enunciation appears which does not fit in [M] and [T], but different from the mentioned ones in the above extract as [?].

[T]	Teacher:	Which variables can we consider in the problem? What happens when the salary is increasing? What will happen? We can discuss this.
[T]	Selma:	It decreases, if the person receives many salaries.
[?]	Maria:	They are an inverse proportion. So, let´s go!
[T]	Selma:	Do you want us to it in relation to the salary?
	Teacher:	As you wish.

[In the sequence, Maria´s utterance was not realized and did not influence in any of the previous utterance]

In this extract, the students are discussing with the teacher the variables to be chosen to the problem. Selma´s enunciation, characterising the relation between the expenses with public transportation and the salary as an inverse proportion, was not heard by her colleagues. It happens like a parallel way, not influencing the building of the mathematical model, being just a link with some previous knowledge. Questions, like "Why?", "What is an inverse proportion?", etc., could be asked and some good mathematical discussions could have been developed.

DISCUSSION

The analyses of the above extracts place new elements to characterise the students' practices in modelling. As previously suggested, the routes of modelling represent the discursive processes produced by the students that play a role in the production of the mathematical model. They can refer to the mathematical, technological and reflexive discussions, as cited in Barbosa (2006a).

Nevertheless, as previously supposed, a lot of utterances do not fit in these categories. It is the case of the ones marked as [?] in the above analysed extracts. The students produced parallel enunciations about the conditions of access to the public transportation by the population. In another moment, Maria produced an utterance about mathematics when she referred to the inversely proportioned values. They didn't change discursive way of the constitution of the model. In the analysed case, they were “invisible”, since they are utterances produced and heard, but not taken into account in the following utterance.

Since the utterance called [?] take place parallel to the route of students´ modelling, I will call them *parallel discussions.* They refer to those that happen in the spaces of interactions of Modelling, but do not belong to the Modelling route, since they do not play a role in building of the mathematical model.

It is not the case when the students begin strategies and abandon them because they consider them as inappropriate. In this case, even if the arrangements do not generate the mathematical model validated by the students, they have it in focus, being part of the Modelling routes.

The data suggest that for the utterance fits in the Modelling routes or parallel discussions, it depends on its use. If Maria´s utterance about the inversely proportioned values influenced, somehow, the building of the mathematical model discussed, it would not be classified as parallel discussion but as part of the Modelling route. Therefore, only after produced, we have how to know the kind of discussion.

The analyses of the extract suggest that the parallel discussions may refer to several domains. A first case refers to the context of the problem, exemplified in the data with comments about the access of the population to public transportation. In this direction, the modelling activity may allow some reflection about social situations, even if they are not reflexive discussions. The parallel discussions, in this case, produced perceptions about the social reality to the mathematics class.

On the other hand, as exemplified in the data, the parallel discussions may refer to mathematical objects. In this case, its legitimacy is based on the context of students´ practice. If they are in the mathematics class, speaking about mathematics is relevant, even if it does not have the purpose to build to the mathematical model. This kind of discussion shows how the modelling milieu may generate problems of pure mathematics. We may take this type of discussion as an opportunity for the conceptual procedural development of the students in the sense described recently by Zbiek and Conner (2206).

The emergency of the parallel discussions has relation to the social and cultural context (Lerman, 2001; Wertsch, 1993, 1998). Since the school and the students themselves belong to a bigger social context, discourses that analyse the questions of life in the society may take place. On the other hand, the students may link mathematical knowledge with subjects previously studied. In both cases, these students´ actions may be legitimated by its context.

In this point, we may recognise the parallel discussions as part of students´ practice in the modelling milieu, even if it is not linked to the purpose of the building of mathematical models. In short, I would like to suggest that students´ practice in modelling may be classified in Modelling routes and parallel discussions. I have underlined here that these latter ones may be opportunities for the development of important discussions to the school environment.

FINAL REMARKS

As I pointed out at the beginning of this paper, many studies about students´ practice in the modelling milieu have focused on the actions that conduct to the building of the mathematical model, using notions as competences, skills and routes of Modelling.

Proposing the notion of parallel discussions to denote the enunciation that do not play a role in the building of mathematical models, I would like to focus bigger attention on the practice of the students in this milieu As underlined previously, the parallel discussions have potentialities in the conduction of modelling activities. It is about opportunities for discussion about the aspects of life in the society and the pure mathematics, that may originate other mathematical activities in the school context and a discussion about aspects of life in the society.

The concept of parallel discussions generates new questions to the agenda of research in the scientific field. What are its conditions of production? Why aren´t they taken into consideration by the group? What other kinds of parallel discussions may happen? What is the way of parallel discussions in generating other activities? The research about these questions may help us build a picture of the students´ practice in the modelling milieu.

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COMPARISON OF MATHEMATIZATION IN MICROCOMPUTER BASED LABORATORY (MBL) AND VERIFICATION –TYPE LABORATORY (VTL) IN PHYSICS

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This study addresses the question of whether and how the MBL features enhance mathematization in physics laboratory classes. For this purpose, two laboratory sessions on Hooke's Law and Newton's Second Law of Motion were conducted twice by the same physics teacher in two different grade 11 classes in the same school. The first time MBL was employed and the mathematics teacher of this grade participated in the two sessions. However, the second time the two laboratory sessions were of the verification type and were conducted in the absence of the mathematics teacher. Results show that MBL has a potential to promote mathematization in favourable instructional environments in physics laboratory classes.

INTRODUCTION

Mastering mathematical skills and concepts is often viewed by curriculum developers and teachers as a prerequisite for understanding physics in the secondary school. Consequently, it is left to students to transfer and apply mathematical concepts and skills in new physics contexts. However, this conception of the pedagogical relationship between mathematics and physics is severely constrained by the domain specificity of mathematics learning in the sense that mathematics learning is specific to the context in which learning takes place (Niss, 1999).

Freudenthal (1991) calls for having students start by exploring phenomena that call for and require organization through the use of mathematics. Physics offers a great variety of such situations that are amenable to be structured by mathematization which encompasses interdisciplinary activities like modeling and representation. According to Michelsen (2005), mathematization entails that "situations from physics are embedded in the contexts to be mathematized – a horizontal linking of mathematics and physics. Also the vertical mathematization must include a vertical structuring, that is the conceptual anchoring of the general model in the systematic and framework of mathematics and physics respectively" (p. 206). In traditional physics laboratories, the use of mathematization is constrained by instructional management factors, particularly data collection and mathematical calculations which consume most of the instructional time. On the other hand, because it provides the capability of real-time collection of data and a menu of mathematical models that may fit the situation, MBL is hypothesized to enhance mathematization in physics laboratory.

Mathematization refers to a process used by students to solve real-life problems. According to the Programme for International Student Assessment (PISA) (2003), mathematization consists of five steps: 1) Starting with a problem situated in reality;

2) organizing it according to mathematical concepts and identifying the relevant mathematics; 3) gradually trimming away the reality through processes such as making assumptions, generalizing and formalizing which promote the mathematical features of the situation and transform the real world problem into a mathematical problem that faithfully represents the situation; 4) solving the mathematical problem; and 5) making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution. (p. 37).

Microcomputer Based Learning (MBL)

Calculator-Based Laboratories (CBL) and Microcomputer-Based Laboratory (MBL) are the most frequently used tools to integrate science, mathematics, and technology in schools. These two tools function as “data grabbers” and consist of two types of hardware: Sensors or probes to collect physical data (such as temperature, humidity, distance, force, etc.) in real time and another device connected to the sensor that digitizes and stores the collected data. Using these tools, students have the opportunity to collect and work on first-hand experimental data. Collecting, representing, and interpreting data collected from experiments using the sensors may provide students with opportunities to work in authentic scientific settings similar to those in which scientists work and attempt to generate generalizations and idealizations reflected in mathematical models of real phenomena (Gillies, Sinclair & Swithenby, 1996).

Research on the role of technology in mathematics education has focused recently on dynamic computerized environments and their effects on student achievement and attitudes (Funkhouser, 2002; Healy & Hoyles, 2001; Arcavi & Hadas, 2000). Kaput (1998) suggested that students can use MBL or CBL to collect scientific data and represent it mathematically allowing them to learn about “both the phenomena represented and the mathematics used to represent quantitative aspects of it, reflecting the connectedness of mathematics with experience and its power as a sense-making tool” (p.4).

This study is a part of a larger project whose purpose was to investigate the barriers teachers face when integrating science and mathematics by using Microcomputer-Based Laboratories (MBL).The study reported here addresses the question of how the MBL features enhance mathematization in physics laboratory classes. This will be done by examining a case in which MBL was used in two Grade 11 inquiry type physics laboratory sessions (henceforth referred to as MBL lessons) taught jointly by a physics teacher and a mathematics teacher as compared to two verification type laboratory sessions.(henceforth referred to as VTL lessons). Specifically, this study will attempt to identify and compare the type and extent to which mathematization activities occurred in the MBL and VTL sessions.

The features of Microcomputer-Based Laboratory (MBL) seem to provide an environment which has the potential to facilitate mathematization. First, as a didactical tool, MBL provides students with the opportunity to collect and work on real-world experimental data in real time. Second, though it does not provide much opportunity for actual model construction (i.e. trimming away reality), MBL provides a menu of mathematical models in the form of function formulas, graphs, and numerical tables that serve as a context for engaging students in thinking about mathematical models that can describe a set of data derived from a real world situation. Third, MBL provides data (mean square error in the model) that may engage the students in evaluating the adequacy and accuracy of a particular mathematical model. Fourth, the real-time data collection and processing provide time-saving that may be used efficiently in exploring and assessing the adequacy of a particular mathematical model.

METHOD

MBL and VTL Lessons

The MBL laboratory sessions involved students in formulating a question, designing an experiment in cooperation with the teacher, collecting data using the MBL set-ups, fitting the collected data into a mathematical model, discussing the modeling process, and drawing conclusions based on the experimental data, data manipulation, and modeling.

The two verification type laboratory sessions addressed the same topics as that of MBL sessions but involved students in collecting and analyzing data to verify content matter taught in class, drawing graph using the collected data, and verifying already taught knowledge.

Data Collection

The study was conducted in two different Grade 11 classrooms in a private co-educational school in a suburb of the city of Beirut in Lebanon in which English is the medium of instruction of science and math. Students in the grade 11 class come from middle to upper socioeconomic families and were following the International Baccalaureate program, a rigorous pre-university course of studies. Prior to conducting the study, the two teachers who participated in the study were involved in a three-day workshop whose purpose was to introduce them to designing experiments, and collecting and analyzing data by using computers. The two participating teachers were enthused by the workshop activities and decided to start using MBL in their classrooms. Consequently, they held secure entry to the school for the researchers.

The study was conducted over the two academic years 2004-2005 and 2005-2006. During the first academic year, Hooke's Law and Newton's Second Law of Motion were taught to grade 11 students using MBL. The two MBL sessions were taught jointly by the mathematics and physics teachers and were videotaped. The following year, Hooke's Law and Newton's Second Law of Motion were again taught to grade 11 students by the same physics teacher, but in the absence of the mathematics teacher. The laboratory sessions in this case were of the verification type. Again these two sessions were videotaped for the purpose of the study.

Hooke's Law and Newton's Second Law of Motion constitute a part of the school physics curriculum and were taught according to the teachers' yearly plans. In preparation for the MBL laboratory sessions, the teachers prepared lesson plans that were discussed in a group meeting with the researchers. Then the teachers implemented the lessons in the school physics laboratory. Two of the researchers attended the MBL sessions without any interference in the lesson proceedings. The VTL session were also conducted in the physics laboratory. However, the physics teachers used the lesson plans he typically uses in such laboratories.

Data Analysis

Prior to data analysis, a DVD copy of each of the four videotapes was prepared. The four lessons were transcribed word by word and time was recorded next to each utterance of the transcriptions to facilitate data analysis. The four DVDs of the four videotaped lessons along with the verbatim transcriptions constituted the raw data. Evidence to support each of the five steps of mathematization was sought from the actions of the teachers and their discourse with the students. The mathematization lens was used to look at and interpret the data.

RESULTS

In this section we shall identify and compare MBL and VTL in terms of the extent to which the elements of mathematization.

Starting with a Problem Situated in Reality

In the two MBL sessions, the physics teacher (PT) always started with questions which encouraged the students to make conjectures about relationships between the physical variables under consideration in that particular laboratory session. For example, the PT started the session on Hooke's law by introducing the apparatus and asking students about the variables involved in the experiment to engage students in thinking about the physical relationships to be demonstrated. However, the questions by themselves do not constitute a situation which can be transformed into a mathematical problem. However, it was not clear whether these questions presented genuine problems to students.

The first phase of the mathematization seems to be almost missing in the VTL sessions. The PT started the session by reminding students of content and asking them to recall the law they studied in class. Soon after, the teacher reviewed all the content that relates to the law covered earlier. In other words, the starting point is memory work and not a real world situation.

Organizing the Problem According to Mathematical Concepts

The mathematical concepts that were used in the two MBL sessions were limited to the concepts of variables and relations among them. This is understandable since the objectives of each of the two lessons (Hooke's Law and Newton's Second Law of Motion) were to have students acquire concepts, principles, and laws in physics. For example, the PT expressed the variables in symbols and even suggested a mathematical formulation of the relationship between the force (F) and the extension in the string (x) in the session on Hooke's law. In the session on Newton's Second Law, the PT identified the variables only in words.

However, the VTL sessions did not seem to provide similar opportunities for students, simply because the concepts had been acquired in class before students attempted to conduct any experiments. Because, there was little room for students to organize the problem according to mathematical concepts, they resorted to formulas they had memorized.

Transforming the Real World Problem into a Mathematical Problem

The features of MBL are critical in this phase of the mathematization process in the sense that they present constraints and opportunities. Using MBL did not seem to provide the opportunity for students to engage in "gradually trimming away the reality through processes… which promote the mathematical features of the situation" (PISA, 2003, p.37). Rather, MBL presents a menu of mathematical models, in the form of a formula, a graph or a table of values, on the screen without any control from either the student or the teacher, thus making the MBL act as a 'black box'. The fact that this feature is a constraint in the process of mathematization is best illustrated in the following example. In the laboratory session on Hooke's Law, discussions led the students to expect that the relationship between the force applied and the extension it produces to be positive and linear. However, the graph that was displayed on the screen was a straight line in the fourth quadrant. This generated discussion (and confusion) for the mathematics and physics teachers as well as the students since they could not rationalize why the software produced a representation that was not consistent with their initial expectations.

On the other hand, the MBL presented an opportunity for mathematization by providing a menu of functional relations (with the mean square error of each) for the student to choose from, based on the concept of best-fit of data. This feature is unique to MBL because it provided students with an efficient method to explore the optimal mathematical model that best fit the data.

VTL sessions did not provide evidence of any transformation of a real world problem into a mathematical problem. The fact that there has been no real world problem initially makes it impossible to observe this feature. Students seemed to start with mathematical representation almost exclusively by recalling already taught formulas.

Working within the Mathematical Model

In the two MBL sessions, once the graph was displayed on the screen, the mathematics teacher (MT) took over the lesson. The general pattern followed by the MT consisted of the following activities: Through a whole-class instructional format, the MT started a series of questions to enable students to choose from the menu the appropriate mathematical model (functional relationship) and rationalize their choice. The MT seized the opportunity to consolidate the students' understanding of the mathematical concepts involved (linear function, inverse function).The MT shied away from explicitly linking the mathematical concepts to the physical concepts.

The display of the mathematical model on the screen provided a context for a lively classroom discourse about the best-fit model. In many cases the discourse was triggered by discrepancies which resulted in cognitive tension between the teacher and students and among the students themselves. An example is the discrepancy between the standard (criterion) for best-fit in MBL and that of the teacher. The following excerpts illustrate each of the discrepancies.

MT: What's the mean square error for the proportional fit?

MT: 0.021, right? And for the linear fit?

Student: 4.87×10^{-4}

MT: OK, if you move a little bit here…can you see the lower part of the yellow line? Do you see where it continues?

Student: Through the origin

MT: Regardless of the square mean error, what should it be?

MT: What's the y-intercept if the line passes through the origin?

Student: Zero

MT: So, it becomes y= mx, proportional fit. Does it make sense now with your physical result? Can you link it to the physical result?

A second discrepancy was between the model expected by the students and the one displayed by MBL. This is illustrated by an episode from the laboratory session on Hooke's law. The students were led to believe that the relationship between the force and the extension in the string was positive and hence the graph would be in the first quadrant. MBL displayed the graph in the fourth quadrant. The MT tried to rationalize the discrepancy in terms of the definitions of the variables by saying:

> Is x the difference between the initial length and extended length or the extended length itself?

On the other hand the PT tried to rationalize the situation in terms of experimental error.

> You see the computer was recording measurements every 0.5 cm but I wasn't sure if Mario was really seeing a displacement equal to 0.5 cm…And apparently there is a slight difference between what the computer is set to record and what actually has been happening at the cart. So there is a small error that the x here did not correspond to the x the computer program was set to have. I'm not sure if the pull by Joe all the time was constant because our nervous system sometimes does not allow us for a long period of time to be able to hold something for a long time exactly the same way.

Both interpretations did not convince the students and left them confused.
The third discrepancy was between the PT and the MT in choosing the model and occurred in the laboratory session on Newton's second law of motion. The experiment consisted of keeping the weight (F, pulling force) constant while varying the mass (m) of the cart by adding mass to it. The PT selected the line representing v=at (v, a, t represent velocity, acceleration, and time respectively), whereas the MT tried to rationalize the choice of the model a=F/m (where F is kept constant). Though both models are tenable, the second one is more appropriate and powerful. Anyway, this discrepancy led to some confusion among students.
The VTL sessions seem to provide an opportunity for students to practice the use of calculators in an attempt to find the values they needed to fill in the tables they were supposed to complete. Once students finished observing the demonstration presented by their PT, they broke up into groups where they used their calculators to find 'T' values (tension) and 'ΔL' values (elongation/compression) during Hooke's Law VTL session. Similarly, students spent a very long time working in groups on a much more complicated procedure to calculate 'F' values (force) and 'a' values (acceleration) in the case of Newton's Second Law VTL.
In both VTL sessions, time was not sufficient for students to plot the required graphs. Thus the PT had to sketch 'the expected' graph on the board and try to discuss it with students. Thus, in this phase students not only spent most of their time on mathematical calculations that were grounded mainly in conversions among units of measurement, but they did not even make use of the data due to time constraints.

Making Sense of the Mathematical Solution in the Real World

In the two MBL sessions, no significant attempt was made to make sense of the mathematical solution in terms of the real situation. Both teachers missed many opportunities to have students see the power of the mathematical modeling in making sense of the physical reality. One such missed opportunity was in the MBL session on Newton's Second Law, when the MT established that the model representing the force (F) and acceleration (a) was a straight line passing through the origin. However, no suggestion was made to actually weigh the mass of the cart and compare it with the slope of the straight line represented by F=ma.
In the VTL sessions, the PT did not relate the mathematical model to the original situation, since there was none to begin with.

DISCUSSION

The results of this study support the hypothesis that physics experiments conducted in an MBL environment provide the kind of real world situations that are amenable to mathematization. Thus mathematization, suggested by Freudenthal (1991) as a pedagogical theory for the meaningful learning of mathematics, may apply for the meaningful learning of physics as well.

Let me start by emphasizing that physics laboratory is intended to teach physics, and mathematization is only an opportunity that physics laboratory may provide. Our intention in this research was to investigate whether the MBL has a comparative advantage over VTL in increasing this opportunity. The data suggest that the features of MBL seem to provide an environment which has the potential to facilitate some aspects of mathematization in physics. However, this potential was constrained by the instructional design used in the MBL sessions. In the following paragraphs we shall identify the potential of each step of the five steps of mathematization in MBL and examine the degree to which the instructional design constrained this potential.

MBL does not have any intrinsic potential in promoting the first step in mathematization i.e. setting a problem situated in reality. This step exclusively belongs to the instructional design used. With or without MBL, an inquiry-based instructional design may promote mathematization by starting with a reality-situated problem. In this research the instructional design of MBL provided students with the opportunity to collect and work on real-world experimental data, however, there was no explicit effort by the physics teacher to pause an authentic problem. On the other hand, VTL provided no opportunity to pause a problem since the student were just verifying principles they had already been taught.

Similarly MBL does not provide a comparative advantage in organizing the problem according to mathematical concepts and identifying the relevant mathematics. However, there were some efforts by the physics teacher to organize the situation by expressing the variables in symbols and suggesting a mathematical formulation of the relationship. VTL, however, did not provide similar opportunities for students, simply because the concepts had been taught in class prior to the experiment.

MBL does not provide much opportunity for actual model construction pertaining to step 3 of mathematization (trimming away reality and transforming the real world situation into a mathematical problem) and step 4 (solving the mathematical problem). However, MBL has the potential of providing a context to engage students in some aspects of model representation and evaluation. The limitation of MBL in model construction is that it does not provide opportunities for students to ‘trim away reality’ but rather provide a menu of mathematical models from which students can choose. Moreover, MBL does not explicitly engage the students in solving the mathematical problem but rather provide an opportunity to reflect on given solutions to select the one that best fits the situation. However, MBL has much potential in this regard. The first comparative advantage of MBL is that it gives the opportunity to manipulate the situation i.e. the experimental set-up, and immediately visualize the impact on the representation of the mathematical model. Second, MBL provides multiple representations (all represent the same situation) of a particular mathematical model in the form of function a formula, a graph, or a numerical table and the facility to navigate from one representation to another. Third, MBL provides a facility to assess the adequacy of a certain mathematical model through examining the mean square error of the best-fit curve. Fourth, MBL has the facility to collect and process data in real time and thus providing time-saving that may be used efficiently in exploring and assessing the adequacy of a particular mathematical model.

The potential of MBL in steps 3 and 4 of mathematization were partially realized because of the instructional design. First, not much use was made of the manipulation facility of MBL. Second, the mathematics and physics teachers conceived their roles as a context to "teach their subject". For example, the mathematics teacher used the model as a context to teach functions and graphs without much reference to the situation they represent. Processing the mathematical model was also ridden with discrepancies which are due to lack of coordination between the mathematics and science teachers and a deficiency in internalizing the power and limitations of the technology.

Two potentials of MBL were reasonably realized. First, Students were engaged in assessing the mathematical models that may best fit the data by looking at the mean square error and, as evidenced by the results, this generated lively discussions and argumentations regarding the best-fit model. Second, the time-saving in MBL and its utilization in exploring and assessing the adequacy of a particular model were substantiated. Time analysis of data MBL session showed that most of the time (48% for each of Hooke's Law and Newton's Second Law of Motion) was devoted to analyzing and discussing the graph that resulted from plotting the data while most of the time (42% for Hooke's Law and 61% for Newton's Second Law of Motion) was devoted to performing the experiment in the VTL session. In contrast to model identification and assessment in MBL, VTL students spent most of their time performing procedural tasks with no or very little thinking- a constraint which weakened vertical mathematization (working within a mathematical model) in VTL.

The MBL has a potential in promoting the last step in mathematization (making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution) in that it allows the learner to test the model in reality. For example, the learner could check the model by measuring the mass of the body and compares with the slope of F=ma (Newton's second law of motion) or identify the limitation of a mathematical model (limiting the range of values may produce a model which is different from the theoretical one). However, the instructional design of both teachers did not make use of this potential of MBL.

A replication of this study, which avoids the pitfalls of the instructional design may hopefully confirm better the claims regarding the potential of MBL in promoting mathematization.

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MODELLING TASKS FOR LOW ACHIEVING STUDENTS – FIRST RESULTS OF AN EMPIRICAL STUDY

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The integration of mathematical modelling into mathematics curricula has been demanded for some time in response to international comparative studies such as PISA. However, low achieving students have more or less been ignored in this regard. In this study, beginner problems for such low achieving students have been developed and evaluated.

THEORETICAL BASIS

For a long period of time, pedagogical discussions have been calling for the integration of applications and modelling into mathematics curricula (Kaiser-Messmer 1986(I), p. 82) so that students can understand and critically analyze their environment, as well as gain insights into the usefulness of mathematics to society (Blum & Niss 1991, p. 42). Numerous modelling problems have already been developed. However, most of these problems are relatively complex and not suitable for low achieving students. Keeping in mind the necessity to be able to master their everyday lives and to prepare for their later working life, these students in particular need to learn how to use mathematics to solve non routine problems.

Most of these students, however, who are taught in a separate school in Germany, have enormous problems in using elementary mathematical knowledge and in reading and understanding texts. Experience shows that teachers think the best way to support these students is by explaining in detail every step they have to do in advance. So these students are not used to working independently, instead they ask for repeated explanations from the teacher.

Word problems are seen as one possibility to support the development of modelling competencies at primary school level and with students of all abilities (Verschaffel 2002). A word problem is a text which describes a situation more or less familiar to the reader and which poses a quantitative question, an answer to which can be derived by mathematical operations performed with data given in the text or otherwise inferred. When trying to develop modelling competencies it is, however, important to choose word problems which do not meet students' expectations and which make students think about the context. (Greer, Verschaffel & De Corte 2002). In this context, Puchalska & Semadeni (1988) differentiate between problems with missing information, problems with surplus data and problems with contradictory data. The first category of tasks, called here under-defined tasks, are in fact - very simple - modelling problems, the second category, here called over-specified problems (as well as the third) may not always be modelling tasks. Their aim is to force students to think about the context and not only to take the numbers out of the context and to connect them with an operator. Students thus learn to proceed from the real world to the mathematical model.

When solving reality-related problems several steps have to be carried out. There are many different diagrams which may be used to describe modelling processes (Kaiser-Messmer 1986 (I), p.82). The diagram selected here takes into account the special interests of low achieving students by identifying the understanding of the situation as a separate step. This appears to be important as it may not be easy for these students to understand the situation due to problems with language (reading) or problems of general understanding (Fig 1, Blum & Leiß 2005).

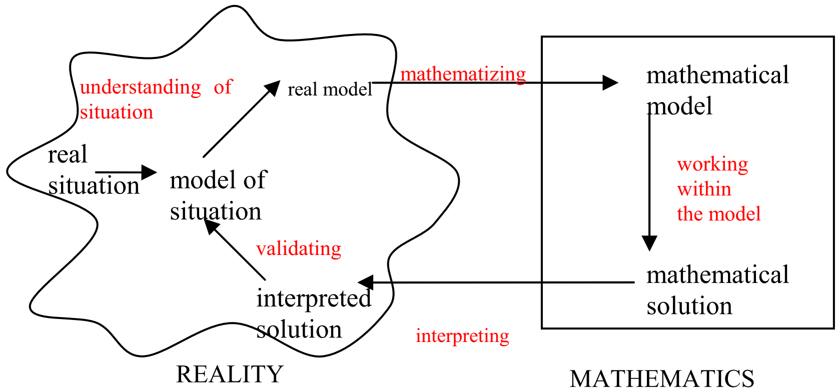

Fig. 1: Modelling process according to Blum/Leiß 2005

In order to carry out mathematical modelling a particular set of competencies is necessary:

1. Sub-competencies to carry out the single steps of the modelling process (Sub-competencies in understanding the situation, in simplifying and setting up a real model, in mathematizing, in working within the model, in interpreting and in validating).

2. Metacognitive modelling competencies

3. Competencies to structure real world problems and to work towards a solution with a sense of direction

4. Competencies to form arguments relating to the modelling process and to write down this argumentation

5. Competencies to see the possibilities mathematics offers to solve real world problems (Maaß 2006, p.139).

Based on this theoretical framework the study aims to develop beginner modelling problems for low achieving students and to evaluate them. Additionally, the study establishes criteria for their development and describes the students' modelling competencies.

METHODOLOGY

The study described here is a qualitative study (Flick 2000). Semi-structured interviews were used for data collection. For every interview one student was taken out of their regular lesson into a separate room. He or she was given a worksheet with 4 modelling tasks and was asked to work on the tasks and to think out loud. Assistance was given as little as possible, with most interviewers' comments being simply motivational remarks ("try again", "you will make it") or to strategic help ("Try to think about the quantities you need!"). Additionally students were asked questions about their views about the tasks and mathematics. Each interview lasted about 1 hour.

The data was analysed according to the qualitative analysis of content (Mayring 2002). Categories for the analysis were partly chosen theoretically from the list of sub-competencies described in part 1. However, the positive attitude towards modelling examples (see sub-competency 5) will not be discussed here due to the limit of space. Metacognition was also neglected as it is unlikely that beginners will have many metacognitive modelling competencies. Additionally, during the analysis further competencies and aspects turned out to be relevant. These included basic mathematical skills (precondition for mathematizing) and understanding the text (precondition of understanding the situation). Also some students merely guessed or gave up without carrying out any step of the modelling process. Based on these categories we wrote a description for each task to show in which areas mistakes occurred and a description for each student. Finally, case-comparing and case-contrasting analyses were conducted and typologies were created. One tool to elucidate the results was the construction of ideal types (see Gerhardt 1990, p. 437).

Overall, about 40 problems were developed and the data from 150 students between the ages of 11 and 16 was collected. The sample group includes students of all grades and is large enough to justify the typology developed. Evaluation and analysis of the data is still in process. The study described is closely aligned to everyday practice in mathematics classrooms and may therefore be of interest and assistance to practising teachers.

Selection and development of problems

In this study under-defined tasks as well as over-specified have been chosen as beginner tasks for low achieving students. In view of the call for an increased emphasis on modelling in mathematics classrooms (see part 1) it seems important that the contexts of problems are as realistic and authentic as possible. However, the easier the problems are the more difficult it is to comply with the demands that they should be realistic. Therefore, compromises must be made. However, the compromises have to be appropriate taking into account the goal of acquiring modelling competencies. Here is an example for an over-specified task:

> Alina would like to cycle to her grandma, who lives about 2 km away from her. On her way she pedals 220 times and cycles 150 – 170 m per minute. How long does it take her to get to her grandma?

This problem has been developed for 11-year-old students. The context is a reality-related situation possibly relating to their everyday life and the problem is useful from a pedagogical point of view. Students can easily understand the described situation and there is only one more quantity given than necessary. The aim of the task is to draw the students' attention to the context and to teach them to differentiate between relevant and irrelevant information. An example for an under-defined task is the following mobile-task developed for students of the same age:

> "Hi, Mike. How are you? I'm on a ferry having lots of fun. Can we meet tomorrow 3 pm? At the town hall? Yours Sarah" How long do you need to text this message?

The context is also from an everyday situation that students are likely to have met and can therefore be easily understood by them. The calculation in some aspects can give a deeper insight into texting and therefore may be of interest to the students.

RESULTS OF THE STUDY

To give an insight into the results we will have a closer look at two students, Mario and Hakan. Due to the length of the interviews only extracts are presented here. Following this, taking into account all data of all students, areas of difficulty for students are identified and a typology of student modellers is developed. Finally, criteria for the development of real world problems for mathematically weak students are suggested.

Two case-examples

Mario seems to have no problems in dealing with modelling tasks although he has never done any modelling before. In the following you see his solution of the *bicycle-task*.

M: I just calculate 2000 m, divided by 150, I think. [calculates]

M: 13, remainder 50.

I: What does that mean?

M: She needs about 13 minutes, I think, I think.

I: Why are you hesitating?

M: I don't know, I don't know, the 220 times are irritating me.

I: And if this information were not given?

M: It would be easier then. […]

I: How did you like the task?

M: It was good.

I: Why?

M: It was easy. […]

The text shows that Mario has no difficulties in identifying the relevant quantities and in setting up a real and a mathematical model. He finds a solution quickly and seems to know quite clearly what to do (sense of direction). Yet, he neither explains nor validates his calculation. The result is interpreted only because the interviewer asks for it. The additional quantity seems to make him feel uneasy. Nevertheless, Mario seems to like the task because he finds it easy. He has no problems with the *mobile-task* either as the following quotation shows.

M: I think, she needs 1 second for 2 letters. [counts]

M: I think about 60 seconds.

I: What did you do?

M: I have calculated, that she needs for one, for two letters about a second and then I have put two letters together and then I have counted. […]

I: Do you like the task?

M: Yes.

I: Why do you like it?

M: Because it is about a mobile. …And in our lessons we have never done anything with mobiles […]

I: Was the task unfamiliar to you?

M: Yes, we have never done such a task in lessons.

I: Was the task easy or difficult for you?

M: I found it easy, because I text very often and I need about 1 second for 2 letters.

Again, Mario understands the context. He finds a model very quickly and calculates a solution in an appropriate way. He seems to know exactly what he is heading for and he interprets his solution. In contrast to the first tasks he gives reasons for his proceeding. Additionally, he is able to combine his personal experiences with the tasks. However, he does not validate his solution. His statement of interest this time relates to the context and not to the fact that he finds the task easy.

Mario's dealing with the modelling tasks identifies Mario as a good modeller. Although there are some small deficits he has no problems in finding a solution. The situation is completely different for ***Hakan***, who has enormous problems with the modelling tasks. Here we see how he deals with the *bicycle-task*.

I: Please tell me, what you are writing.

H: How many kilometres she needs?

I: Why are you writing: how many kilometres? […] In the task a different question is given.

H: Oh yes, how long does it take her to get to her? […] 220, the calculation is 220 minus 150.

I: Why?

H: […] Minus 150 or 170.

I: Why do you want to calculate this?

H: Because we can find the result this way. […]

I: Why?

H: We can also calculate 150 plus 170.

I: Yes, you can calculate this, but do we get the answer by doing so?

H: Yes.

I: Do we get the time then how long it takes her?

H: Yes, she has to pedal 70 times.

I: Please think about it again. Which quantities do you need to calculate how long it takes her to get to her grandma. […]

H: Or 150, no, 170 minus 150. No.

I: What does 150 – 170 mean?

H: Whether she 170 und 100, about 170 and 150 pedal, pedal. […]

H: We can also calculate 2 times 170. […]

I: Do you like the task?

H: No. […] I haven't understood what to do. […]

I: Was the task unfamiliar to you?

H: Yes […] We have never calculated such a task.

Hakan does not seem to be able to distinguish between relevant and irrelevant quantities and so he is not able to set up a model and calculate a solution. He seems to have problems in understanding as well. At first, he does not read the question carefully, later it shows that the meaning of 150 – 170 m is not clear to him. He chooses numbers randomly, connecting them by any operator without reason. He is not able to identify how to proceed and is therefore unable to justify any procedure he might undertake. Reasons for this may be a lack of understanding of a mathematical term (150-170 m) or a lack of competency such as being unable to set up a mathematical model. Hakan seems to have enormous problems with the *mobile-task* as well.

H: [reading the task, thinking] … This cannot be calculated.

I: Do you think so? Why?

H: Because there are no calculations in it.

I: How long do you need to text one letter?

H: 2 seconds.

I: Can you calculate now?

H: I guess, she needs about 6 minutes to text this message. […]

I: But if you say she needs 2 seconds for one letter: Can you calculate how long she needs?

H: Yes, I could calculate then. [counts the letters] […]218.

I: What is the answer?

H: She needs 218 seconds.

I: Do you like the task?

H: Yes […] it was easier than number 1.

At first, Hakan wants to give up, because he does not see what can be calculated. Although the interviewer informs him about the relevant quantity he does not use it at first. He tries to guess a result. When the interviewer intervenes again, he starts calculating and finally is able to give a result, however he makes mistakes in counting. This tasks again shows that he seem to have a lot of difficulties in setting up a model. Moreover, he does not validate his results and he does not give any reasons for his choice of procedure. He interprets his results only when asked.

The two extracts of the interview show Hakan's problems in dealing with modelling tasks. The most important issue seems to be that he has enormous problems in setting up a model even when he seems to more or less understand the situation.

Looking at the solutions of all students, the bicycle problem seemed to be really difficult for most of them and therefore met with negative reactions. This may be due to the students not understanding the meaning of the two quantities "220 times" and "150-170 m". Some students described the problem as not familiar and suggested leaving out the unnecessary details. The mobile-task did not seem to be as complicated as the bicycle-task and so was more popular with students with some being interested in the context. However, almost all students described the task as very unfamiliar, because no numbers were given. Quite a few students tried to solve the task by guessing.

Areas of difficulty

Altogether, the analysis of all students and all tasks showed the following areas of difficulty in the students' attempts to solve the problems:

1. Understanding the situation: The most important reason for such problems were difficulties in understanding the text. Partly, students did not understand single terms of the text and partly they read too quickly to understand the situation.

2. Setting up a model: Many students had problems developing a model. Often, students were not able to extract the relevant quantities (see, for example, Hakan). Others recognized which bits of information were relevant, but were not able set up a model.

3. Working within the mathematical model: Some problems could not be solved, because the students lacked the required elementary mathematical knowledge.

4. Interpretation mistakes: The students succeeded in interpreting the results of easy tasks. However, they mainly interpreted results when demanded or prompted to do so by the interviewer.

6. Validating: The study shows that students generally did not validate their results.

7. Sense of direction: Some students seemed to calculate anything just to get a result without really knowing what to do.

8. Argumentation: Overall, it became clear, that many students were weak in providing justification for their method and answers. Some simply avoided this probably because they are not be used to doing this in mathematics. Others seemed unable to provide such argumentation at all.

9. Guessing: Some students try to solve a problem through simple and unfounded estimating.

10. Giving up: Some students were quick to give up as soon as they had difficulty in proceeding with a problem.

The last two areas of difficulty were often easily overcome by motivational assistance from the interviewer.

The students' reactions to the problems

The majority of the students reacted positively to most of the realistic problems because they were often a welcome change to the type of tasks they commonly meet in mathematics. However, whether or not they understood the problem seemed to have a big influence on how the students liked them (see Mario and Hakan). If students said something about the context, then the comments were mainly positive. Especially, those tasks in which the sense of the calculation was evident to them were regarded as positive. Contexts, in which the students were unable to imagine the situation, led to negative reactions. In addition, the presented problems often seemed to be too unfamiliar to the students because of either a lack or an excess of information provided in the problem. Overall, under-defined tasks seemed to be more unfamiliar to students than over-specified tasks.

Types of modellers

Case-comparing and case-contrasting analysis let to the following ideal student-types concerning problem solving methods for modelling:

1. The good modeller figures out the answer to a modelling problem independently. This type of student is also interested in the problems (see, for example, Mario).

2. The unsure modeller needs assistance or a hint before he or she can solve a problem. Careless mistakes may appear in all areas.

3. The weak modeller has deficits in all areas of modelling. He or she tends to reject modelling problems.

4. The student who does not understand the context has trouble in understanding the formulated situation.

5. The situation-non-transformer: This student either cannot transform the given situation into a sensible calculation or he focuses merely on calculating by taking any numbers without being faithful to the meaning of the problem (see, for example, Hakan).

In the examined sample group, the unsure modeller and the situation-non-transformer could be reconstructed quite often. Weak and good modellers were found less frequently, but were nonetheless evident. The student who does not understand the context was very rare.

Criteria for the development of real world problems

Criteria for the development of real world problems for weak students without experience in modelling can be set up based on the results of the study. These criteria form a complex network.

The *mathematical knowledge* that is necessary to solve a problem should be analyzed carefully and should be adjusted according to the competency level of the students. The *text* of the real world problem should not be too complex to be understood by the students. Students seemed to be more comfortable with *over-specified* tasks than with *under-defined* tasks, maybe because they are more comfortable with discarding information than with estimating information. The problem should be written in a *context that is familiar* to students and so that they understand the purpose of the task.

CONSEQUENCES

The most important result of the study is that *low achieving students are in fact able to solve modelling problems*: this ability was demonstrated by both good and the unsure modellers. The study shows quite clearly that the goals related to the integration of modelling into mathematics lessons can be achieved for low achieving as well as for more capable students.

However, the study also identifies some of the problems some students have. The analysis of mistakes carried out here could inform diagnosis of students' weaknesses during class and therefore make useful intervention possible.

The study also identified criteria that should be considered when developing modelling problems for beginners and aspects that need to be considered when using such problems in class. As the understanding of the task seems to be the dominating criterion for students modelling tasks need to be chosen carefully to ensure accessibility for students so as to avoid an initial negative reaction. Additionally, students are more likely to be able to solve the problem if they understand the purpose of their calculation and if they can relate personally to the problem. Gradually tasks should be introduced that are more difficult in order to allow low achieving students in mathematics to make progress in modelling.

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TRACING STUDENTS' MODELLING PROCESSES IN ELEMENTARY AND SECONDARY SCHOOL

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This study examines 6th and 8th grade students' mathematization processes as they worked on a mathematical modelling problem. We report on an analysis of the mathematization processes and developments of two groups of students, one 6th and one 8th grade, as they worked the problem, with special emphasis on the similarities and differences between the two groups. Results provide evidence that all students developed the necessary mathematical constructs and processes to actively engage and solve the problem through meaningful problem solving. Among the differences between the two groups, 8th grade students were involved in higher level of mathematical communication, projected and effectively employed higher order mathematical concepts and processes and reached better and more refined solutions.

INTRODUCTION

In the present study we aim to study how students in elementary and secondary school work on modeling problems. According to our knowledge, so far only a limited number of research studies focused on students' developments through their work on modeling problems in elementary school (English, 2006; Doerr & English, 2003). Findings from these studies indicated that students in elementary school can effectively work in modeling problems and therefore, researchers stressed that modeling problems should be included in elementary school's mathematics. The present study aims to build on these prior findings, by tracing similarities and differences between two groups of elementary and secondary school students, while working on a modeling problem. The identification of these similarities and differences is expected to further contribute to the appropriate introduction of modeling problems in elementary and secondary school mathematics.

THEORETICAL FRAMEWORK

An increasing number of mathematics education researchers have begun focusing their research efforts on mathematical modelling, especially at the school level. This is evident in numerous research publications from groups of researchers in Australia (English, Galbraith and colleagues), Belgium (Verschaeffel and colleagues), Denmark (Niss, Blomhøj and colleagues), Germany (Blum, Kaiser and colleagues), Netherlands (de Lange and colleagues) and the U.S. (Lesh, Schoenfeld and colleagues), among many others.

A very promising idea, coming from Blum and Niss (1991) documented the importance of modeling as a problem solving activity. Blum and Niss (1991) reported that there is a strong need to implement worthwhile modelling experiences in the elementary and middle school years if teachers are to make mathematical modelling a successful way of problem solving for students. Recent research indicated that student work with modeling activities assisted students to build on their existing understandings, and to develop important mathematical ideas and processes that students normally would not meet in the traditional school curriculum (Zawojewski, Lesh, & English, 2003; Lesh & Sriraman, 2006). As students work in these activities, they engage in important mathematical processes such as describing, analyzing, coordinating, explaining, constructing, and reasoning critically as they mathematize objects, relations and patterns (Mousoulides, Pittalis & Christou, 2006).

A number of researchers stressed the appropriateness of modeling activities for elementary and middle school students (English, 2006; Doerr & English, 2006). English and Watters (2005) reported that there was considerable evidence, in their research with young learners, that students' mathematical ideas had improved after they worked in a sequence of modeling activities. Since students' work in modeling activities is not narrowed only in working with ready made models, students need to construct models in a meaningful way for solving a real problem. This construction can lead to conceptual understanding and mathematization (Lesh & Doerr, 2003; Lesh & Sriraman, 2006). Doerr and English (2003) reported that modeling activities provide opportunities for elementary school students to explore quantitative relationships, analyze change, and identify, describe, and compare varying rates of change, as recommended in the Grades 3-5 algebra strand of the Principles and Standards for School Mathematics (NCTM, 2000). In addition, English (2003) pointed that elementary probability ideas emerging when young students linked the conditions and constrains of problems.

Another important parameter is students' use of their informal knowledge in solving modeling problems. Mousoulides and colleagues (2006) reported that students' informal knowledge helped them relate to and identify the important problem information (e.g., understanding and interpreting the conditions for the solution of a problem). The interplay helped students in finding solutions for the assigned problems and refining solutions accordingly to meet the necessary real world restrictions and criteria (Doerr & English, 2003; Zawojewski, et al., 2003).

As a concluding point, modeling activities provide a pathway in understanding how students approach a mathematical task and how their ideas develop; these activities appear to provide a strong basis for teachers to interact with students in ways that would promote their learning (Kaiser & Sriraman, 2006; Doerr & English, 2006). The latter is among the core aims of mathematics education.

THE PRESENT STUDY

The Purpose of the Study

The aim of the present study is to explore the similarities and differences between elementary and secondary school students, while they work on an authentic modeling problem. To this end, it is expected from both groups of students to work with authentic mathematical problems, using their prior mathematical knowledge to investigate, make sense and understand these problems. In other words, in the focus of the present study is the tracing of the aforementioned similarities and differences in an attempt to explain why these similarities and differences might appear and to explore possible reasons for that. The results of the study are expected to contribute to current research in the area of introducing modeling as an appropriate and successful approach in mathematical problem solving for elementary and secondary school students.

Participants and Modeling Activities

Thirty seven students (22 females and 15 males) from two intact 6^{th} and 8^{th} grade classes in two urban schools in Cyprus participated in one modeling activity, presented below. All students had little experience in solving problems in a mathematical modeling context, since both classes are participating in a larger project on the effectiveness of mathematical modeling in problem solving.

For the purposes of this study, student work on one modeling activity will be presented, namely the "The Best City" activity. The activity is a modified version of one activity derived from a list found in Lesh and Doerr (2003). The purpose of the activity was to provide opportunities for students to organize and explore data, to use statistical reasoning and to develop appropriate models for solving the problem. Additionally, the activity provided a setting for students to focus and work with the notions of ranking, selecting, aggregating ranked quantities and weighting ranks.

The application of the "The Best City" activity (see Figure 1) followed three stages: (a) the warm-up stage in which students read an article with the purpose to familiarize themselves with the context of the modeling activity and to answer readiness questions through a whole class discussion, (b) the modeling stage in which students were engaged in constructing models to solve the activity, and (c) the presentation and discussion stage in which students presented their solutions and reflect on other student solutions.

Procedure

The students spent around 160 minutes (four 40 minute sessions) in completing the modeling activity. The activity started with a whole class discussion on the warm-up task and readiness questions on the related article (this stage lasted around 20 minutes for both groups of students). The second part of the modeling activity was the modeling stage. During this stage, which lasted around 80-90 minutes (for both groups), students worked in groups of three or four to provide solutions for the activity. After completing their work, each group presented its solutions to the rest of the class for questioning, comparing with others' solutions and constructive feedback. Finally, a whole class discussion focused on the key mathematical ideas and processes that were developed during the modeling activity. This last stage of the activity, namely the presentation and discussion stage lasted around 45 minutes for the 6^{th} grade group and around 60 minutes for the 8^{th} grade group of students.

Data Sources and Analysis

The data for this study were collected through (a) videotapes of students' responses during whole class discussions, (b) audiotapes of students' work in their groups, (c) students' worksheets and final reports detailing the processes used in developing models and solutions, and (d) researchers' field notes. Videotapes and audiotapes were analyzed using interpretative techniques (Miles & Huberman, 1994), for evidence of students' mathematical developments towards the mathematical concepts appeared in the modeling activity. The analysis of the data was completed in several steps. First, all transcripts were reviewed by two researchers to identify the ways in which students interpreted and understood the problem, their approaches to selecting, categorizing, and aggregating the different factors, and their mathematization processes as they quantified factors, transformed factors, such as the next year's budget, and combined factors for creating "super factors" for kids and adults (see transcripts in results session). Second, all of the students written products in their worksheets were analyzed to identify and compare the mathematization processes used in their model development to obtain solutions to the problem and to compare solutions among the two groups (6^{th} and 8^{th} grade) of students.

Due to space limitations, we mainly present the results of one group of students in each grade, as they worked on the modeling stage of "The Best City" activity. Each group of students in each grade was selected on the basis of their provided solutions and their whole work. We have to report here that the selected groups were representative of the two classes, in a way that other groups in their classes reported similar work on the provided problem.

Use the data in the table below to find the best city, Anastasia can live in. When you reach an answer, write a letter, explaining and documenting your results, to Anastasia.

	Parks	Nursery Schools	Schools	Cinemas	Restaurants	Shops	Road quality (%)	Next year budget*
Lakecity	2	2	7	1	3	23	45.5	Same
Relaxcity	3	1	4	3	12	16	36.8	More
Safecity	2	4	5	4	4	26	57.2	Less
Dreamcity	0	5	10	0	6	12	19.7	Less
Nicecity	3	2	8	2	5	20	25.8	Less
Livecity	4	3	7	3	8	15	76.2	More

* Next year's budget is compared to this year's budget.

Figure 1. The Modeling Stage in "The Best City" activity.

RESULTS

The results of the study are presented as follows: First, consideration is given to a "microlevel analysis" of the developments displayed by each group of students in working with the modeling activity. Following, this fine-grained microlevel analysis, a "macrolevel analysis" of the mathematization processes displayed by both groups of students is presented, to obtain possible similarities and differences between the different grade groups.

Microlevel analysis

Identifying and clarifying factors

Both groups commenced the question for finding the best place Anastasia could move on, by brainstorming on the factors presented in the table (see Figure 1), questioning the meaning and importance of these factors. In 6th grade group, a student pointed that parks and cinemas are important for a person: "Dreamcity neither has parks nor cinemas. I think that Dreamcity is the worst place for Anastasia". Similarly, students in 8th grade group reported that Dreamcity was the worst place for Anastasia, since: "No cinemas, no parks, few shops and bad roads". On the contrary, while 6th grade students did not discuss the meaning of increased or decreased next year budget, there was a long debate among 8th grade students to clarify what is the meaning of this factor, and most importantly, how this factor is related with other factors and most importantly how budget can influence other factors: "Look at Relaxcity. There are few parks but only one nursery school … this could change, since Relaxcity's next year budget will increase […] People there can use these money to improve city's facilities".

A long discussion between the members of the 8th grade group questioned the representativeness of their ideas and solutions, related to the importance of certain factors. One girl pointer out that "having parks is it not important for me…having shops and cinemas is more important". The same girl highlighted that Anastasia was a college graduate and therefore "many schools and nursery schools are not as important for her as shops, restaurants and cinemas".

Beginning mathematization

After their first impressions, one student in the 6th grade group suggested that the group should focus on comparing two cities at each time. In doing so, students compared one factor every time to find out which city was "better" than the other one. "Dreamcity has more restaurants than Safecity. Its streets are, also, better than Safecity's". There were also attempts to compare more than two cities: "Livecity has more parks than all other cities and the road quality in Livecity is much better than road quality in other cities".

On the other hand, students in 8th grade group presented more sophisticated ideas, such as "Adding horizontally the numbers for each city" and "finding the number of buildings and facilities in each city". In doing so, students made simple calculations and compare their results: "Let's sum the total number of buildings and facilities for each city. This is a way to find which city is the best one for Anastasia". It has to be reported here that students attempted to take into consideration road quality and budget factors: "Safecity's budget will be decreased next year and look at its roads. Quality is only 25%. Road quality can not improve, since they will not have more money to spend on it".

Working with factors

As seen above, 8th grade students started mathematization earlier than 6th grade students. However, it was clear that both groups of students experienced difficulties in their efforts to work with factors such as next year's budget and the quality of roads and to combine these factors with other factors. At a later stage, one 6th grade student reported in his worksheet: "We added the buildings in each city. Budget is an important factor. It means what they will do next year. We decided to keep this factor by itself, since we could not add it with buildings and roads". Similarly, 6th grade students kept the road quality factor as it was, but in their discussions they referred to road quality by reporting that "Dreamcity has bad quality of roads and Livevity has good quality of roads".

A number of differences appeared in 8th grade group's discussion. As the group discussed the meaning of budget and tried to summarize the number of buildings in each city, their next attempts focused on trying to recode the road quality data.

Students categorized road quality as “above average”, “average” and “below average”. This was helpful in transforming the different numbers into three categories and therefore in making use of this factor, in contrast with 6th grade students. More specifically, they added 15 points to the total number of buildings if city’s road average was above average (>60%), 10 points to average cities (from 40% to 60%) and 5 points to below average cities (<40%).

An interesting strategy was presented by 6th grade students after the first presentation of their solutions to the whole class. Instead of finding the total number of buildings for each city, they grouped factors as those being important for young people and those that are more important for adults. Therefore, factors like cinemas, restaurants and shops were categorized as “adult factors”.

Macrolevel analysis of mathematization processes

In this level of analysis, we primarily focus on the mathematization processes projected by both groups of students, during their work on the modeling activity.

Categorizing and Merging factors

As reported in more detail above, b oth groups of students categorized factors as either related to *buildings* (schools, restaurants), *facilities* (parks, roads) and *budget*. Of importance is the sub categories reported by the 6th grade group, who assigned labels as *“buildings/facilities for children and teenagers and for adults”*.

The approach presented by 6th grade students suggested that: “We need to find a way to merge all these columns (referring to the table). My idea is to sum the first 3 columns for each city and then the last 3 columns. If we use this method we will have 2 factors; the first will refer to children/teenagers and the second one to adults”. This idea was adopted and students were able to refine their prior solutions, since: “the second factor is more important for Anastasia; she is probably single and she does not have children”.

An interesting strategy was presented in 8th grade group’s discussion. They transformed, for example, the existing number of parks to numbers from 1-6, by assigning 1 to the city with the maximum number or parks, 2 to the second city and 6 to the city with the least number of parks. Accordingly, they assigned numbers from 1-6 for each factor, except for next year’s budget. Students continued, by adding these numbers to obtain a general factor. They clearly stated that: “the best city is the one with the *minimum* number”. When students were encouraged to also include next year’s budget, one student reported that: “Our solution would be much better, but this is not easy. How can we change this qualitative information (more, less, same) into numbers”? A second student added that it was not the case to add some points for cities with increased budget and subtract from cities with decreased budget, because “more” for one city’s budget is not necessarily the same like “more” for another city’s budget.

Aggregating and Ranking factors

In ranking the factors, 8th grade students first applied a point multiplication system, which was changed two times during student work in the modeling problem. The first system that was applied in an attempt to rank the different factors was to multiply each factor by a number from 1 to 8 (since they had to consider eight factors). The most important factor was multiplied by 8, the second most important by 7 etc. At a second attempt, students decided to “group” factors in terms of their importance. As a result, students grouped the eight factors in three groups and assigned a system similar to previous one, multiplying by 1, 2, and 3, considering the importance of each factor. More specifically, in this “cycle” of possible solutions, they multiplied by 3 the “building group”, by 2 the “facilities group” and by 1 the budget.

On the contrary, 6th grade students ranked the different factors only in a qualitative manner; they considered some factors being more important than others, but that distinction was done only in a qualitative way. For example, one student from the 6th grade group wrote: “Livecity is a better place than Dreamcity. Livecity might have fewer nursery schools, schools and shops than Dreamcity, but these things are not so important. Budget is important, Livecity’s budget will increase and Dreamcity’s will decrease”.

DISCUSSION AND CONCLUDING POINTS

There is a number of aspects of this study that have particular significance for the use of modeling in mathematical problem solving in elementary and secondary school mathematics. First, primary and lower secondary school students can successfully participate and satisfactorily solve mathematical modeling problems when presented as meaningful, real-world case studies. As presented earlier, the activity did not narrow students’ freedom and autonomy to approach and analyze the problem taking into account their prior and informal knowledge. Modeling problems, like the one used in this study, enable different trajectories of learning, with students’ mathematical understandings developing along multiple pathways. Students present different trajectories of learning and because modeling problems can be solved at different levels of sophistication, students can use a diversity of solution approaches; as a result, students of different achievement level can contribute to, and benefit from, the learning experiences these modeling activities offer.

A second aspect of the study is located in the similarities between the two groups of students as they worked in the modeling activity. Students’ work in both groups was impressive; they analyzed the problem using different viewing angles, set and test hypotheses, evaluate, modify and refine their models and solutions. Quite important was students’ engagement in self evaluation; both groups were constantly questioning the validity of their solutions, and wondering about the representativeness of their models.

The third significant aspect lies in the differences between 6th and 8th grade students in using and sharing their mathematical ideas and understandings. Although modeling problems are valuable because they provide a rich framework for developing and presenting students' mathematical skills, only 8th grade students explicitly presented and communicated a number of mathematical concepts and processes, and effectively applied them in solving the problem. The 8th grade group sufficiently used weighting and aggregating data, ranking factors and assigning scores in subgroups of factors. On the contrary, although 6th grade students presented implicitly a number of mathematical processes, they did not manage to effectively apply them in solving the problem, but they partially use them without much success.

Another difference between the work of the two groups of students was from the perspective of communication and assessment. Although in both groups, students adequately communicated their ideas and solutions, it was clear that in 8th grade group, students progressively assess and revise their current ways of thinking. As a result, by listening to and reflecting on their peers' suggestions and models, they undertake constructive assessment. The latter helped students to reach better and more refined solutions. It is not the case that 6th grade students did not communicate sufficiently, but this communication was mainly focused on subsets of information and on discussions on single factors, and therefore was not productive in terms of refining and improving student models.

In preparing students for being successful mathematical problem solvers, both for school mathematics as well as beyond school, teachers need to implement rich problem solving experiences starting from the elementary grades and continue to lower and higher elementary grades. Results from research work like this study that provide both teachers and curriculum designers with details on how students at different grade levels access higher order mathematical understandings and processes.

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A META-PERSPECTIVE ON THE NATURE OF MODELLING AND THE ROLE OF MATHEMATICS

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Abstract *This is a theoretical paper, offering an analysis of some dimensions of the process of fitting mathematical models to situations. The analysis looks at problem characteristics from a meta-perspective that involves determining the degree of decision making that is expected of the solver. It offers to classify problems by degree of explicitness of the mathematical model in the problem, and by problem type of context, i.e. scientific, social, etc. In the latter case problems are found to differ in the degree of freedom involved in fitting a mathematical model and in the role of this model, e.g. describing and predicting results in scientific contexts. The analysis counters traditional conceptions in social context, legitimizing alternative solutions not just on realistic grounds.*

Theoretical background

Changing goals (or not)

In their review of (then existing) literature on modelling and related subjects such as problem solving, Blum and Niss (1991) discuss the change in arguments that were used to promote modelling and applications in mathematics curricula. Earlier arguments talked about the development of general problem solving strategies, critical thinking, and primarily on the use of modelling to increase motivation and develop richer concepts. Later arguments viewed modelling itself *not merely among tactical devices to improve the situation for traditional mathematics instruction, but as an integral part of the discussion of mathematics education as a whole* (ibid, p. 47).

It remained, though, a difficult challenge to convince curriculum writers and teachers to change their perspectives. Since teachers viewed the construction of mathematical concepts as their ultimate goal, they could accept the implementation of modelling tasks as a tool but found it hard to accept modelling as a goal.

These beliefs have not changed much since decade-ago studies. Although, mainly due to the Pisa study, teachers and curriculum writers are using more modelling tasks, many still do it more as a required "preparation" for Pisa than through new convictions or goals.

According to comparative studies the situation is different in different countries. Thus, for example, Ikeda and Kaiser (2005) report that while teachers in the UK use modelling tasks more meaningfully, both German and Japanese teachers use

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real world and modelling examples mainly for introducing, exercising and illustrating mathematical concepts rather than acquiring modelling skills. Even teachers who use modelling tasks and declare their belief in the importance of modelling, might, in fact, prefer simple tasks or view the process mainly as an application (Kaiser, 2006). Similar findings are reported by Boaler (2004) in observing teachers with identical explicit reform beliefs but a wide range of implicit beliefs. Boaler (ibid) describes Mr. Life as a prototype of a teacher who does believe in giving challenging problems and yet worries that children will have difficulty solving them as such, and therefore proceeds by giving hints and more guided instruction.

Indeed, children, and as a matter of fact pre-service and in-service teachers as well, do find it difficult to solve inquiry type problems when encountering them at the first time. Possibly, one of the reasons for why there is such little change in teacher goals and attitudes towards modelling lies in the fact that it takes time for children to develop and exhibit modelling behavior and modelling habits. As can be observed in a three year study reported by English and Fox (2005), there was a big difference in the quality and nature of children modelling behavior between their third year and earlier years in the project. Observing such changes in children would probably be a necessary, though not sufficient, cause for triggering teacher change.

As concluded in research on teacher beliefs about modelling (e.g. Schorr & Lesh, 2003; Kaiser, 2006), the situation calls for special efforts in teacher education of in-service and pre-service teachers in order to change teacher beliefs. In an answer to this call, this study suggests that we might start with a theoretical epistemological analysis of modelling. It proposes a meta-perspective analysis leading to insights on the nature of modelling through an examination of the relationship between mathematical models and situations.

Instructional goals and definitions

In discussing the two instructional goals that are central to this work, constructing mathematical concepts and developing modeling skills, I will use the term *mathematical model*. Researchers use the word *model* in many different combinations and connections, resulting in similar terms that have different meanings. Therefore I will start by clarifying what I mean by *mathematical models* and add definitions of other types of related models to emphasize the difference.

Mathematical concepts together with a variety of mathematical tools (e.g. graphs) will be called *mathematical models*. This term is used here in order to call attention to the role of mathematical concepts as organizing tools and to promote this perspective.

The construction of mathematical concepts that serve as *mathematical models* involves the use of *didactical models*, i.e. routes that are combinations of didactical objects and instructional trajectories as defined by Thompson (2002).

Once an initial repertoire of *mathematical models* is constructed, they can be used to analyze situations and solve problems. These actions that include organization of situations and their mathematization with the help of mathematical models and tools define the *modelling* process. The engagement in such actions facilitates the development of modelling skills.

The two goals, constructing mathematical models and developing modelling skills, are related and support each other. In order to analyze situations mathematically, some initial mathematical models and tools should be available. The child makes choices from her mathematical repertoire and sometimes integrates different mathematical concepts. In turn, this act of modelling contributes to the further development of meaning and integration of the concepts applied in this process.

Still, there is also some tension between these two goals, especially when it comes to determining the nature of problems or tasks constructed to achieve them. On the one hand, when a specific concept is learned, tasks are designed with the purpose of demonstrating the use of this concept and therefore often consist of 'end of chapter' applications of that concept. There might even be some effort to teach a repertoire of examples that would serve as prototypes for using that concept. On the other hand, when the goal is to develop modelling skills, tasks have to provide the opportunity to choose among variety of mathematical models.

The common desired problem solving activity for the first goal involves an almost automated application of a mathematical model. As described by Peled and Hershkovitz (2004), the application of a mathematical model might be carried out without much deliberation over the rationale for using this model. Such a behavior would not be considered a good habit for the purpose of developing modelling skills, i.e. good decision making processes that involve educated argumentation about mathematical choices.

Problem explicitness and mathematical decision making

It might seem quite trivial to offer a problem classification that would differentiate between traditional problems and modelling problems. Nesher (1980) noted long ago that traditional problems have only the essential information needed for solving them, and therefore have a sterile stereotypical nature. However, while one might tell traditional problems from modelling problems by looking at the style of the narrative and on features mentioned earlier (everyday situations, inquiry, etc.), the purpose of this classification is to focus the attention on the following main issues: Does the problem give the child an opportunity to go through a decision making process in choosing mathematical models and tools?

Problems can range from being very explicit about the mathematical model that can be used to not giving any hint about it. On one end of the explicitness scale we have problems that explicitly say what mathematics to use and thus the solver has no say at all about it. In the following problem, for example, the solver is explicitly told to apply a mathematical model of ratio:

> *When the animal-lover, Mr. Henry, died he left 240 thousand dollars to be divided among two animal shelters using a 2:3 ratio between the amount that Cat Best Home gets and the amount that Dog House gets. How much money should each shelter get?*

On the other end of the scale we have problems that do not give any explicit clue as to the mathematics that could be used:

> *Haifa (whole-week) film festival offers viewers a discount in the form of a fifth free ticket for every 4 paid tickets. Would it be worth my efforts (money-wise) to convince a friend to buy tickets together?*

In between the two extremes there are problems with different degrees of explicitness. However, just as in discussions on the difficulty of a given problem (e.g. Hiebert et al., 1996), the degree of explicitness, too, depends on children's experience. For example, in the following problem the use of ratio and proportion is implicit. Still, in many cases students would have solved problems of this type, and thus they might recognize it immediately as a case of proportion:

> *When I joined the party I could sit at a table where (taking myself into account) 3 pizzas were served to 7 friends, or a table with 4 pizzas and 9 friends. Given that I love pizza, where should I have chosen to sit?*

Still, the above example can be said to be closer to the "very explicit" end of the scale. Other problems, such as many of the Pisa tasks, are closer to the Festival problem, but more explicit than it. Although these tasks require some understanding and organization of the situation, they often lead and expect the solver to use certain mathematical models. This is not surprising in view of the fact that Pisa tasks have to be scored.

That's how it is in real life and this is how you do it in math

One of the more obvious features on which traditional problems, realistic tasks (Verschaffel, Greer, & De Corte, 2002), Pisa tasks or model-eliciting tasks (Lesh and Doerr, 2003) differ from each other is in the nature of introducing real life situations in the problems.

Realistic problems introduce reality using a minimalistic narrative. They look very similar to traditional problems, and indeed, unless one (a child or a teacher) undergoes some special instruction, one would often come up with an "automatic" solution as expected in standard problems. Taking, for example, the following item from Verschaffel, Greer and De Corte (2002), a chapter in which the authors review their own studies and others' replications on this subject:

John's best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometer?

Most students and teachers in these studies solve the problem by multiplying 17x10, disregarding the probable fact that John cannot keep up his best speed for the entire kilometer.

Following the introduction of these problems with my own students (including in-service and pre-service teachers), they would often react by saying: *This [taking fatigue into account] is how you do it in real life, but this [using automatic multiplicative structure] is how you do it in math.*

I have found this reaction to be indicative of some deep misunderstanding of the role of mathematics. Such an attitude can also be found in Koirala (1999) where teachers are given a problem in a much less traditional form. The problem deals with a situation where two shoppers buy 3 pairs of shoes altogether getting the cheapest pair for free, and have to decide how to split the cost. Teachers solved the problem by offering a variety of realistic solutions. However, the article's message (and title) seems to be that one should take care of not loosing the mathematics. The attitude is that these are possible realistic solutions, but there is only one best solution to the situation, namely, offering the same percent reduction to both shoppers.

While math educators do not agree with teacher reactions towards a realistic solution in the runner's case and have tried to change them, the attitude towards realistic solutions in problems, such as the shoes case, where there is some expected good mathematical solution, are different. In the latter case realistic solutions are accepted with some understanding on the grounds that: *that's how you behave in life,* but are not considered as sound *mathematical* solutions.

In the following section I will argue against this conception in an effort to show that realistic solutions of this type deserve the same *mathematical* status as the expected mathematical solution. I will also develop an analysis that is geared towards understanding the role of mathematics in problems such as the runner's problem.

Context and degrees of freedom in modelling

While in an earlier section I have dealt with the extent of telling solvers what math to use, this section intends to call our attention to situations where certain mathematical models are imposed creating an impression that such model fitting is engraved in stone. The ambivalent attitude towards realistic solutions evidently stems from such impositions.

As has long been claimed in problem solving literature, it is often easy to tell which mathematical model a curriculum writer believes should be used in a given problem simply by looking at the title of the chapter. Such 'end of chapter' problems can be found, for example, in a chapter on arithmetical average:

> *The Grades Problem: Ms. Mollekula, the chemistry teacher, is preparing her students' grades for their report cards. She goes over Oren's grades in class exams during the last semester realizing that he got two 6-es, one 8, and four 9-es. What will Oren's grade be?*

Similarly, in a chapter on ratio and proportion one might find a problem such as:

> *The Lottery Problem: Two friends, Anne and John, bought a $5 lottery ticket together. Anne paid $3 and John paid $2. Their ticket won $40. How should they split the winnings?*

Apparently, the problem composer believes that the answer to the first problem should involve calculating a weighted average of the grades, and that the lottery ticket solution should involve splitting the winnings into two parts according to the purchasing ratio. In accordance with the previous discussion on teacher attitude, it is expected that other solutions might, in the best scenario, be accepted as good realistic solutions, but nevertheless not what the teacher and the textbook would accept as a relevant answer.

However, it is this article's claim that both problem composer and teacher, who view these problems as having one ideal mathematical solution, are wrong. To establish this claim, another problem will be introduced:

> *The Lemonade Stand Problem: During the Country Fair Abby and Bill put up a lemonade stand. Bill bought disposable cups for $5 and Abby bought some concentrated lemon-juice cans for $10. These were all their expenses. They sold lemonade for a total of $150. How should they split the money?*

The Lemonade problem would usually be solved by reimbursing expenses to each partner, i.e. $5 to Bill and $10 to Abby, and then splitting the remaining amount evenly between the two.

Looking at the general problem situation, we can see that in both Lottery problem and Lemonade problem the two partners invested a certain amount of money. Yet, in the first case the common answer is proportional sharing of the winnings and in the second the profits are evenly split.

The reason for this difference lies in a moral argument: In the lottery ticket case there is a hidden assumption that the partners should get the same amount of profit for each invested dollar. However, these partners could have reached another agreement, and it would not be less 'mathematically correct'. Similarly, the Lemonade stand partners could have viewed the expenses as an investment and could have decided to split the profits proportionally. In fact, when I gave the Lemonade problem to my students immediately after the Lottery problem, most of them were influenced by this priming and solved it by using proportion…

What is the nature of these problems that enables us such mathematical freedom? To answer this question the above problems will be compared with the following problem:

> *The Mixture Problem: Having mixed 3 cans of yellow paint with 2 cans of blue paint, Ron got a nice shade of green. When he ran out of paint and needed 40 more cans to finish painting his fence, he wanted to get the same shade of green by mixing yellow and blue again. How many of the 40 cans should be yellow, and how many should be blue?*

The Mixture problem can be solved by applying a mathematical model of ratio and proportion. The rationale for this application lies in our knowledge about the way color mixing 'behaves'. The following points summarize some of the differences between the first three problems: Grades, Lottery and Lemonade and the Mixture problem:

- The mixture problem describes a scientific phenomenon that could be observed and learned, while there is no phenomenon to observe in the first three problems.
- In the mixture problem the solution can be viewed as an experimental prediction that can be tested by making the mixture. In the three problems there is no meaning to talking about 'a prediction'.
- The first problems, present a social-moral situation where the participants can make any decision that fits with their values. The mathematical model 'comes into action' after such a decision has been made. Before that, it can only act as 'an idea', i.e. familiarity with existing mathematical models might influence the set of models from which the choice is made. Thus, the solver has quite a lot of freedom in fitting a mathematical model.
- In the mixture problem there are, in the sense of this discussion (see reservations in the next comment), no degrees of freedom in choosing the mathematical model. It is the phenomenon that 'dictates' the mathematical structure.
- It should be noted that in a scientific case the observer might not have a good mathematical model available, or might go through a developmental cycle in fitting the mathematical model as described by Lesh and Harel (2003). In addition, the model fitting might depend on the 'experimental' conditions. For example, the behavior of the colors and hence the mathematical model might be different for large quantities.

- Letting the textbook dictate the use of a specific mathematical model in problems such as the three social-moral problems, actually means that we let math *determine* life events rather than *describe* them.

This analysis brings to mind Blum and Niss (1991) differentiation between normative models used for economic items such as taxes and cases involving value judgments, and between descriptive models used to describe physical phenomena. Indeed, different types of models should be used for these different types of contexts: a normative model for social-moral context and a descriptive model for a scientific context.

Our analysis shows that these different types of models carry with them different constraints. While descriptive models are expected to best fit a phenomenon, normative models depend on social agreements. This difference results in different degrees of freedom in fitting a mathematical model in these two types of problem contexts.

It should be noted that only two types of problems have been discussed here. However, similarly to having problems at different points on the explicitness scale, problems can also be found at different locations on a degree-of-freedom scale. For example, in Peled and Bassan-Cincenatus (2005) we give an example of a context which is more extreme than a scientific case, leaving no (zero) degrees of freedom. Further analysis will be done in a more extended paper.

Coming back to Koirala's (1999) 2+1 shoe sale example, there is no justification in saying that one mathematical model fits the situation better than the other. There is no observed phenomenon, and the buyers can decide on their own criterion. One might discuss the rationale for the criterion, and prefer one rationale over the other, but this does not change the mathematically symmetrical status of the offered solutions.

Concluding Remarks

This paper has started establishing a tool for analyzing the nature of the modelling process from the perspective of explicitness of the mathematical model and the freedom one has in mathematizing the situation as a function of the type of context.

This offered classification can be used with teachers and curriculum writers for several purposes. For example, it can be used to evaluate the modelling challenge in given problems, and it can be used to discuss the meaning and the roles of mathematical models.

The analysis of degree of explicitness is expected to increase teacher awareness of the (small) amount of challenge in problems. Existing text books, even in programs declaring themselves as adapting new trends, still offer a lot of traditional problems. These problems are sometimes disguised as reform problems by being more complex or using everyday context. The decision making criterion should help reveal the actual amount of mathematical thinking that is left for the child.

With the help of this classification, my graduate students, most of whom in-service mathematics teachers, were asked to identify problems of different degree of explicitness. Using commonly used math textbooks, the students reported on finding mainly problems of very low degree of decision making, i.e. high degree of explicitness.

The classification of problems by type of context is expected to change teacher attitude towards children's solution and increase both teacher and children understanding of the role of mathematics. For example, in the runner's case teachers and children are expected to view the role of mathematics as describing and predicting the observed phenomenon.

Further analysis can benefit from looking at other fields and contexts where mathematics is being used. For example, in our work on proportional reasoning (Peled and Bassan-Cincenatus, 2005) we found Talmoodit laws that deal with inheritance and offer a mathematical model for money allocation that is not based on proportional sharing. We have also found a variety of other cases that look similar to problems that are solved using proportion and yet, using economics, operations research, or game theory models, alternative solutions are offered.

To summarize, this study offers a theoretical analysis of modelling which is relevant to researchers, curriculum writers, teachers and students. Going back to the need, stated in the introduction, to change teacher goals with regard to modelling, the basic assumption taken here is that this analysis would increase teacher awareness and facilitate teacher understanding of modelling. Dealing both with modelling and with the role of mathematics, it would result not only in goal change but also in a change of conceptions about modelling and about mathematics.

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THE ROLE OF MATHEMATICAL KNOWLEDGE IN A PRACTICAL ACTIVITY: ENGINEERING PROJECTS AT UNIVERSITY LEVEL

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This paper deals with mathematics education in engineering School. In particular, we are interested in the role these mathematics play in the professional work of engineers. We have chosen to realize this study in Engineers Professional Institute in France. This Institute uses an educational model closely related to the industrial world. During their formation students are required to study a practical question. This activity intends to reproduce the industrial engineer working context. Therefore, our research is focused on these so-called « projects ». In this paper, we present the inquiring methodology and we give an idea of the first outcomes.

The research problem

What place should be given to mathematics in the engineers' formation? Which contents should be approached in this formation and how should they be approached and articulated with other domains of the formation?

These questions have already been asked and treated in different institutions. For example, Belhoste et al. (1994) who studied the formation given by the French *Ecole Polytechnique* between 1794-1994, have shown that the designing of mathematics syllabus recurrently gave rise to discussion.

One of these questions was to decide whether the most important mathematics domain to the engineering formation would be geometry or calculus. This point was clearly related with the most general debate, theory (general method) vs practice (application):

> Pour Lagrange, l'analyse est une méthode générale qui s'applique à la géométrie et à la mécanique, et, sur ce point, il est en accord avec Monge. Mais pour Monge, ce sont les applications qui donnent la vérité de la méthode… » (Belhoste, 1994)

Nowadays, these questions are modified by the technological development, technology taking an increasing place in the engineers' work:

> Before the advent of computers, the working life of an engineer (especially in the early part of his or her career) would be dominated by actually doing structural calculations using pen-and-paper, and a large part of the civil engineering degree was therefore dedicated to giving students an understanding and fluency in a variety of calculational techniques. For the majority of engineers today, all such calculations will be done in practice using computer software. (Kent, 2005)

In other words, the development of powerful software changes the mathematical needs because this software encapsulates some of the usually taught mathematics. Mathematics may even appear to be useless to some engineers.

What is the importance and which role given to the mathematics education~~s~~ in this new context?

During last years, various researches concerning the nature and the role of the mathematical knowledge in the workplace have been realized (Noss et al., 2000; Kent & Noss, 2002; Magajna & Monagan, 2003; Kent et al. 2004). These researches focused on mathematics and the engineers' work is just a limited amount of those, but from them, we got an outline of the state of the art in this domain about the questions considered as crucial and the theoretical frameworks developed to approach them.

These works point out the existence of gaps between the educational programs and the real world in which the engineers work. For example, the institutional speech asserts that undergraduate engineers need a solid mathematical education, but the researches show that for graduate engineers mathematics is of little use in their professional work.

> Once you've left university you don't use the maths you learnt there, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers in this firm, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again. (Kent and Noss, 2002)

In the same way Prudhomme's research reference recognized the difference between mathematics education in the engineering school (or college) context and mathematics in a workplace context. According to Prudhomme (1999), this gap between educational and working contexts can be explained by the fact that in the first case mathematics follows a disciplinary logic: *The knowledge and its use are built for a disciplinary aim, to answer a prescription of the teacher, without knowing if they really become means to resolve real problems because the solutions have been virtual.* (Prudhomme, 1999). The workplace logic says that knowledge built in the first logic can't be used in the second.

It appears that knowledge built in disciplinary logic is inappropriate for the working context. Noss and Al. (2000) have developed this point in a research focused on mathematics in workplace:

> From a mathematical point of view, efficiency is usually associated with a general method that can then be flexibly applied to a wide variety of problems. This is clearly not the case in the workplace. Even if a number of tasks could potentially be solved with a similar approach, practitioner prefer to use different approaches for each task, partly based on the resources at hand. The crucial point is that orientations such as generalisability and abstraction away from the workplace are not part of the mathematics with which practitioners work.

In our research we intend to contribute to the analysis of the observed gaps and to investigate the role that educational practices and technology play in these gaps. We especially study how one innovative practice in a French engineering Institute intends to articulate theoretical and practical knowledge.

In this paper we mainly focus on an innovative practice, the so-called 'engineering projects' and the methodology that we have implemented to observe it. In the same way we present some data obtained during the observation. The analysis of the obtained data is in process, but we think that the above mentioned analysis can be realized in the frame of the Anthropological Theory of Didactics (ATD) proposed by Chevallard (1999). In this paper we briefly justify our election.

Institutional study

In order to realize our study, we have chosen the Institut Universitaire Professionnalisé d'Evry. This Institute uses an educational model of practical education closely related to the industrial world. This educational model is very interesting for our work, because it is possible to study mathematics taught in an engineering school (college) and at the same time mathematics in use within the practical part of the formation which is closer to the industrial world.

In the first part of our study, we realized an institutional study, for this we have used the Prudhomme's classification, which he introduces in order to analyse the engineering college curricula. In this classification two types of knowledge are considered: purely scientific knowledge and technological knowledge. This last mentioned includes the knowledge related to the technique, in the theoretical or practical sense. For example, Robotics, Mechanics.

Taking into account Martinand's terms reference, Prudhomme organizes these types of knowledge as "disciplines of service" for the purely scientific knowledge and "disciplines of formation" for the technological knowledge. Hence mathematics is taught as a purely scientific domain and yet as a discipline of service, which will become operational in the disciplines of formation.

In our research, we will specially focus in the disciplines of formation, not only in the mathematics as a discipline of service. In particular, we focus on an innovative practice, the so-called *Projects*, which intends to strongly connect the official educational universe of disciplines and the professional world of engineers.

The projects

The projects are realized by a group of three or four students, very independent, respecting a didactical organization which tries to reflect the real organization in workplaces.

The **engineering projects** are carried out by teams of students in their fourth year of engineering school, over five weeks. The subject of every project is open; there is no previous requirement established by client. The final production and the route towards it have to be built together in the same process. Therefore students have to organize and plan their work, to look for solutions; this generally supposes that they adapt or develop their knowledge.

The projects are realized in two phases. After the first one the students write an intermediary report; in this report they describe the pre-project which is in general justified by a study of the subject. They present the technological solution they have chosen among those they have found during their exploratory work. In the second phase the pre-project must lead to a concrete product.
In this kind of projects, the manager is a college teacher, who plays the role of a client who requests a product from a student's group. All the terms and conditions of the project are described in the schedule of conditions (cahier des charges) which is negotiated between the client (teacher) and the distributor (students). The students are supposed to work on their own to come up to the client's request.

The project is assessed from on a double point of view, combining workplace and engineering school requirements. The client must be convinced that the technological solution is the best. But this evaluation is also academic; the students present their work to a jury composed of college teachers. The jury evaluates the use of tools in relation whit knowledge taught in the engineering college. Moreover the students are often asked to justify some of their claims.

Projets Observation methodology

We have realized two observations of the projects. To realize the observation of projects, we used Dumping methodology. In the first phase of project (two weeks) we carried out questionnaires and semi-structured interviews with the students and the clients – tutors. After this phase, we collected institutional data, specifications (document), intermediary reports and documents used for the development of projects. This allowed us to get familiar with projects.

For the second phase we chose only three projects, our aim to be able to realize a deeper and precise observation. To select these projects, we based on the intermediate reports following two criteria: 1) the presence of explicit mathematical knowledge and 2) the project domain such as aeronautics, mechanics, electronics, etc.

In the third week of the project, we met with the students' teams (three teams for three projects) for an interview about the intermediary report; the aim of this interview was to understand the project and to investigate on the role of the identified mathematical contents. We asked the students to do a brief exposition of their project. The aim of this exposition was to identify the role that they were giving to the mathematical content expressed in their intermediary report.

From this, we identified the work division inside the team, and we realized that only one student has the responsibility to develop the mathematical activity. After these meetings, interviews were realized with each student individually.

Some Data from projects observation

In this part, we will present the data collected in the first and second observation in each one of their phases. As well the methodological tools used to obtain the data.

The data from the first and second observation are very different. We consider that this difference comes from the methodology we have employed. The first observation allowed us to assess the methodology; for the second observation the methodological changes allowed us to collect more interesting data.

In the first place we'll present the data obtained in the first phase of the projects, which in our research correspond to the identification of the mathematics used in an explicit way in the first part of the project. We'll do it in a comparative way between the first and second observation; afterwards we show a classification of mathematics identified in this phase.

Later on we'll present the role of the computer software, strongly used in the second phase of the projects. With special attention to what we call 'intermediary element'. The reason of that name is because these kind of elements could be placed in element between the mathematical knowledge taught at College of Engineering and the one used in the practical activity (project).

First Phase

In the first phase of the projects we identify the mathematics used in an explicit way. Here we present the data collected. In order to identify them we have done questionnaires and semi-structured interviews.

Elementary Mathematics (First Observation)

In these projects, mathematics is used in an explicit way, and is relatively elementary such as: functional relations, trigonometry and the application of formulae, calculations. We noticed this use of elementary mathematics in student's answers to questionnaires and in the intermediary reports. Within this observation we did the first interview with 6 teams, the aim (or objective) was to know the possible use of mathematics. It was our first contact with students and projects.

Q1 There will be mathematics in the project?
-mathematics, not, only of **empirical formulae**
- **Calculations** of resistance

Q2 what it is empirical formulae?
-the empirical formulas it is those which one uses in practice
-where it is not necessary make the demonstrations
-You learn how to use them with the experience
- It is not necessary to understand them, you should only know-how,
-and how it is applied

After the interview, we choose 3 teams where the topics were diverse and to observe more than 3 teams was not possible. We did an analysis of intermediary reports within the Anthropological Theory. Other than to relate the techniques described by the students to task out of our field was not easy.

Elementary mathematics and complexes mathematics (Second observation)

In these projects, elementary mathematics is explicitly used but also some more complex mathematics such as: Differential Equations, transform of Fourier, elements of dimensional analysis, finite elements. In this observation we firstly met 16 student's groups and we studied 16 intermediate reports. The students' answers showed the complexes mathematics used in several projects.

Type of knowledge, tools and skills	Acquired experience in the class
Software used	**Solidworks** –CAO **Ansys for structure calculations**
Calculations made	Fluids Mechanics, vibrations, finite elements
Use of formulae, graphics representation, geometrical representation	Formulae, graphics, abacs, schemes
Other mathematics (functions, linear algebra, differential equations, probability, statistics…)	Functions, Differential Equations
New knowledge for the next step	Yes, documents provided by the tutor concerning new topics
Useful university knowledge for the project	Mechanical Conception, RDM, vibration, Fluids Mechanics
Also useful mathematics lessons for the project	No, Probability don't seem useful at the moment

A questionnaire was done in the first contact with the students. It was the same for the 16 teams and the questions were much more precise. After this questionnaire, we realized some semi-structured interviews based on questionnaire with some of the teams.

In this observation, the intermediate report analysis allowed us to choose three teams of the second phase. We choose three projects of the same domain or field that was important to be able to understand better the projects and the role of the mathematics in these.

In the same way we identified explicit mathematics in the intermediary reports. For example, this extract is from the project: Design and realization of a system of measurement for the blower. The differential equations are used in order to calculate the flexion of blade. This method of calculations is theorized in structural calculus, which is a discipline of formation.

En partant de la formule de la flèche suivante :

$EI_z y'' = Mflz$?

En intégrant, on obtient la formule suivante :

$$y'' = \frac{Mflz}{EIz} \qquad \text{avec } Mflz = F(L-x)$$

$$y'' = \frac{F(L-x)}{EIz} \qquad E = \text{module d'Young du matériau}$$

$$Iz = bh^3/12 \quad \text{(moment quadratique)}$$

$$y' = \frac{FLx}{EIz} - \frac{Fx^2}{2EIz} + C_1 \qquad \text{détermination de } C_1 \text{ et } C_2 :$$

$$y = \frac{FLx^2}{2EIz} - \frac{Fx^3}{6EIz} + C_1x + C_2 \qquad \text{les conditions aux limites nous donnent :}$$

en $x = 0 \Leftrightarrow y' = 0$ et donc $C_1 = 0$

de même en $x = 0 \Leftrightarrow y = 0$ et donc $C_2 = 0$

on obtient donc la formule de la flèche en $x = L$:

Where does mathematics used in the projects come from?

In both observations, we observed that in certain subjects belonging to the sciences of the engineering field such as the Resistance of Materials, Mechanics or Electronics, there is a strong implication of mathematical elements which is not recognized as such by the students, because they use software that works for them.

After the first observation we classified the different types of mathematics knowledge used in projects.

Mathematical knowledge out of context it is the whole of school mathematical knowledge, i.e. knowledge which is given during the courses of mathematics.

Mathematical knowledge in the context of the engineering sciences is mathematical knowledge present in the constitution of engineering sciences, such as mechanics, dynamics, the structural analysis, the resistance of materials, etc.

The ‘meta-tool’ (Bissell, 2004) belongs to this kind of mathematical knowledge. This term refers to highly sophisticated tools from the mathematical models, used in the fields of electronics, telecommunications and control engineering.

Mathematical knowledge in the context of the engineering practice is that which is used in a systematic way. They are presented in the form of method, like a process recognized socially to solve specific tasks.

Mathematical knowledge in the technological context is that which the use of software mobilizes i.e. mathematical knowledge necessary to uses software.

In fact, we noticed that computer software like RDM (the Resistance of Materials), Solidworks, Catia and others, work on the basis of advanced and complex mathematics. These computers software play a fundamental role in the project development, they allow users to realize calculations, simulations and mechanical systems designs.

In second observation the computers software such as: Matlab, Ansys, Labview are used.

Uses of MatLab

We identified the use of MatLab in the project: Development of a conveyor belt for the aerodynamic study of a light ultra vehicle. To study the aerodynamics' phenomena of a vehicle, it is necessary to reproduce the real conditions. In this project the aim was to build a conveyor belt to reproduce the velocity floor. The group of students designed a conveyor belt as figure 3. The students use Matlab to simulate the system or one of their parts, in particular the simulink option. This option allows to manipulate a system (figure 4).

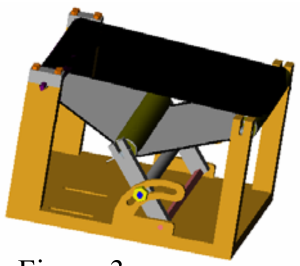

Figure 3

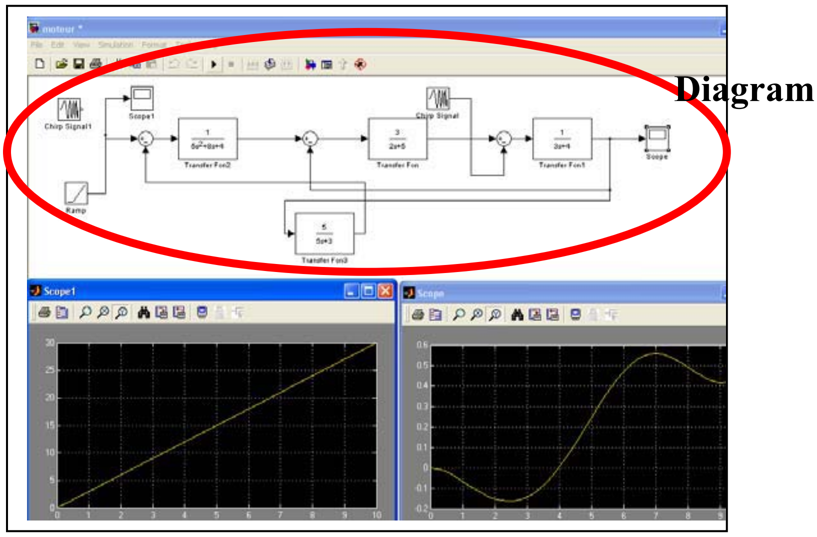

Figure 4

The diagram that appears inside figure 4 is a mathematical model. In this case the diagram models the electric motor of the system. This diagram comes from a discipline of formation: Course d'asservissements. This diagram could be an intermediary element between the mathematical knowledge taught at College of Engineering and the one used in the practical activity (project).

At this time, our work is in the process of analyzing the obtained data. To realize this data analysis according to an institutional study we have decided to use the Anthropological Theory of Didactics (ATD) proposed by Chevallard (1999). For this theory, the knowledge is considered like an emergent of institutionally located practices. The ATD proposes the praxeologies or disciplinary organisation as a tool that allows to model knowledge.

This approach allows us to treat our questions in terms of relations between institutions and to characterize the relations that each one of these institutions have with mathematics, across the notion of institutional relation. This notion allows us to describe and compare the way in which mathematics live in the different devices of formation with the help of the mathematical and didactical praxeologies notions.

Conclusion

The project's observations allow us to study a practical activity closer to professional world within scholar world.

The data obtained show the complexity of the engineer's practical activity. The projects are realized with high technical knowledge and those are a confluence of knowledge and tools. With regard to mathematical knowledge, we noticed that there are several kind of mathematical knowledge:

1) mathematical knowledge out of context

2) mathematical knowledge in context of the engineering sciences, included 'meta-tools'

3) mathematical knowledge in context of the engineering practice

4) mathematical knowledge in technological context

In regard to our previous diagnosis, we need a deeper analysis to assume that the role of the different kinds of mathematical knowledge in the projects is technological in Chevallard's sense, and the mathematical knowledge is the theoretical support of the projects. We stressed that we don't considered the mathematical knowledge in the classical sense.

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DERIVATIVES IN APPLICATIONS: HOW TO DESCRIBE STUDENTS' UNDERSTANDING

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The research described in this paper is part of a larger study, in which we will follow students for a period of three years (from grade 10 until grade 12) and describe the development of their understanding of derivatives in applications. A framework and instruments have been developed for this research. The framework is based on work by Zandieh (2000) and Kendal and Stacey (2003). We developed a framework that focuses on representations as part of understanding derivatives, but includes applications as well. To validate our framework a task-based interview has been administered to six students of different grades. The results show that our framework offers advantages in describing students' strategies, but there are also some limitations.

INTRODUCTION

In the Dutch mathematics curriculum for secondary schools, the role of applications increased over the past 15 years. When the concept of the derivative is introduced to students in grades 10–12, most textbooks provide students with opportunities to learn the concept in different contexts. In grade 10, the first chapter on the rate of change often starts with a variety of contexts related to velocity, steepness of graphs and, for example, increasing or decreasing temperatures. Textbooks provide tasks on the average rate of change, average velocity and the slope of a secant line. The step towards instantaneous rate of change is kept intuitive, as most textbooks avoid the use of the formal limit definition, or only mention it on one page without using the notation with a 'limit'. After this introduction the power rule is introduced. There are exercises with the power rule, but also applications about, for example, velocity of cars. In grades 11 and 12 many differentiation formulas are introduced, and most chapters on derivatives contain applications. The expectation of the curriculum designers was that this context-based approach provides students with a better understanding of the concept of the derivative and enables them to use their knowledge and skills flexibly in different settings. Aim of our research is to describe the evolution of students' knowledge and skills in grades 10–12 within this context-based learning environment. To measure this evolution an adequate framework and corresponding instruments were required. The main question of the research presented in this paper is: *How can students' understanding of the concept of the derivative in different applied contexts be described in a structured way?*

THEORETICAL BACKGROUND

Understanding the concept of the derivative

The concept of the derivative is multi-faceted, and it is complex to determine to what extent a student understands the concept. Many publications on understanding concepts use words like scheme, structure, connections and relations. Hiebert and Carpenter (1992) describe understanding in terms of the way, in which information is represented and structured. The degree of understanding depends on the number and strengths of connections between facts, representations, procedures or ideas. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or a larger quantity of connections. Tall and Vinner (1981) used the term *concept image* for the 'total cognitive structure that is associated with a concept'. Anderson and Kratwohl (2001) defined conceptual knowledge as the interrelationships between the basic elements within a larger structure that enable them to function together. Thus to describe students' understanding of derivatives, we have to investigate which connections and relations between relevant concepts are made by the student. A better understanding might be reflected by more and better connections.

Connections between representations

Many researchers have emphasized, that not only the distinct types of representations are important, but that translations between these and transformations within these, are also important (Dreyfus, 1991). Zandieh (2000) developed a framework for analysing students' understanding of the concept of the derivative. One of the components in her framework is the use of multiple representations. The five representations in Zandieh's framework are: (a) symbolical, such as the limit of the difference quotient, (b) graphical, such as the slope of the tangent line, (c) verbal, such as the instantaneous rate of change, (d) paradigmatic physical, such as speed or velocity, and (e) other representations. Kendal and Stacey (2003) also emphasize the importance of analysing how students work with different mathematical representations. Central in their 'derivative competency framework' are the numerical, symbolical and graphical representations. Both frameworks highlight the importance of different mathematical representations, but in Zandieh's framework translations between representations and non-mathematical applications cannot be visualized well (Roorda et al., 2006).

In line with the above mentioned research we judged that we needed a framework in which *translations between representations can be visualized* (Roorda et al., 2006). In our framework three categories of representations are used: (a) symbolical, (b) graphical; (c) numerical. We rejected verbal ways of representation as a separate category, because it turned out to be redundant: a student can talk about derivatives from a formulae viewpoint (such as difference quotient, derivative), from a graphical viewpoint (slope, steepness), or from a numerical viewpoint (such as average increase, instantaneous rate of change).

Connections within representations

As mentioned above, not only translations between representations are important but also transformations within one representation (Dreyfus, 1991). Zandieh's framework uses three object-process layers. The three object-process layers are ratio, limit and function, as can be seen in the symbolic definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$; f' is a function whose value at any point is defined as the limit of a ratio. For each layer a student can mention these objects and processes. When a student knows how to use a certain object, but he or she does not know the underlying process, Zandieh speaks of pseudo–objects. An example of a pseudo-object is that a student knows the value of a limit, the end result of a process, without the recognition of the process that leads to that result. A difficulty is that many problems involving the concept of the derivative can be solved using only pseudo-object understandings (Zandieh, 2000).

A requirement for our framework was to *visualize transformations within a representation*, and not only transformations, which are connected to the process – object pairs, but also transformations which are connected to pseudo-structural understanding.

Connections between applications and mathematics

In our country, the derivative and differentiation rules are part of the mathematics curriculum for secondary school students in grades 10–12 in the non-vocational streams (approx. 40% of all). Simultaneously with calculus, in the physics lessons there are tasks about velocity, acceleration or radioactive decay. In the chemistry lessons students learn how to calculate reaction rates, and during economics lessons students work on maximising profits, using marginal costs and marginal revenues. Many students have difficulties to transfer knowledge and skills between subjects. However, physics teachers frequently complain that students cannot use what they have learned in their mathematics classes (Basson, 2002). To cope with the transfer of knowledge, Zandieh (2000) included a column named *physical* into her framework. She argued that the context of motion serves as a model for the derivative. For example, 'velocity' is the derivative if the function is 'displacement', and 'acceleration' is the derivative if the function is 'velocity'. Kendal and Stacey (2003) linked the physical representation to the numerical representation. Building further on these extensions, in the new framework for understanding the concept derivative *we want to visualize the connections between applications and mathematics* more explicitly than was done in previous frameworks.

Framework

In our framework we start from the three mathematical representations and we connect these to representations from other subjects: economics, chemistry and physics. In each representation there are four 'layers'.

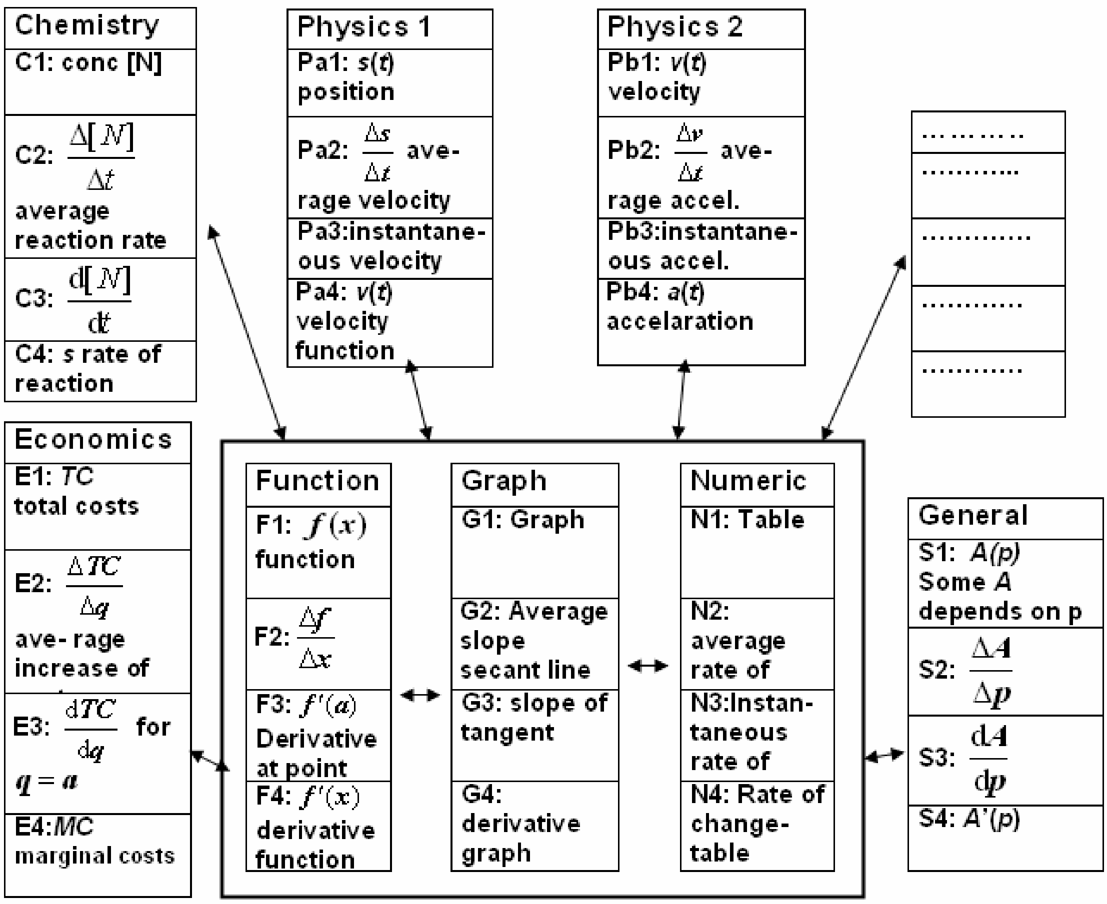

Figure 1: Three mathematical representations and different applications

As mentioned before, understanding of the concept of the derivative is displayed through the quantity of connections and relations between procedures, facts, representations and applications. In our analysis, we will use arrows (as connectors) to visualize the connections in the scheme above. Arrows can exist between representations and within representations, because during the problem solving process a student may switch, for example, from a function (F1) to the derivative function (F4) to solve a problem. In figure 1 only one arrow between each application and the mathematical representation is drawn. However, in fact there are many arrows. If a student for example in a economics problem, focuses on the graph, draws a tangent line, and calculates the slope which he then interprets in terms of the original economical context we will denote this as E1→G1→G3→E3.

When a student only mentions a strategy without executing it, we will apply a star. For example, "*I think I can use a tangent line*" is denoted as G3*.

METHODOLOGICAL DESIGN

The framework was used to design tasks and questions for students. In the fall of 2005, task-based interviews were carried out with six students, (two grade-10, two grade-11, and two grade-12 students), each interview taking 60 minutes. The interviews were videotaped and transcribed. We used our framework to analyse the work of these students with the aim to evaluate its usefulness.

In this paper we will report on two tasks, named *Emptying a Barrel* and *Tickertape.* These tasks were selected because of their similarity from a mathematical point of view, in particular task 1 and task 2b.

Task 1: Emptying a barrel: A barrel is emptied through a hole in the bottom. For the volume of the liquid in the barrel you can deduct a formula by using Torricelli's law. The formula is $V = 10(2 - \frac{1}{60}t)^2$. In this task a graph is also given. The question is to calculate the highest out-flow velocity.

In terms of our framework the task contains a general situation (S1), a graphical (G1) and a symbolical model (F1).

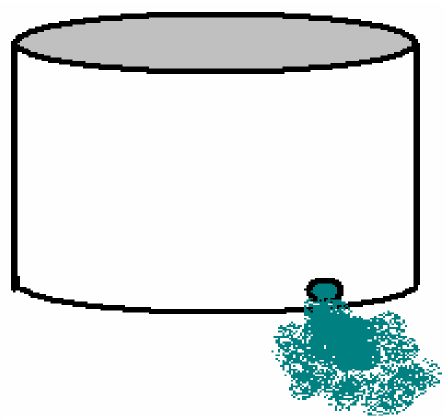

Figure 2: A barrel with a hole

Task 2 Tickertape: This task is about a small car, which is pulled by a falling weight. A person used tickertape to make a time-distance table for this car. The resulting table is given in the task. In the table the distance is given with increments of 0,08 second. In addition, a graph of these measurements is given, together with a formula for the distance as a function of time ($s = 100t^2$).

The first question is to calculate the average velocity (task 2a); the second question is to calculate the velocity at $t = 0,6$ (task 2b).

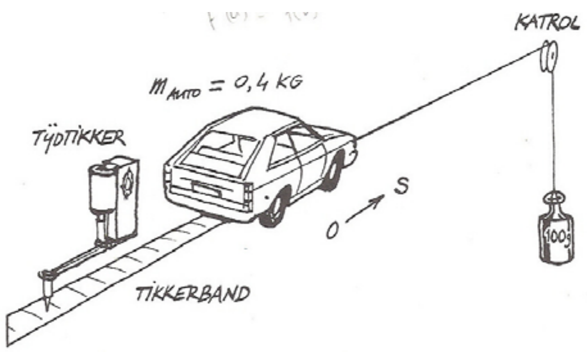

Figure 3: Tickertape

This is a physics task (Pa1), which includes a graphical (G1), a numerical (N1) and a symbolical (F1) representation.

RESULTS

In this section, the analysis of students' strategies in terms of our framework are presented. For the two presented tasks we describe the strategy and the observed transitions made by the students. The three most interesting cases (Hans, Edward and Jill) are discussed.

Hans' strategies: In *task 1* Hans calculated immediately the derivative $V'(t) = -\dfrac{2-\frac{1}{60}t}{3}$

Hans: This [points at the derivative] is the out-flow velocity. If this number [points at t] increases, this number [points at the formula] will decrease, so in the beginning he's going fastest. And you can see it from the graph, too.

Interviewer: Why?

Hans: Because the graph is steeper here [points at the first part of the graph] than here [points at the last part].

As we observe, Hans used the graph only superficially (G1→G2*). By substituting t=0 into the derivative he found the out-flow velocity (in our analysis: S1→F1→F4→F3→S3). When asked to double-check his answer, he calculated the volume at t=0 and t=1 (S1→F1→N1→N2→S2). Hans did not mention a limiting process to get a more accurate answer, nor did he use options of the graphic calculator to find the slope of the tangent line.

In *task 2a* Hans first calculated s(1,20), the distance after 1,2 seconds. Then, the average velocity is 144/1,2 =120 cm/s. When asked to double-check his answer, he turned to formulas from his Physics lessons and calculated the acceleration using $F = m \cdot a$. The formula $v = a \cdot t$ with t = 1,2 gave him the velocity at the end of the tickertape. This number is divided by 2, to obtain the average velocity.

In *task 2b* the formula $v = a \cdot t$ gives him the velocity at $t = 0{,}6$. On his paper however, he made a small error by writing 1,76 m/s instead of 1,176 m/s (figure 4).

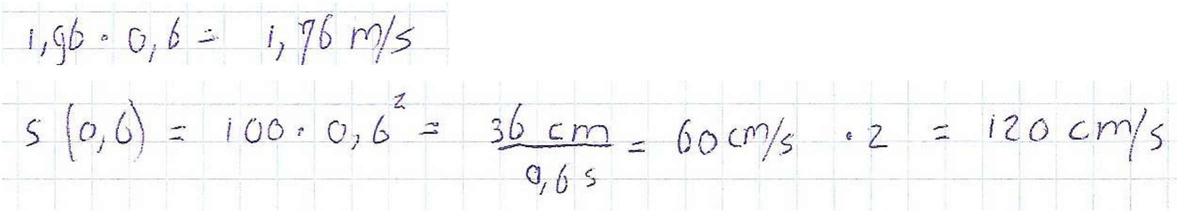

Figure 4: Selection of Hans' work

When asked to double-check his answer through another calculation method, Hans used again an argumentation from Physics, calculating the average velocity on the interval [0; 0,6] and then multiplying this number by two. This method yields a correct answer because of the uniformly accelerated movement. Hans was confused because of the difference between the two answers. When the interviewer asked *"If you had money which answer would you bet on?",* Hans choose the first one. So his strategy was entirely physical. Hans' work can be displayed as Pb4→Pb1→Pb2 and Pa2→Pb3 with his use of physical formulas. He ignored the graph and the table totally. Although the interviewer frequently offered him opportunities, he did not talk about derivatives at all. Summarizing Hans' activities: he often used formulas, but rarely used his graphic calculator. Hans could easily switch between representations. We were surprised that, in the second exercise about the speed of a car he hardly used the formula and did not use the graph at all.

Edward's strategies: In *task 1* Edward differentiated V and substituted $t=0$ (S1→F1→F4→F3→S3). He double-checked his answer by drawing a tangent line on his graphic calculator and he read off the slope (S1→F1→G1→G3→S3).

In *task 2a* an interesting strategy appeared. To calculate the average velocity, Edward first differentiated $s=100t^2$ and then found the average velocity by the physical formula $\bar{v}=\dfrac{v_1+v_2}{2}$ (figure 5). When asked to check, Edward hesitated for 2,5 minutes and in the end he said *"I don't see a shorter way"*.

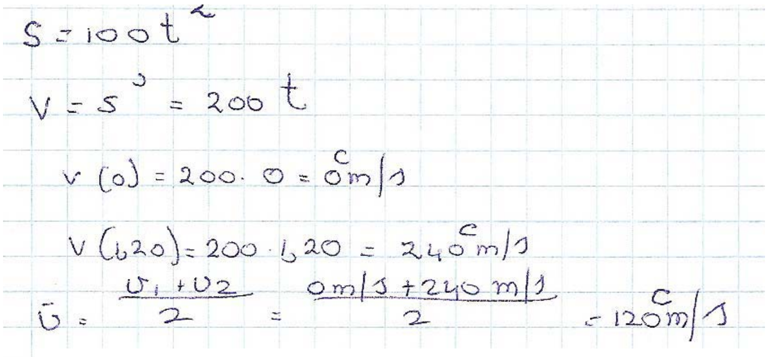

Figure 5: Selection of Edward's work

Task 2b is solved in the same way as task 1: $s'(0,60)$ for the velocity at $t=0,60$, and Edwards denoted this as $v(0,60)$ (Pa1→F1→F4→Pa4→Pa3). He double-checked by drawing a tangent line, which is an option of his graphic calculator (Pa1→F1→G1→G3→Pa3). Edward again concluded that 120 cm/s is the correct answer, because the velocity halfway equals the average velocity. Summarizing Edwards activities: In task 1 en 2b Edward immediately differentiated the given formula. In both tasks he checked his answers with the graphical calculator, which has an option 'Tangent'. He ignored the table in task 2, and the table-options in his calculator. Although Edward is using mathematical techniques in tasks 1 and 2b, in task 2a, on the average velocity, he only mentions physical formulas.

Jill's strategies: In *task 1* Jill calculated immediately the derivative, and substituted $t=0$. (S1→F1→F4→F3→S3). She made an error, instead of multiplying $40\cdot -\frac{1}{60}$, she is subtracting, and her answer is $-39\frac{59}{60}$. She wonders if this is the right answer.

Jill: this means that the velocity at the start is fast…….

Interviewer: why do you have doubts about your answer?

Jill: Because it seems to be wrong, but perhaps the tangent is indeed nearly vertical…[….]

Interviewer: Do you know another strategy or can you double-check your answer?

Jill: ehm…..I only know differentiation, because how much it decreases, is the tangent and you calculate that by differentiation…no I don't know another strategy.

Then Jill detected her error. After her correction ($-\frac{2}{3}\mathrm{m}^3/\mathrm{min}$) the interviewer recalled the reasoning about the tangent. Jill is arguing that $-39\frac{59}{60}$ could have been the right answer, because the time is in minutes and at $t=0$ the tangent can be nearly vertical for a second. The interviewer asked for the last time:

Interviewer: Can you check if your answer ($-\frac{2}{3}\mathrm{m}^3/\mathrm{min}$) is correct?

Jill: [thinking]..perhaps I can use half-life (has to do with radio-active decay), something from Physics…I don't know….perhaps I have to rewrite the formula …..no I don't know.

In her argumentations Jill thought about the tangent, and also about the change of the volume in the barrel during one second. In our framework we describe this as S1→G1→G3* (a graphical argument, without a calculation), and S1→G1→N2*→G3* (a numerical/graphical argument without a calculation).

Task 2a was solved straightforwardly by Jill, using the table (Pa1→N1→N2→Pa2).

In task 2b Jill first tried to work with the formula $s=\frac{1}{2}at^2$. The formula presented in the task is $s=100t^2$. Jill concluded that $a=50$ (an error because $\frac{1}{2}a=100 \rightarrow a=200$). Then she calculated the velocity with $v=a\cdot t$, but of course the wrong acceleration results in a wrong answer. To double-check she said *"the velocity is de derivative of this function ($s=100t^2$), so $v=200t$"* (Pa1→F1→F4→Pa4). Jill was confused and chose her first answer to be the correct one. Summarizing, Jill heavily relied on the symbolical representations. She knew that one can calculate the 'tangent'(she is not talking about slope) with a derivative. But even when she is not sure about her answers she didn't use any kind of graphical of numerical calculations to double-check her (wrong) answers.

As a result we present the connections of these three students in terms of our framework by an 'arrow-scheme'. For example, we can compare task 1 and task 2b which are equivalent from a mathematical viewpoint.

Hans: *Task 1*: G1→G2*, S1→F1→F4→F3→S3; S1→F1→N1→N2→S2; *Task 2b*: Pb4→Pb1→Pb2 , Pa2→Pb3.

Edward: *Task 1*: S1→F1→F4→F3→S3; S1→F1→G1→G3→S3; *Task 2b:* Pa1→F1→F4→Pa4→Pa3, Pa1→F1→G1→G3→Pa3.

Jill: *Task 1*: S1→F1→F4→F3→S3; S1→G1→G3*; S1→G1→N2*→G3*; *Task 2b:* Pa1→F1→F4→Pa4

CONCLUSIONS

In her doctoral dissertation Zandieh (1997) stated that

> "As a student solves a problem involving the concept of derivative, the student makes choices about what context or representation will be helpful in solving the problem. The

> student also makes decisions to work with the derivative as a function or at a point, and a student may use a difference quotient to estimate a derivative value. In these ways a students' concept of derivative is observable through his or her problem solving choices". (p. 219)

With our framework we tried to analyse students' concept of the derivative by looking at the strategies to solve application problems. The framework presented in figure 1 and the resulting arrow-schemes describe the strategies in a structured way by indicating patterns between cells. This facilitates the interpretation of students' statements and operations. Our framework also gives a clear description of transitions between applications and mathematical representations which students make during problem solving. These transitions are an important feature of understanding derivatives (Dreyfus, 1991).

The arrow-schemes can, for example, be used to compare students. The three students all went along the arrow-scheme F1→F4→F3, using the derivative. Task 2b led to different strategies; for example Hans is only using knowledge he learned in physics, ignoring completely the graph and the table in this task. Another way of looking at the arrow-schemes is to compare tasks. Task 1 and 2b are similar from a mathematical point of view. Edwards' strategies to solve these two tasks look very similar. In both tasks he prefers to differentiate the formula (F1→F4), and check his answer by drawing tangents on his graphic calculator and reading off the slope of the tangent line (G1→G3). Hans treats the tasks in a totally different way. Jill uses derivatives in both tasks, however in the first task she mentions a graphical and a numerical argument, as well.

Patterns in the arrow-schemes can be highlighted. For example, the arrow-schemes from Jill and Hans show their preference for working with the symbolical representation, because the arrow-schemes contain many 'F'. Graphical or numerical arguments were scarce in their calculations. However, our arrow-schemes also reveal a limitation of our framework. Although Hans scarcely used a graphical or numerical calculation, we cannot conclude with certainty that this part of the framework is not part of his understanding of derivatives. We can only conclude that Hans did not apply graphical knowledge in these two application tasks; he might use it in other contexts.

Our framework includes both frameworks discussed in this paper. The knowledge is visualized by arrows, and although all arrows look similar, they have different connotations. Some arrows correspond with process-object pairs, such as described in the framework of Zandieh (2000), for example the arrows F(n)→F(n+1). Other arrows show if the student can use a procedure, for example F1→F4. There are also arrows that show students' abilities to switch between representations, for example F(n)→G(n). The framework also highlights important connections between mathematical representations and applications.

Although there are limitations in the framework presented in this paper, it enabled us to highlight major transitions in students' thinking during a one hour task-based interview. Of course, we can only describe parts of their understanding, which were visible to the researcher. Nevertheless we can conclude that with our framework many aspects of understanding derivatives can be visualized in a structured way.

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The functional algebraic modelling at Secondary level

Noemí Ruiz[1], Marianna Bosch[2], Josep Gascón[3]

ABSTRACT:

We adopt the definition of algebraic modelling formulated by various authors in the Anthropological Theory of the Didactic as a basis to define functional-algebraic modelling. We then analyse both the conditions needed to teach and learn this activity at the end of secondary level (16-18) and the constraints that hinder its development in the classroom. The analysis is supported by a teaching experimentation carried out in a "workshop of mathematics modelling" centred on the study of a business situation (how to make money by selling T-shirts?) using the symbolic calculator Wiris. The conclusions presented concern the difficulties originated by the modelling activity as well as those coming from the use of symbolic calculators as a normalised tool of the mathematical activity.

1. INTRODUCTION

At compulsory secondary school (12-16), letters generally only play the role of unknowns (in equations) or of variables (in functional language). They rarely act as parameters. And when they appear in *formulas*, for instance in geometry, statistics, etc., they only function as rules to carry out computations, a kind of shorthand of verbal expressions. They never appear as the result of algebraic work nor do they act as "algebraic models" in which unknowns and parameters are exchangeable. This absence on interplay between parameters and variables is one of the indicators of the *pre-algebraic character* of mathematics studied at secondary school (Bolea, Bosch, Gascón 2001) and can be related to the interpretation of elementary algebra as a generalized arithmetic which is dominant in school institutions (Gascón 1993, 1999). Following Bolea *et al.* (2001), we consider that elementary algebra has to be introduced at school not as a delimited body of knowledge, but as a generic modelling tool. In this sense, it has to be used as a tool to:

(a) Describe the relation between different types of problems or techniques;

(b) Formulate and approach questions related to the existence and uniqueness of the solution of a certain type of problems and the structure of the solutions set;

(c) Generalize the techniques used and the results obtained.

In short, the proposal is to introduce elementary algebra not as an object of study in itself but as a tool to develop, enlarge and interconnect previously studied mathematical organisations-MO (with their types of tasks, techniques, technologies and theories), that is, to study them in depth. In this sense, algebra can be

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considered as a tool of *progressive mathematisation*. Actually, the multiple *transpositive constraints* that hinder the process of algebrization and impede carrying out an algebraic modelling activity (Bolea, Bosch & Gascón 2004), can be considered as constraints to the *process of school curriculum mathematisation.*

2. DEVELOPING THE ALGEBRAIC TOOL: THE FUNCTIONAL ALGEBRAIC MODELLING

Despite the strong pre-algebraic character of school mathematics, it must be pointed out that the mathematical activity becomes fully algebrized from a certain educational level onwards. The development of the mathematical activity requires the complete functioning of the algebraic instrument, although it may remain implicit. For this reason we must assume the existence of a process of algebrization of school mathematics starting at primary school, continuing through compulsory secondary school and culminating at university.

In this work we introduce the notion of *functional-algebraic modelling* as a development of the algebraic instrument, i.e. as a development of the instrument that allows enlarging the mathematical organisations which appear throughout secondary school, especially in the passage from compulsory school to college and, more particularly, in the relations between algebra and differential calculus. Thus functional-algebraic modelling allows:

(a) Unifying certain types of problems thanks to models that can be formulated in terms of families of functions.

(b) Using new mathematical techniques to answer questions that go beyond the calculation of a particular solution to a problem.

(c) Proposing new types of problems involving the reciprocal incidence between the changes of variables that define the underlying modelled system as a family of curves. Questions related to the ratio of variation will appear, preparing for the introduction of the differential calculus.[4]

Let's not forget that the algebraic instrument stems from Pappos' classical analysis which, according to Descartes, consists in the method to find the dependence between all the variables that intervene in a problem, leaving aside whether they take on a known or an unknown value. The development of this instrument leaded to establish a close relation between "geometric" and "algebraic" problems with the creation of analytical geometry. The fundamental principle of analytical geometry consists in the discovery that undetermined equations (in principle, algebraic ones) with two unknowns: $f(x,y) = 0$ correspond to geometrical places determined by points, the coordinates of which satisfy the given equation. This interpretation does not only introduce *analytical geometry* but also the fundamental idea of algebraic variables, essential for the development of calculus as it occurred

[4] The characterization of functions has to be found in its "type of variability". This is the reason why differential equation models are so important. A natural way of understanding functions as models is to consider them as *primitives* in the general sense of the word, that is, a solution to a differential equation. (García 2005)

throughout the 17^{th} century (Urbaneja 1992). We can consider this type of modelling, where the algebraic variables x and y measure any kind of magnitudes, as the seed of the functional-algebraic modelling. The progressive interplay between parameters and variables will help this type of modelling develop into the rationale of the differential calculus.

To clearly explain the general notion of functional-algebraic modelling, we will consider three levels corresponding to three progressive steps of mathematisation.

2.1. First level of functional-algebraic modelling

One of the characteristics of the pre-algebraic character of school mathematics comes from the separation between "algebraic" and "functional" language, which is especially made clear in the role played by formulas and the fact that they are rarely interpreted as "functional" models to study properties of the modelled objects. In fact, the strict separation between school algebraic language (confined in formulas) and functional language is a consequence of the process of didactic transposition that organises the mathematical knowledge to be taught into different blocks of contents and makes it difficult to integrate mathematical objects coming from different "themes" or "domains" (Chevallard 1985, Bosch & Gascón 2006).

We will call the *first functional-algebraic modelling level of a mathematical organisation (MO)* the one that materializes in models expressed by means of isolated functions of one variable and the corresponding equations (or inequalities). For instance, if a product is sold at a unitary price of 6 €, it's unitary cost is 2,5 € and there is a constant production cost of 150 €, at this first level the benefits of the situation can be modelled by the function:

$$B(x) = 6x - (2{,}5x + 150).$$

The kind of mathematical tasks included in these models are the ones that require an analysis of the relations between the components of an isolated function and of the global behaviour of the function: What value of x gives $B > 1000$? How to interpret the constant term -150? Etc.

2.2. Second level of functional-algebraic modelling

We have already mentioned that, at secondary school, the letters which are part of an algebraic expression only play the role of unknowns (in equations) or the role of variables (in functional language), while parameters are hardly existent. In any case, the systematic interplay between their different roles is completely ignored. Moreover, the activity of nominating and re-nominating variables, that is, the introduction of new letters while working with algebraic expressions, essential in the algebraic work, only appears in some activities completely stereotyped as a "change of variable" (for example, in solving bi-squared equations). This situation continues throughout the last year of secondary education. It makes the step from working with algebraic expressions to studying families of functions and to using these families as models of relations between magnitudes extremely difficult.

We call *the second level of functional-algebraic modelling of a MO* the one that materializes in models precisely by means of a family of functions of one variable and the corresponding parametrical equation. At this second level of modelling, there exists a clear distinction between "parameters" and "variables" in such a way that their roles cannot yet be considered exchangeable. Families of one-variable functions are being studied but not functions of two (or several) variables. Some examples can be suggested by the following expressions:[5]

$$B_c(x) = 6\,x - (cx + 150); \qquad B_L(x) = 6\,x - (2{,}5x + L)$$

This kind of models includes the tasks and techniques necessary to study real functions of one variable and to solve equations and inequalities with a parameter.

2.3. Third level of functional-algebraic modelling

We will call *third level of functional-algebraic modelling of a MO* the one that materializes in models expressed by means of families of functions of 2 or more variables and the corresponding algebraic formulas. It is this third level of modelling in which the roles of "parameters" and "variables" are exchangeable. How the joint variation of 2 or more variables has an effect on the variation of a function is being studied. Examples of this third level are the following:

$$B(c, L, x) = 6\,x - (cx + L); \qquad p(x, c, L) = \frac{2000 + L}{x} + c$$

As functions of 2 or more variables have explicitly been put aside from secondary education, the mathematical activity to be carried out to construct, use, study and interpret this kind of models is completely absent from current secondary school.

To sum up, in Spanish secondary education there are very few techniques to carry out the mathematical activity we have called functional-algebraic modelling. Till 10th grade, the only thing that exists is the algebraic manipulation of elementary formulas and some techniques to solve equations and inequalities (if they are easy), stereotyped techniques that are limited to represent graphs and even more limited to interpret formulas and to connect them to graphs. Eventually, it turns out that functional-algebraic modelling is practically inexistent at this level and that, in current conditions, only activities belonging to the first functional-algebraic modelling level could be carried out.

3. THE DIDACTIC PROBLEM CONSIDERED

We consider the didactic problem of how to modify the ecology of the mathematics teaching system to make students capable of carrying out a functional-algebraic modelling activity throughout post-compulsory secondary school (11th and 12th grade). We thus intend to locally create the conditions that allow using functional-algebraic modelling as a study technique to enlarge and deepen the questioning of school mathematical organisations.

[5] They all are models of the form $f_p(x, y) = 0$ where f_p is a family of 2-variable functions with p as a parameter and where one variable can be isolated.

To progress in that direction, the work we present puts forward both the design and conclusions taken from the experimentation in different courses of a study process based on the combination of two didactic strategies:

(a) Propose the study of a problematic question that arises in an economic system, initially defined by means of some fixed values. The study of the system requires the students to progressively convert data values into parameters and to take into consideration the functional relations between 4 variables of the system (sales, costs, incomes and benefits).

(b) Use the symbolic calculator Wiris (www.wiris.com) to instrumentalise the mathematical techniques needed to approach the types of problems that arise during this activity. We try to take advantage of the resources of Wiris to make the work of creation, graphic representation and manipulation of algebraic expressions and functions easier for the students, bearing in mind the interpretation of all these manipulations in the context of the system.

The experimentation was organised as a "mathematical modelling workshop" in 5 different schools of Barcelona's metropolitan area, four classes of 10th grade and one of 11th grade. It took place during the first term of 2006 and went on for 10 or 11 sessions of 50 minutes. The workshop teacher was the usual teacher of each class and one of the members of our research team did the observations of almost all the sessions, which were audio and video recorded.[6] In all cases, most of the sessions took place in the "computers room" of the school, even if some sessions, especially the first and the last ones, took place in the normal classroom. We will present an outline of the experimentation taking the successive mathematical models involved as a guideline of the description.

4. HOW WAS PERFORMED THE MODELLING PROCESS

The "worshop" starts with a question raised in an *economic system* (the production and sale of T-shirts) about how to obtain a given benefit. The study of this question gives rise to a functional-algebraic modelling process in which the interplay between parameters and unknowns is essential.

Initially, the system can be characterized considering 4 variable magnitudes: the number of produced and sold items (x), the incomes (I), the production costs (C) and the benefits (B). The three functions $I = I(x)$, $C = C(x)$, $B = B(x)$ can be defined and related through the equality: $B(x) = I(x) - C(x)$.

If we consider a single product sold at a constant unitary price p, the incomes function is given by $I(x) = p \cdot x$. As far as the costs function is concerned, different models are possible. The simplest hypothesis is to consider costs linearly depending on the number of produced items $C(x) = c \cdot x + L$, where c is the unitary cost and L other possible fixed costs (the rent of the workplace, for instance). It is also possible to consider that the unitary cost is not constant but increases linearly with x, which gives rise to a quadratic function of the form: $C(x) = (c + \alpha x)x + L$.

[6] A preliminary experimentation took place with two of these classes in the first term of 2005.

The coefficient α here indicates a "low" value of the type $1/K$ with $K > 0$ so that, for x values much lower than K, αx can be neglected making costs almost linear while quadratic increasing remains only for "big" sales.

The workshop was divided into two parts. The first starts with a consultation from a youth association that wants to earn 3000 € by printing and selling T-shirts (linear costs). The second one is about a similar proposal made by a sports firm printing and selling bigger amounts of T-shirts (quadratic costs). Only the case of the linear cost function is being considered here and we will describe the modelling process followed by the students through a sequence of mathematical organisations where each one models the previous one and is modelled by the following one.[7]

Youth association: buying and selling T-shirts

The students are given the following chart about the costs and incomes obtained by a youth association selling T-shirts during May, June and July:

MONTH	May	June	July	August
Sold T-shirts	100	329	264	
Total costs (€)	550	1122,5	960	
Total incomes (€)	520	1710,8	1372,8	
Benefits (€)	-30	588,3	412,8	

The aim of the work is to write a report for the youth association telling them what strategies can lead to the desired benefits of 3000 €.

The initial conditions of the business can be obtained from the given data: a constant unitary cost c = 2,5 €, a constant unitary price p = 5,2 € and a constant fixed cost L = 300 €. The first question to consider is:

> $\boldsymbol{Q_0}$**:** In the given initial conditions, is it possible to obtain a benefit of 3000 € in August by selling a reasonable number of T-shirts?

With the help of the teacher, the situation is modelled by the one-variable function $\mathrm{B}(x) = 5{,}2x - 2{,}5x - 300$ and the associated inequality $5{,}2x - 2{,}5x - 300 > 3000$. However, the students first spontaneous strategy is to answer the question working in OM_{eq}, the first-level algebraic MO of one-unknown equations, solving $5{,}2x - 2{,}5x - 300 = 3000$ and deducing the "limit value" $x = 3300/2{,}7 \approx 1223$, which is a big amount certainly not "reasonable" for the youth association.

The negative answer to $\boldsymbol{Q_0}$ creates the need to modify some data of the initial situation, turning them into *parameters.* The following question $\boldsymbol{Q_1}$ comes up:

> $\boldsymbol{Q_1}$**:** Is it possible to obtain the desired benefits by changing only one initial condition of the situation: unitary price, unitary cost, fixed cost (rent)?[8]

It can be specified into the following questions:

> $\boldsymbol{Q_{11}}$ **($\boldsymbol{p}$ as free parameter):** Suppose the costs are constant (c = 2,5 y L = 300). How much does the unitary prize p have to increase in order to obtain a benefit of 3000 € selling a "reasonable" number of T-shirts ($x < 450$)?

[7] In the case of a quadratic cost function the sequence of praxeologies is similar and can be found in Bosch, Gascón & Ruiz (2005).

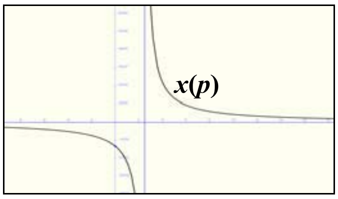
[8] After discussion, $p < 8$, $c > 1$ and $L > 100$ are considered as "reasonable" values.

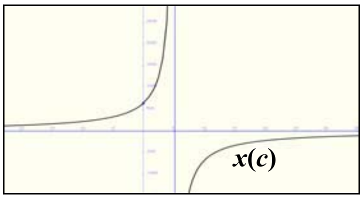
And the same with the other two cases: c or L as free parameters.

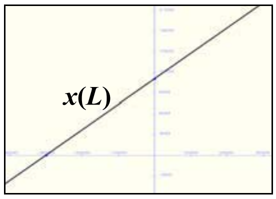
The spontaneous answer given by the students consists in considering the "extremal" cases ($c = 1$, $L = 150$ or $p = 8$), which let them remain in $\mathrm{OM}_{\mathrm{eq}}$. To force the passage to $\mathrm{OM}_{f(x)}$, the teacher has to refer to the consultancy situation ("we cannot expect the youth association to always find these extreme conditions", etc.) and introduce a functional technique and the corresponding reformulation of the result obtained:

$$3000 = (p-2{,}5)\cdot x - 300, \qquad 3000 = (5{,}2-c)\cdot x - 300, \qquad 3000 = (5{,}2-2{,}5)\cdot x - L$$

$$x = \frac{3000+300}{p-2{,}5}; \qquad x = \frac{3000+300}{5{,}2-c}; \qquad x = \frac{3000+L}{5{,}2-2{,}5}$$







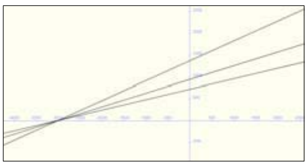
To obtain more than 3000 € of benefits by modifying only one parameter, we have:

$$p = 8 \Rightarrow x > 600; \qquad c = 1 \Rightarrow x > 786; \qquad L = 100 \Rightarrow x > 1149.$$

The work in $\mathrm{MO}_{f(x)}$ establishes that the desired benefit cannot be obtained by changing only one parameter. A new question $\boldsymbol{Q_2}$ arises based on the necessity to modify two parameters:

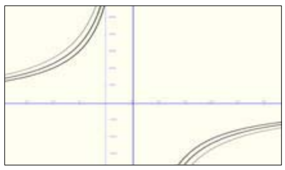
> $\boldsymbol{Q_{21}}$ **(p and L as free parameters):** If $c = 2{,}5$ remains constant, what *relation* between p and L is needed to obtain a benefit of 3000 € selling a "reasonable" number of T-shirts ($x < 450$)? Does a "reasonable" pair of values p and L exist?
>
> And the same with the other two cases: (c,p) or (c,L) as free parameters.

Here the work takes place in $\mathrm{MO}_{f_p(x)}$ and the study of the relations:

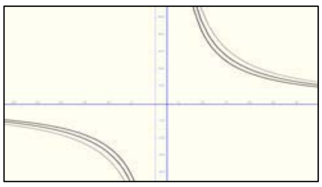


Isoquants of $c = c(x,L)$.

$$3000 \le (5{,}2-c)\cdot x - L, \qquad x \le \frac{3000+L}{5{,}2-c};$$

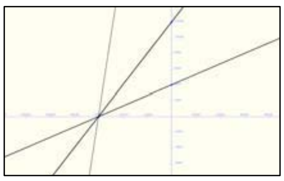


Isoquants of $L = L(x,c)$.

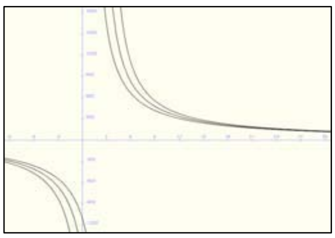


Isoquants of $p = p(x,L)$.

$$3000 \le (p-2{,}5)\cdot x - L, \qquad x \le \frac{3000+L}{p-2{,}5};$$

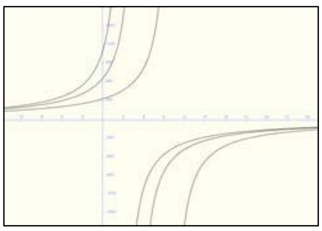


Isoquants of $L = L(x,p)$.



Isoquants of $p = p(x,c)$.

$$3000 \le (p-c)\cdot x - 300 \qquad x \le \frac{3000+300}{p-c}$$



Isoquants of $c = c(x,p)$.

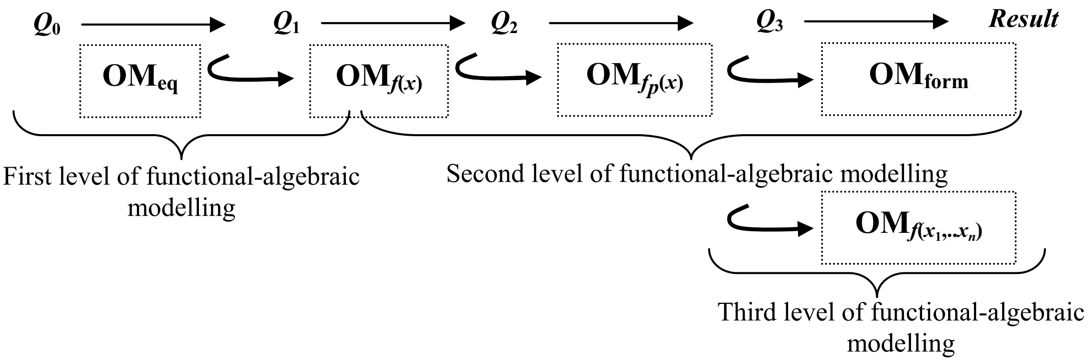
Admissible solutions can only be reached when p and c are taken as free parameters. Thus $p-c$ *appears to be the interesting parameter* to consider.

A last a more general question can then be formulated in the following terms:

> ***Q_3* (all three parameters free):** What changes can be applied to the initial conditions (c = 2,5, p = 5,2, L = 300) to obtain 3000 € of benefit selling a "reasonable" number of T-shirts ($x < 450$)?

An exploratory study with Wiris through the variation of graphs shows that a reasonable answer to ***Q_0*** requires the condition: $p - c \geq (3000 + L)/x$.

Due to the fact that, at the considered level, students are not able to use *analytic* techniques to approach ***Q_3***, the study cannot go beyond the transformation and interpretation of this formula. The following schema summarizes the sequence of questions and praxeologies that may appear throughout the study of ***Q_0***:



During the design and carrying out of the instruction, the symbolic calculator Wiris is considered as an instrument that decreases the difficulty or the cost of certain problematic tasks such as graphing functions, solving equations (with and without parameters) and doing algebraic transformations of formulae. It thus facilitates the exploratory and experimental dimensions of the study. Furthermore, Wiris has appeared as a useful tool to distinguish between the system considered and the mathematical model of the system, encouraging both the formulation of questions about the models used and the interpretation-justification-evaluation of the results obtained, in short, the work of "coming back" to the system initially considered.

5. RESULTS AND OPEN QUESTIONS

(A) *The need to study long-term mathematical questions*

A basic characteristic of an algebrized mathematical activity and, in particular, of functional-algebraic modelling, is the requirement of long-term goals that can only be reached through systematic work extended in time. This was a trait of our instructional proposal: starting with a question that was not immediately solvable but required the construction and progressive enlargement of the models considered. This requirement appeared to be an important obstacle to the experimentation and can be explained, partially at least, as a consequence of the way the study of mathematics is interpreted in secondary schools, that is, in terms of the dominant cultural notion of *study*. It is in fact a conception clearly compatible with the limited, rigid and isolated character of mathematical organizations studied in such institutions (Bosch, Fonseca, Gascón 2004). It is also reflected in the way students carry out limited tasks and change the activity several times, even during the same session. On the contrary, our proposal is based on an activity that needs to integrate different mathematical objects in a functional way, working with a set of MOs that usually appear in an incomplete and isolated way. Thus, the type of study designed, with its long-term objectives, the non-definitive answers and the rising of new questions, go against the dominant epistemologic and didactic conception of the teaching and learning of mathematics.

(B) *The connection between numerical and functional language*

To carry out the passage from an arithmetic solution to the construction of a functional-algebraic model, it is necessary to turn "numerical questions" into questions the answer of which is not a concrete number but a *relation* between variables. At the beginning of the workshop, the students had real difficulties with this, always trying to find "the number solution", as they are used to doing. It was the teacher who had to clearly highlight the new didactic contract that was being established, using the "reality" of the situation to justify it. In any case, it seems that it is necessary to work more on the "question of the questions", ie the kind of problem that is really approached, the kind of solution that is "receivable", etc.

(C) *Management of the study process and students autonomy*

The development of a functional-algebraic modelling activity requires a bigger degree of autonomy from the students than is usually needed. This kind of activity (even at the first levels) wants the student to take his/her own initiatives related to the kind of questions to solve, the tools to use and, even, the direction that the study process can take at a given moment. How to organize an appropriate new sharing of responsibilities (didactic contract) between the teacher and the students remains an open problem at this level. How can we determine the optimum degree of responsibility students have to be assigned with at every educational level? How can the teacher manage this new kind of study process? Etc.

(D) *The role of Wiris to facilitate functional-algebraic modelling*

The symbolic calculator Wiris was used during the workshop to use mathematical techniques in a more fluent way and with less difficulty than their "paper and pencil" versions. Wiris was thus supposed to help students carry out a richer exploratory activity. It was used to carry out a lot of trials and the exploration of different cases (different values of the parameters). However, we did not succeed in making students question the techniques used (their scope, economy or efficiency). This is an essential point because any modelling activity requires the systematic interpretation of intermediate results and the questioning of the adequacy between model and system. What did appear, with the help of Wiris, was a certain degree of *articulation* between the algebraic work with formulas and the graphs obtained by considering any letter of the formula as independent variables.

Because it goes towards the didactic organisation of current secondary schools, our proposal has highlighted various constraints that hinder the study of a long-term question and the use of functional-algebraic models to deal with it. A symbolic calculator like Wiris helps to overcome some of these constraints. Anyway, our experimentation has shown that, to go beyond the "second level of functional-algebraic modelling", a profound change in the didactic contract prevailing in secondary schools is necessary. This change also seems essential to give sense to the differential calculus that is taught at the end of secondary school and at university level.

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MATHEMATICAL MODELLING IN SCHOOL - EXPERIENCES FROM A PROJECT INTEGRATING SCHOOL AND UNIVERSITY

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Abstract: *The paper describes a project about modelling in school hold jointly by mathematicians and mathematics educators at the university. Core of this project are modelling examples carried out at school by students who are supported by future teachers. The paper describes the frame of the project and two modelling examples concentrating on the students' attempts of solving it. Finally, some experiences of modelling in school are discussed in the context of the modelling examples described in the paper.*

1. Introduction

In the recent didactical discussion on mathematics education modelling is regarded as one way to satisfy aims of a mathematical education providing more than just algorithms and schematic calculation. Mathematical modelling offers students the chance to understand mathematics as an area of knowledge which is relevant for everyday life. Furthermore, through working on modelling problems students can develop the ability to use mathematics in real life situations. This aim, as for example formulated in the PISA study, is obviously one of the main aims of mathematical education integrated in a sustainable concept of general knowledge.

One way to establish modelling in school is the integration of modelling courses into pre-service teacher education. A project carried out at the University of Hamburg is following this direction. Core of this project is joint work of students and future teachers in school on authentic modelling problems over a longer period of time. In addition to that, the future teachers are supervised by staff of the faculty of education and the faculty of mathematics.

This paper reports on experiences made during several cycles of the project. First, in the second chapter the project is described in general. In the third part two examples of problems worked on in school during the project are described in detail while concentrating on the students′ attempts of solving. Starting from these examples, more general experiences about modelling in school are described in the fourth chapter.

2. Description of the project

The project "Mathematical modelling as bridge between school and university" was established in 2000 and has been completely conducted four times since then, followed by an overall extensive evaluation (for details see Kaiser, Ortlieb, &

Struckmeier, 2004, 2005). Currently, the fifths round is carried out. Each seminar is planned and offered as a joint seminar of the department of mathematics and the department of education and lasts for two semesters and consists of several parts.

The theoretical framework of the project has already been described extensively elsewhere, thus we restrict ourselves to a few central remarks (see for example Kaiser, & Schwarz, 2006). The project relates to the current debate on modelling in school and calls for the inclusion of real world examples or modelling examples in mathematics teaching in order to foster the following goals: Real world or modelling examples

- should enable students to understand the relevance of mathematics in everyday life, in our environment and for the sciences;
- should enable students to acquire competencies which help them to solve real world problems coming from everyday life, our environment or the sciences.

The project emphasises the usage of authentic problems in order to show and to persuade students that mathematics is really used in the world outside school. This implies a holistic approach in which the students carry out a whole modelling process (for details of the discussion concerning holistic versus atomistic approach see Blomhoj, & Jensen, 2003). In detail it means that the students are working through the various phases of a modelling process, such as describing a real world problem from a real situation, developing a mathematical model on the basis of a real world model, generating mathematical results which then have to be discussed in the context of the real world model and to be validated in the real situation (for a more detailed description see for example Blum, 1996 or Kaiser, 1996).

The aims of the project are twofold: On the one hand, the project aims at introducing modelling into ordinary mathematics teaching in school (limited to upper secondary level; for experiences with modelling in lower secondary level see Maaß. 2005). On the other hand, the project wants to offer future teachers the chance to gain experiences through teaching modelling in school.

In more detail, it can be stated that the project aims at various targets concerning students and future teachers. On the one hand, students as well as future teachers are to learn about mathematical modelling by doing it themselves. This does not only mean dealing with the different phases of the modelling process but also experiencing how to handle open problems in general. In relation to that it also means learning to stand the phases of uncertainty and ambiguity which are characteristic for mathematical modelling. Furthermore, the students' and future teachers' beliefs and attitudes towards mathematics of are to be broadened; they should realize in how many different ways mathematics forms part of our everyday's life. On the other hand, the university seminar is a supplementary course in the area of mathematics didactics and mathematics for future teachers. The aim here is to offer an opportunity to future teachers to develop experiences with teaching in school with coinstantaneous possibilities to reflect theoretically their work. Beyond this, of course, the seminar is an attempt to establish modelling in school in two ways: bringing it directly into school during the time of the project and to raise the future teachers' awareness for modelling in teaching mathematics.

Core of the project is the common work of future teachers and students in school. Normally, two to four students attend a mathematics course in the upper secondary level of German schools, year 11 or 12 (17-18 year-olds), preferably an advanced mathematics course for two lessons a week. During these lessons the students work on one modelling problem for about 2-3 months. The project starts with a meeting of all participants at the university where the problems the students will work on are presented. Often this presentation is done by applied mathematicians working in industry by describing problems from their current work or a project the applied mathematicians is currently involved in. At the end of the project all participants meet again and the students present their results in front of the whole audience or with a stall in an exhibition. Furthermore a university course is offered for the future teachers. Here mathematical questions of modelling and didactical questions are discussed and the future teachers have the opportunity to reflect and discuss their experiences in school with the other future teachers and the university staff.

The problems are throughout authentic questions and often solutions are unknown not only to the students but to the experts as well. In addition, the problems are not or only little simplified which makes working on them even more difficult for the students. In some cases only a modelling situation was presented and the students had to develop their own questions to work with.

3. Description of two modelling examples

In this section two modelling examples are presented in detail. For this, the descriptions of the solutions do not focus on complete prime solutions but on attempts really developed by the students.

3.1. The pricing of an airline

The first problem deals with the pricing of the low-price-airline "Air Berlin". This airline sells its tickets basically by an online booking system and obviously the prices are not constant or subjected to defined and published algorithms. So the question was how Air Berlin fixes its prices. This problem means reconstructing an algorithm or a formula by given data, here the prices of flights which can be observed by internet research. Alternatively, the students could solve another problem thematically related to this. This second problem was to find an own price system for an airline. The following ideas were worked out at the Gymnasium Harksheide in Norderstedt (a town near Hamburg) by an advanced mathematics course in year 12 (students' age 17 -18) with 11 students participating.

At the beginning the students decided to try to reconstruct the price system of Air Berlin. For this they collected data by observing the development of prices for several flights offered by the airline. The prices as expected seemed to vary not regularly which made the solution of the problem quite difficult. The students developed a list of factors which could affect the price of a flight. These factors were among others:

- date/departure time of the flight
- date of booking (referring to the flight-date)
- number of free seats
- place of departure and destination
- market situation

The problem was that not all of these variables could be observed by the students. The market situation for example might have a strong impact on the prices but can not be measured or researched. However, the students tried to develop an algorithm for predicting the prices. Doing this, they divided themselves into two subgroups. One group tried to find a formula for the prices depending on the time of booking, the other group concentrated on the factor of free seats. The first group was quite successful in developing a model describing the prices of national flights in relation to the booking time. The other attempts did not lead to results fitting to the collected data. Because of this disappointment the group decided to change the problem and work on the question of developing their own price system.

For this, the group split again into two sub-groups and they followed two different attempts: The first group decided to describe the development of flight prices by means of an exponential function, due to experimental considerations with real values. Thus, as model for the development of prices this group developed the function $f(x) = e^{cx} + b$ with *b* as initial price and *c* as description of price behaviour (meaning a steep or slow rise). For the determination of the parameter c, the students, assisted by the prospective teachers, referred to the mean value theorem of integral calculus formulating the following equation: $\int_0^t (e^{cx} + b - 1)dx = a \cdot t$, in which *a* represents the average price of a flight. The meaning of the factor -1 cannot be explained by us. We suspect that the students introduced the factor in order to adapt their description to the data.

The second sub-group agreed on a description of price trends starting 30 days before departure by means of the exponential function $f(t) = basic\ \ price \cdot a^{30-t}$, (where t is the number of days until flight time), with the assumption that the basic prices vary between 25€ and 125€ and that no flight shall cost more than 300€. As growth factor *a,* based on the collected data, the group experimented with factor 1.024 and 1.03. Then, this attempt was modified further and generalised by taking into consideration the amount of remaining free seats by inserting *m*(10-s) as additive factor with *s* as number of remaining free seats and *m* as multiplicative factor representing a not yet fixed lump sum. The basic price was fixed depending on flight distance for which they referred partly to the real flight distances of Air Berlin and further assumptions with $\frac{z}{16} = basic\ \ price$, where z is the distance of the flight in km.

Both attempts were carried out simultaneously with two modifications:

- Consideration of dependence on season, modelled by means of a cosine function: $0.25 \cdot \cos(\frac{2\pi d}{100}) + 1.25$, where d is a factor depending on the time of flight which was determined by analysing older data. Obviously this term can never be smaller than 1 what makes it meaningful as a multiplicative factor for the complete price.
- Rounding up to the following decade

The first sub-group produced as model for the development of prices:

$$f(d,c) = (0.25 \cdot \cos(\frac{2\pi d}{100}) + 1.25) \cdot \left(e^{cx} + b\right)$$

The second sub-group produced as model for the development of prices:

$$f(d,t,s) = \left(0.25 \cdot \cos(\frac{2\pi d}{100}) + 1.25\right) \cdot \left(\frac{z}{16} \cdot a^{30-t} + m \cdot (10 - s)\right)$$

The students did not use the notation with several variables and it is unclear whether they realized the problems resulting from their model containing more than one variable or not.

3.2. The pricing of an internet café

The second problem which was to be introduced here, "Pricing of an internet café", deals with the question how to calculate - reasonably and with the aim of profit - the prices of an internet café. The solution presented here is developed by a group of students in year 11 from the Albert-Schweitzer-Gymnasium in Hamburg within the framework of the project.

Generally, the students' solutions were characterised by far-reaching and to some degree strongly simplifying assumptions. On the one hand, this stimulates questions about the model's proximity to reality, but on the other hand, it makes executing an entire modelling circle process possible.

As result of a first discussion within the group, the students compiled the price influencing factors such as marketing, rent, maintenance, internet access and so on, described as regular cost, and non-recurring costs such as equipment, licenses and so on. It is remarkable that only the costs to be covered exclusively by the entrepreneur were listed as price influencing factors. The students did not find out autonomously but only after being asked further aspects like competition and demand which surely influence a product's price decisively too. Taken together, within the framework of a real model three primary price influencing factors were determined: competition, costs to be borne by the owner and demand.

To calculate the range of costs, the students first did some research by using the internet but finally decided to estimate the costs just roughly. Still on the level of creating a real model, the students furthermore decided not to take the second factor, competition, into account in order to keep the problem simpler.

Especially for being able to create a reasonable relation between the third factor, the demand and the price as well, as for being able to construct a model adequate to the context, basic knowledge of economical structures and relations was necessary. Contrary to the usual proceedings of this project, where students usually work independently on a problem, in this case the technical knowledge was introduced by the university students in conventional lessons during a specific learning unit. The learning unit comprised topics like price response function and break-even-point which is the moment the internet café starts being profitable. Even in this learning unit the students turned out to be active participants; they analysed critically the presented contents and already related them to the modelling problem. This demonstrates that the students really adopted the problem.

Starting from what they have learned in the specific learning units, the students then formulated two modelling assumptions for the relation of price and demand on the level of the mathematical model:

- With a price of 0 € per hour for using the internet there would be a demand of 10,000 hours
- With a price increase around 0.10 € for an hour online the demand decreases respectively about 5%.

These modelling assumptions mentioned above are very strong but correspond with simple economic methods. Starting from these assumptions the students were able to construct the so-called price response function within the framework of a mathematical model:

$$N(x) = 10{,}000 \cdot 0.95^{10 \cdot x}$$

in which x indicates the price for one hour internet access in €/h and N(x) is the number of demanded hours at a price of x €/h. The exponent was chosen for being able to indicate prices in € per hour.

The sales volume is calculated based on the amount of demanded hours multiplied with the price for one hour internet access. With S(x) as sales volume at a price x one receives the so-called sales volume function:

$$S(x) = x \cdot N(x) \Leftrightarrow S(x) = x \cdot 10{,}000 \cdot 0.95^{10x}$$

At this point the students did not make any statement about what time period the calculation refers to, meaning during which time the 10,000 hours with a price of 0 € would be demanded and during which time the sales volume would be achieved.

However, in any case, this information is insignificant for the further solution; it only becomes important if the point of break-even has to be defined.

Based on the context described above, the optimal price was to be determined as the point at which the sales volume function reaches its maximum. Although the students had experiences with differential calculus, they did not know the derivation of an exponential function and so here they asked actively for help from the university students. Afterwards, the students determined the maximum of the sales volume function on their own as $x = \frac{-1}{10 \cdot \ln(0.95)}$, thus the optimal price becomes approximately $x \approx 1.95$€.

While validating their results, some of the students were not satisfied with the obtained results because in their opinion the assumptions and as a consequence the related results were made too arbitrary. So, in the further development of the project the students modified their assumptions and tested the influence of various modifications on the result.

This criticism indicates that the students have achieved learning success not only concerning the related concrete contexts or mathematical contents but also in the field of modelling methods which should be valued as an important and satisfying result of the project.

4. Analysis of the two described modelling problems

In order to analyse the different modelling approaches developed by the students and as a mean to describe our experiences made we use a description of the modelling–cycle proposed by Blum (1996) or Kaiser (1996), which in various studies has been proven to be a useful tool for these kinds of analyses. In the following section the description of experiences of modelling in school is structured by three elements of this modelling cycle. The corresponding examples are taken from the modelling approaches described n chapter 3, but the reflections described are not bound to these modelling problems.

(1) Developing a real-world-model / mathematical model

Students tend to omit the separation of the phases of developing a real model and a mathematical model or they even do not consider the real model. The distinction between a mathematical model and a real model is obviously useful for didactic-diagnostic purposes but mostly not for a first attempt in school. Instead one can often observe one of the following three approaches:

i) The students mix together the two phases and develop descriptions on the level of a real world problem and related mathematical concepts at once. That approach can be seen while tackling the internet-café-problem: the description of possible price behaviour and the mathematical description of it are formulated coevally.

ii) The students do not formulate a real model, but instead develop a mathematical concept with implicit reflections on the real context. This mathematical model covers essential structures of the real situation. For example one group of students started solving the Air-Berlin-problem by formulating an exponential function to describe the price behaviour. Although this approach was not justified it obviously seems to be a reasonable attempt because of the idea of rising prices.

iii) The students develop a mathematical concept not under the criterion of fitness to the question but under regard to their current present mathematical knowledge. A good example is the usage of the cosine–function for modelling the seasonal influences on flight times. Although there are probably times of bigger and smaller demand for flights depending on the time of the year the usage of the cosine–function is a very far-going presumption because of the regularity of the cosine–function. But this function is a familiar mathematical concept to the students and therefore was used.

In all three cases but especially in conjunction with the third case, another phenomenon can be observed obviously resulting from students´ experiences with conventional mathematical lessons: when working with authentic modelling problems students sometimes even expect the modelling problem to be just an enclosure of concepts they have already learned and which are given to them to apply exactly this knowledge. So from the very beginning they search for familiar mathematical concepts fitting more or less to the given situation. Furthermore, this sometimes explains the abandonment of developing a real model. It seems necessary for the teacher to clarify this miscomprehension and to refer to the openness of the question and its various possible solutions.

This openness of solutions on the other hand can lead to a problem concerning the development of an adequate model. The students often try to capture all factors which could be relevant for the problem from the very beginning and by this they acquire quite complicated und almost insolvable problems. This tendency is commonly intensified by extensive internet-inquiries conducted by the students. It has proved of value to encourage the students to radically simplify the problem in the beginning. The problem can always be refined afterwards when the students have reached a higher survey over the complete problem and its modelling cycle while in contrast a very complex system of assumptions as a starting point often inhibits successfully modelling.

(2) Solving the inner mathematical problems

During this phase of the modelling cycle the students naturally often come up with particular mathematical questions. This especially emerges when the students have not developed a mathematical model under requirement of their mathematical knowledge. In both modelling examples described above, for example the students reached a point where they needed and asked for the help of the future teachers to solve such a question. This is not a problem as long as the students ask for help by themselves; in contrast asking for special mathematical concepts can only result from long analysis of the mathematical problems and can be taken as a sign for intensive work carried out by the students. It is only important to make sure that the students are not pushed into a special mathematical concept by the teacher. Rather after dealing with the mathematical model the students should be able to outline what kind of mathematical appliance they require. This appliance then can be taught by the teacher.

Often during this phase of solving an inner mathematical problem, the students´ approaches are strongly influenced by the experiences the students gained from traditional mathematical lessons. This means that students are often only satisfied with their solution when they can produce a clear result in the sense of a single number. This can clearly be seen within the modelling approach of the students tackling the internet-café-problem: here the students tried to calculate a distinct price of an hour internet usage.

(3) Validation

This phase is often largely neglected in the framework of mathematical modelling in school. Furthermore, the students sometimes interpret the validation of a model and so its critical reflection as decrying their work afterwards. It is up to the teacher to show them that validating is an essential part of mathematical modelling and that realizing the insufficiency of a model in the context of a problem is just a good work as formulating a model is.

During the phase of validation the whole modelling cycle should be reflected again. With the survey of the complete cycle the particular assumptions and their influence on the result can be analysed and the effects of changing the assumptions can be regarded. Good examples are the students´ thoughts after modelling the internet-café-problem: the assumptions seemed to be arbitrarily to the students and so they reflected the dependency of the change of the assumptions and the change of the results.

5. Conclusions

Taking a first look on the problems of the project "Modelling in school" yields the impression that these problems are far too difficult to be dealt with in school. But as we pointed out they are worked on with quite remarkable results. This is an experience of the project in general. Modelling problems with a true reference on reality can be handled in school not only by more gifted students but by average ability students. In addition the project shows that this kind of authentic complex modelling examples can be dealt with in ordinary classroom lessons during normal teaching, although one has to consider that mathematics education becomes more challenging for all participants, students and teachers. Carrying out such teaching units requires good preparation from the teacher and puts higher demands on the students. In addition, such kind of projects are quite time-consuming, a problem which must be taken seriously in times of central examinations.

Concerning university teaching the evaluation of the project (see Kaiser et al., 2005) shows clear advantages of the project for both participating groups – future teachers and students. The future teachers have the chance to gain experiences with teaching and working in classes. This was welcomed by most of the participating students. The students often changed their image of mathematics. While impressions of mathematics as a static and schematic science were dominant before the project afterwards impressions of process and connections to reality are named much more frequently. And furthermore some skills not directly connected to mathematics have been trained such as organizing teamwork or preparing a presentation. The acquirement of these abilities was highly regarded as an important part of the project by many students.

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PERSONAL MEANING IN RELATION TO MODELLING PROBLEMS

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The paper reports about a study examining the question which meaning students see in learning and doing mathematics. Because the term meaning is used for very different concepts, the author's understanding of meaning is explained and a relational framework of personal meaning is developed. Furthermore, the design of the study as well as two modelling problems that were used for the data collection are described. Finally, some preliminary results of the study are presented.

INTRODUCTION

Many students think that the mathematics learnt at school is meaningless to them because they do not see where they can apply mathematics in their lives. Showing students some applications is one aim connected to the introduction of modelling problems in school. This can be done by pointing out how strongly mathematics is used in society and in the students' everyday life. As a result, their motivation to learn mathematics can increase. However, up to now there is neither any systematic investigation about the realisation of these aims nor is it known which aspects of modelling problems are meaningful to the students. The study presented seeks to find out which meaning students see in doing and learning mathematics and in which way solving modelling problems can promote or hinder the students in constructing a personal meaning. In this paper the theoretical background and the design of the study will be presented. Finally, preliminary results will be presented.

THEORETICAL BACKGROUND

The study is embedded in the context of the Graduate Research Group on Educational Experience and Learner Development located at the University of Hamburg. Beside the theory of application and modelling, the concept of personal meaning (German: *Sinnkonstruktion*) developed by members of the Graduate Research Group is of high relevance for this study. Therefore, this concept will be explained and related to other concepts which are important for learning. After that, it will be described why and on what conditions modelling problems may have an influence on the students' construction of personal meaning.

Terminology

In the context of the Graduate Research Group, the German term *Sinnkonstruktion* is used for what will be called *personal meaning* in this paper. *Sinnkonstruktion* is difficult to translate because it is based on deep epistemological and philosophical reflections. Although the English expression *personal meaning* does not cover the whole

German debate, the expression was chosen due to the lack of a better term. Like the English term *meaning*, the German term *Sinnkonstruktion* is used very often in various contexts. Therefore, the term has to be specified in order to be able to reconstruct the influence of modelling problems on students' personal meaning. So in the following the understanding of personal meaning will be described (for further information see Vollstedt, in press).

As mentioned above, the study is embedded in the context of the Graduate Research Group on Educational Experience and Learner Development. For reasons of readability, not the English translation 'Educational Experience and Learner Development', but the German term *Bildungsgangforschung* will be used in the following.

Personal Meaning

As mentioned in the introduction, one of the aims of modelling and application in school is to give the students a possibility of finding out in which way mathematics can be meaningful to them. As Howson (2005) remarks, one has to distinguish at least between two different aspects: “those relating to relevance and personal significance (e.g., 'What is the point of this for me?') and those referring to the objective sense intended (i.e., signification and referents)." (Howson, 2005, p. 18). In this study, only the first aspect, the relevance and personal significance, is dealt with under the term personal meaning. In other words, the students' kind of personal meaning is the answer to the question, which relevance the students see in learning and doing mathematics for themselves and for their lives. This is exactly the meaning of meaningfulness as Mitchell uses the term: "Meaningfulness refers to students' perception of the topics under study in their mathematics class as meaningful to them in their present lives." (Mitchell, 1993, p. 427). Because we assume like Thom that “'meaning' in mathematics is the fruit of constructive activity” (see Thom, 1973, p. 204) we use the term *construction of personal meaning* to stress the constructive part of this process.

Although the importance of meaning for learning is well known (Krapp, 2003; Mitchell, 1993), hardly any empirical studies exist on the students' personal meaning in certain mathematical issues and the conditions which can influence this process. The study described below is based on the following theses about personal meaning and its construction. They have been developed from existing literature from mathematics education, *Bildungsgangforschung* and educational psychology.

- Things and events have no implicit meaning. This implies that everyone has to construct his or her own personal meaning so that it is possible that students develop different kinds of meaning concerning the same mathematical task or problem.
- Although everyone has to construct his or her own meaning, the kind of meaning students construct is not arbitrary. It depends on the one hand on offers of meaning given by teachers, parents and society as well as, on the other hand,

on the students' personal experiences, abilities, dispositions, their wishes and intentions.

- Neither the construction of personal meaning nor the product of this process are necessarily conscious efforts. It is, however, possible to reconstruct them from oral data.
- The kind of personal meaning students develop when dealing with a situation can differ from the one they construct after dealing with the situation. So the kind of personal meaning may change.

As mentioned above, there are different factors which influences the construction of personal meaning. These can be captured by concepts like mathematical beliefs (see for example Maaß, 2004), mathematical thinking styles (see for example Borromeo Ferri, 2004), different kinds of learning motivation (see for example Wild, Hofer and Pekrun, 2001) and developmental tasks (see for example Trautmann, 2004). As shown in figure 1, these concepts arise from the three fields of mathematics education, *Bildungsgang*-didactics and educational psychology.

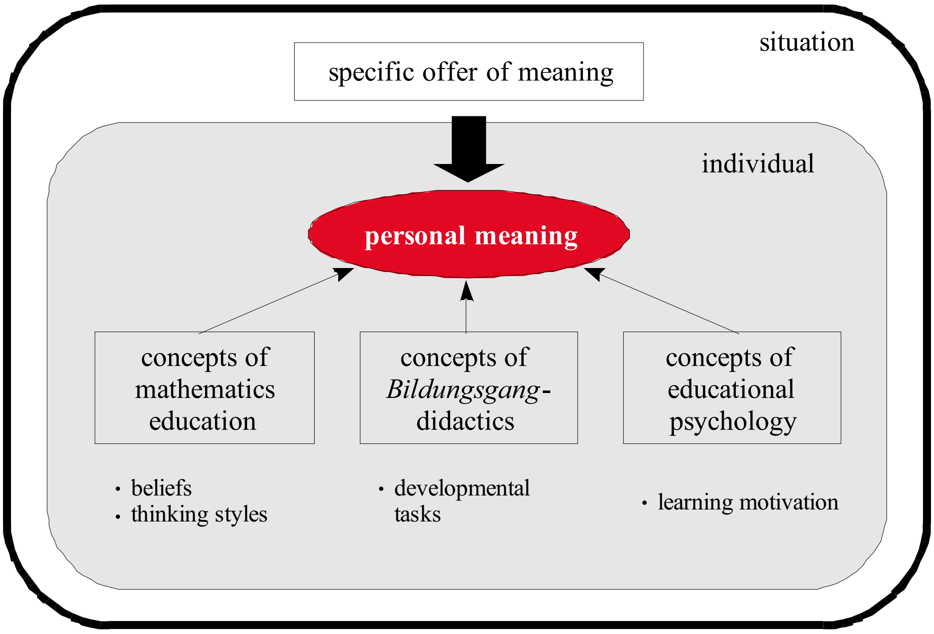

Fig. 1: Relational Framework of Personal Meaning

Empirical Results

Jahnke-Klein, states that students always consider mathematical issues meaningful if the tasks and problems appear as 'worth to think about it'. The results of her extensive empirical study suggest that the construction of personal meaning can be supported and promoted by appropriate offers (see Jahnke-Klein, 2001). Here the question arises, in what respect and under which conditions modelling problems can be such worthwhile tasks, and thus, favourable for the individual construction of personal meaning.

This question cannot be answered yet. Different empirical studies show, however, that apart from the interest in the task context there are various other factors that affect the students' handling of modelling problems (and thereby also their construction of personal meaning). So in addition to the task problem, the respective context of the task was determined as an important influencing factor. Depending both on the students' previous knowledge and on their subjective evaluation, the task's context is understood differently (as is the task itself) so that the students work on it differently (see Busse, 2000).

In addition, the respective mathematical thinking styles and the mathematical beliefs of the students have effects on the handling of reality-based tasks. Maaß reconstructs in her extensive empirical study that students who have a schematic and/or formalism-oriented belief system show a rather rejecting attitude when working on reality-based tasks (see Maaß, 2004, p. 283f.). In addition, Stillman observes that some students solve tasks without engaging with the task context (see Stillman, 1998). Finally Busse shows that students dealing with reality-based tasks argue on different levels (see Busse, 2005). A further important result of Stillman's study (1998) is that there is generally no direct connection between the students' arguments with the special context (commitment) and the obtained achievement (performance) of the students.

Especially the different impacts of reality-based modelling problems on students' modelling approaches suggest that these tasks can influence the students' construction personal meaning in a similar way. Whether this influence actually exists and whether it is to be judged positively or negatively is to be analysed in this study.

METHODOLOGY AND METHODS

In order to answer the research question, qualitative methods are used for two reasons. The first one is that there are various things that can influence students' construction of personal meaning. This high number of different influencing factors makes it difficult to use quantitative methods. The other reason, which is particularly more important, is that there does not exist a theoretical conceptualisation the construction of personal meaning yet. So at the moment it is not really possible to find out different kinds of students' personal meaning because nobody knows which different kinds there are.

As mentioned above, it is assumed that there are different kinds of personal meaning and that students themselves often do not realize their particular way of constructing a personal meaning. This is why the different kinds of meaning-making in normal mathematics lessons as well as those dealing with reality-based tasks have to be reconstructed.

For data collection five classes of the tenth grade (age 16-17) of two different German higher achieving schools (so-called Gymnasium) were chosen. One reason for choosing students of this age is that these students are capable of reflecting on their experience during mathematics lessons. Further reasons, not less important, are that usually students at this age cannot apply the mathematical methods they have learnt in their mathematics lessons in every-day life and that students of this age often think that there are much more important things than school.

The first step taken in the process of data collection was to ask the students to fill in a questionnaire with eight open questions. Most of them were handed back in. The intention of the questionnaires was, on the one hand, to get a general impression of the students' attitude towards mathematics and mathematics lessons and, on the other hand, to find out who was interested in and had time for being interviewed. As the interviews were done in the students' free time, students had to volunteer.

After that, in each class a normal mathematics lesson was videotaped with two cameras. The only advice given to the teachers for this lesson was that the lesson should be normal for the students. In the afternoon (or in the afternoon of the next day), a stimulated recall (see Gass and Mackey, 2000) and an interview was done with the volunteering students. The number of students varies from 2 to 6 students per class. During the stimulated recall they were shown the whole lesson and asked to comment on it. Especially they were asked to tell what they had been thinking during the lesson. After that, they were asked further questions about what they had said during the stimulated recall and about their answers in the questionnaire. They were also interviewed about their attitude towards mathematics, mathematics lessons, their feelings during mathematics lessons, and whether they thought they needed or would need mathematics outside school.

As a next step, four modelling problems were created. As mentioned above, each of these tasks has another context and different mathematical methods are necessary to solve these tasks. The tasks were embedded in contexts and problems that seemed to be important or interesting for the students.

After developing the tasks and testing them in another class, they were given to the teachers of the five classes having participated in the first part of the study. The teachers were requested to act in the following way: At first every student should read all four tasks and decide for herself or himself which one she or he would like to work on. Then students who had chosen the same task were to work on this task in a group of not more than five. The teachers were asked only to assist if the students asked for help. After a while the students should get the possibility to present and discuss their results in class. The group work of the students interviewed the last time was videotaped and another stimulated recall and interview was done with them. In total, there are stimulated recalls and interviews of 17 students relating to two different mathematics lessons.

In order to find out different kinds of personal meaning and the influence of reality-based tasks, the data will be analysed by coding according to the rules of Grounded Theory (see Strauss and Corbin, 2003). Strauss and Corbin were first of all interested in actions and not in biographical developments. Therefore in order to be able to analyse personal meanings (which are subjective mental processes), the coding paradigm will be modified.

TWO EXAMPLES OF MODELLING PROBLEMS

As mentioned above, four tasks which differ in context and in the necessary mathematical methods were created. To give an impression of theses tasks two of them will be presented here. Because both tasks are based on newspaper articles which are too long to be presented here, the tasks are not shown in their original (translated) version, but are summarised. A short comment on the context as well as a report of some students' reactions will also be given.

The personal meaning the students construct while solving the chosen task need not be in relation to the mathematics used. So the personal meaning does not have to be in relation with the special task but can also be in relation with the characteristics of modelling problems in general.

Pension

Students who chose this task had to deal with the question, which amount of money they had to save each month in order to have 1000€ of today's purchasing power per month when they are pensioners.

For answering this question they got an article in which the effect of compound interests as well as the effects of annual rate of inflation and return were mentioned. If asked, the teachers gave them further information about life expectancy.

It was assumed that at least some of the students had heard about the problem of empty pension funds in Germany and the need for joining a private pension scheme for younger people. Because this modelling problem was given to sixteen year old students and some of them were soon doing their intermediate school certificate and starting their vocational training, this complex of themes was assumed to be interesting for them or at least should be.

The reactions concerning this task were very different. Some of the students who did not choose it thought that, in contrast to the other tasks, this one only consists of numbers and had not as much to do with their lives as the other tasks. However, many others argued that they do not have enough money at the moment to save some, or they simply did not want to think about financial security but live their lives. One girl commented that she would rather spend the money for new shoes than for her pension. So, many of the students were not interested in the task context. Nevertheless, some students were interested. Those students mostly had talked about this topic with their parents or other members of their families, or they had got an offer for a private pension scheme. Now they were interested in calculating on their own.

Noisy Snorer

Coming from a newspaper article in which decibel values of different sound sources like breathing, screaming and snoring were given, the students were asked to calculate the number of people who must talk at the same time to be as loud as a snorer. This was the only task in which different kinds of formulae were given as assistance for the solution. The first one was a rather complex one for calculating the sound of sources in decibel, which seems to be very difficult and incomprehensible for most students at first glance. The second one was a rule of thumb for calculating the decibel value of two equal sound sources. If asked, the teachers gave the students values of further sound sources like quiet and normal talking.

Many students like listening to loud music and use their mp3-players. Therefore they are often told about possible damages resulting of loud music. Furthermore, they often go to parties and hence know that it is often hard to understand each other on such events because of the volume of the music. So the aspect of volume is familiar to them. It was assumed that, in addition, the close connection to aspects of physics lessons could be interesting for at least some of the students. However, this connection did not seem that important to the students who chose this task. On the other hand, rather the aspect of volume and the consequences for health were interesting for them, as will be described in the next chapter.

PRELIMINARY RESULTS

The questionnaires, stimulated recalls and interviews have not been analysed yet in depth, so final outcomes cannot be presented here. However, as a first step of the analyzing process, every interview was summarized and some of them were coded openly. Therefore, some preliminary results in the form of case studies of two students can be described. These are presented in the following two sections.

Robin

Robin is a rather extraordinary student for he enjoys busying himself in detail with mathematics and physics in his free time. He especially likes reading physics texts and says that he 'does not like reading a text and not understand the content only because it contains something like logarithm'. Further examples mentioned by him make clear that Robin tries to explain many occurrences in his life with the help of mathematics. Robin constructs different personal meanings for doing mathematics. Apart from exercising logical reasoning, preparing for future life, and improving his mathematics marks, understanding the world is the most important one for him.

Robin decided to choose the modelling problem 'Pension' because it 'seems not to be that easy' and he likes calculating with interest and compound interests.

For understanding his judgement concerning the task, it is important to know that for him mathematics lessons are the place where he can learn mathematics. During these lessons he is requested to fulfil special rules. One important rule is that students are asked to solve tasks in the right way. Because he only knows tasks which only have one right answer, he is convinced that students have to find this one correct answer.

Due to this conviction, he values the modelling task negatively because he understands that for solving this problem everyone has to make assumptions and therefore will get different solutions. His remarks concerning the task context indicate that he has busied himself with investment in different forms before and that his reflections about this context would have been more precise und subtle if he had dealt with this context in his free time and not during mathematics lessons.

Working on this special modelling task therefore did not influence Robin's personal meaning of learning and doing mathematics.

Larissa

Unlike Robin, Larissa seems to be a student who can be found more often in German classrooms. Her mathematics marks are relatively bad as well as her relationship with her mathematics teacher. For her, the meaning for learning and doing mathematics consists in pretending to be competent, not necessarily being competent. The reason thereof is her strong desire to move up to the next class level and to get a new mathematics teacher. The more astonishing (especially for her teacher) were the enthusiasm and the involvement with which she, together with the members of her group, tried to solve the chosen modelling task. the 'Noisy Snorer'. Her decision for this task was influenced by different factors: One was the assumed low level of difficulty, the other was the task context: Larissa had just heard a report about a comparison between the volume of an mp3-player and an aircraft turbine which she could not believe. Furthermore an eardrum of some of her friends had burst some time afore and the friend claimed that it had burst just because he was boxed on his ear. She could not believe neither the report not the description of her friend.

For her enthusiasm for and involvement in working on the task, Larissa gave the following reasons: After a long time of feeling incompetent during mathematics lessons, she got the impression of being able to solve a task, because she did not just have to find the right formula and use it in the correct way. Concerning this task it was necessary to find a way for solving the task and to discuss it with others. Hence, not only the high-achieving but also the poor-achieving students got the possibility of finding a suitable answer to the problem. Furthermore, because of the task context and the given information, Larissa was enabled to value the report and the description of her friend in a new way.

So Larissa's personal meaning was influenced by the modelling problem at least temporarily: Pretending to be competent was not that important any more, but, on the contrary, getting the feeling of being competent was way more important. The experience to be able to value reports and descriptions with the help of mathematics pointed out a new kind of personal meaning for learning and doing mathematics to her. Whether these different kinds of personal meaning are permanent or only temporary can, however, not be said on the basis of the existing data.

FINAL REMARKS

As shown above, no negative consequences were ascertained resulting from working on modelling problems, although Robin refuses the task itself. For Larissa, in contrast, the chosen modelling problem offered a meaning for learning and doing mathematics.

Modelling tasks can, however, influence the students' personal meaning of learning and doing mathematics. The question whether these influences are merely temporary or by which methods they persist permanently cannot be answered currently.

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INTERPRETING VELOCITY AND STOPPING DISTANCE; COMPLEMENTARITY, CONTEXT AND MATHEMATICS

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In the Netherlands, mathematics education is intertwined with applications. For example, in advanced calculus in secondary school many exercises contain a formula, which is set in a real-life context. In our research on application skills we confronted students from the science & technology stream with an interpretation exercise on the stopping distance as a function of a car's velocity. Students were familiar with the context and they had been exposed to all underlying concepts. We analysed students' difficulties with the interpretation exercise in light of the complementarity of the mathematical world and non-mathematical contexts. Out of four bright students, only one was able to reconcile mathematics with the context. This paper describes how complex the activity of 'interpreting' is, as students are drawn into contexts and need heuristics to stay within the mathematical world.

INTRODUCTION

In the Netherlands mathematics education has been inspired by Hans Freudenthal and his colleagues, who developed a treatise known as Realistic Mathematics Education (RME). In the earlier decades, Freudenthal (1973) characterised mathematics as being an integral part of real-life, perceiving it as an activity and not as a set of rules. As such, mathematics is a creative and organising activity in which unknown regularities, relations and structures are to be discovered.

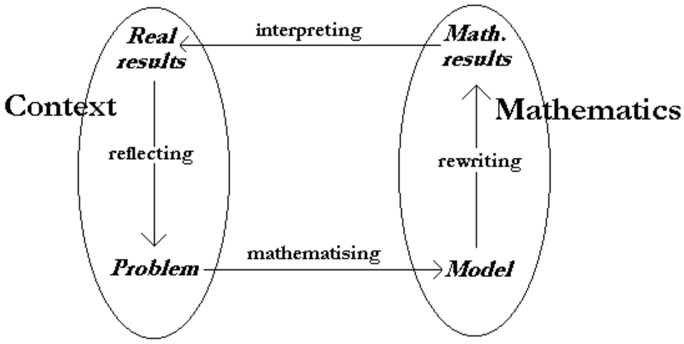

Figure 1: Modelling activities

Treffers (1987) described RME in terms of activities that connect contexts and mathematics. When starting from a problem set in a context, the student has to strip it from its details and find relations and regularities that result in a formula, a graph or a table. This activity is called mathematising and the resulting sketch, formula, graph or table is called: a mathematical model. The model is part of the mathematical world, and needs rewriting, restructuring and refining to obtain a mathematical result. This activity is called reformulating. Returning to the context, the (mathematical) answer needs to be interpreted in its context. Finally, the result needs reflection whether it offers an answer to the initial problem.

Since the late nineties, under the consecutive RME influences all Dutch mathematics curricula at primary and secondary level contain many application exercises. However, not all instructional strategies do comply with the original RME intentions, and some RME-based researchers have questioned an indiscriminate use of applications in textbooks (Wijers, Jonker & Kemme, 2004).

In this paper we focus on the calculus curriculum at pre-university level in the science and technology stream. This domain contains many applications, not only in textbooks but also in the national exams, which students need to pass for entry into tertiary education. Generally, an exercise starts with a real life situation followed by a formula and then students have to find certain characteristics (a maximum, the rate of change at a certain point, etc.). Occasionally, they have to adapt the given model to suit a similar, but slightly different situation. The questions are often embedded within the context in such a way that the mathematical terminology is concealed. We have translated, as an example, the first question on the national exams for pre-university Advanced Mathematics in May 2005:

> At a medical check-up people are asked to fully breathe out and then inhale deeply for five seconds. While inhaling, the amount of fresh air in the lungs is a function of time. For healthy people we use the following model: $L(t) = 3.6\ (1\text{-}e^{-2.5t})$.
>
> Here L is the amount of fresh air in litres and t is the time in seconds ($0 \leq t \leq 5$). According to this model, the maximum amount of fresh air in lungs of healthy people is 3.6 litres.
>
> [a figure shows the graph of the function]
>
> Calculate after how many seconds 90% of the maximum amount of air is inhaled. (..)
>
> What is the rate of inhaling (in litres per second) at $t = 0$?

Analysing this exercise in light of the modelling cycle, we see that it omits the mathematisation activity: the students do not have to create the formula from collected pneumonic data. Also, the questions do not logically emerge from the context: why should one want to know when 90% is reached? Even the interpretation of the answers and reflection thereupon is deleted, although this could have been justified [1]. The sole activity within the modelling cycle is entirely within the mathematical world. As such, the exercise is not a modelling exercise, but an application exercise, in which the context is irrelevant to the mathematical activities, and in which the mathematics is not well-connected to the context. Students are asked to move into the mathematical world and they are not asked to move out of it again.

The above exercise is only one example of current Dutch practice, and the CERME5 paper by Roorda, Vos and Goedhart (2007) contains two more exemplary exercises, to which the same analysis can be applied. We could have selected many others. These exercises do not comply with the original objective of mathematics being a creative and organising activity, in which unknown regularities, relations and structures are discovered. Our critique is in line with the conclusions drawn by other researchers, who describe the over-abundance of contexts in Dutch mathematics textbooks, the irrelevance of many contexts and the omission of the mathematisation activity (Wijers, Jonker & Kemme, 2004). Dekker *et al.* (2006) even speak of *nontexts*.

Set within the Dutch debate on a sensible use of contexts in mathematics education, we decided to start a series of *surplace* studies. We intend to study what and how students learn when being torn between mathematical concepts and non-mathematical contexts. We want to obtain a better insight into how our Dutch students learn mathematical concepts with and through contexts, how contexts can assist or hinder in concept construction, and what learning difficulties do arise when students move from contexts to concepts and back; that is, when mathematising and interpreting. Therefore, we have set up a range of studies on applications in mathematics education, in particular with respect to calculus in senior secondary schools (grades 10-12). The present paper describes an exploratory study on 'interpreting', which is one of the activities that connect contexts and mathematics.

THEORETICAL FRAMEWORK

We already introduced Treffers' diagram for modelling with its four activities: mathematising, reformulating, interpreting and reflecting. The diagram bears resemblance to diagrams used by Blum (2004) on modelling in mathematics education. The cyclic diagram is also used for modelling in science education, following system theory that contains a feedback mechanism. These frameworks contain a loop from the concrete to the abstract and back.

Fundamental in these frameworks is the lack of overlap, and even the apparent distance between the real world (contexts) and the abstract world (mathematical concepts). How do contexts and concepts relate to each other in mathematics education? In the present paper, we will explore their relation in terms of *complementarity*. Complementarity as a notion has its origins in the work by Niels Bohr, who worked on a dilemma in physics. In the quantum world he needed to reconcile two perceptions of light, one as a particle and the other one as a wave. The two offer significantly different perspectives, but they can co-exist for explaining theories. As such, complementarity differs from notions such as dichotomy, or duality. The idea of complementarity was used in the epistemology of mathematics by Otte (1990) in describing how mathematicians think. He described the complementarity between intuitive and axiomatic reasoning.

In the present paper, we build on Vithal (2006), who explored the notion of complementarity of concepts and contexts in order to explain her research data collected in South African mathematics classrooms. In the educational setting, complementarity means that concepts and contexts can hinder, but also strengthen each other. We will use this notion to analyse how students switch between the real world and mathematics while working on application exercises. Are they able to look through the contexts and see the mathematics, and are they able to look through the mathematics and see the contexts? Does the context obstruct or reinforce the mathematics and vice versa? In our study, we exposed four students to a particular exercise, set in a context, in which they had to interpret mathematical representations. Thus, the exercise asked students to move from the mathematics to the context.

Other researchers also studied how students move between mathematics and the real world. Crouch and Haines (2004) studied in particular the activity of mathematising (or abstraction), confirming the idea of a complementarity when they stated that *"most students cannot keep the needs of the real world and the mathematical model in mind at once"* (p. 203). They recommend that students need many strong experiences in building connections between the real and the mathematical world. In a similar vein, Maull and Berry (2001) also found that students do not reconcile well the given context with the mathematical model that they produce. Adding to these studies, we felt a need to focus in particular on the activity of interpreting as part of the modelling cycle. By giving students an interpretation activity, we hoped to study the complementarity of mathematics and the real world from a new angle. In many modelling exercises, the activity of interpretation is not explicitly asked for.

THE EXERCISE

To assess the activity of 'interpretation' we used one particular application exercise that differed considerably from the exercises that our students are exposed to in the current curriculum. It was set within a clear, familiar context but it did not contain an explicit formula, graph nor table. It asked for the interpretation of certain representations (symbols), of which all underlying mathematical concepts were familiar to the students. The exercise is on the stopping distance of a car, from Bezuidenhout (1998). For our students we translated the exercise into Dutch, with *remweg* being our word for *stopping distance*. Hence, R is the dependent variable:

> The stopping distance $R(v)$ (in metres), which a vehicle covers after applying its brakes, is a function of the vehicle's velocity v (in km/h). Assume that the maximum velocity of a vehicle is 200 km/h. Interpret the following equations in terms of stopping distance and velocity of the vehicle (also indicate the units that are applicable)
>
> 1. $R(100) = 80$
> 2. $R'(80) = 1.15$

The exercise is clearly not a modelling exercise. The questions do not emerge from the context, there is no requirement to mathematize (to develop a formula, a graph, etc.), and the exercise does not even ask for calculations. Nevertheless, the exercise implicitly contains a modelling activity. Although unmentioned, the context can be associated to the world of traffic safety, where speed limits are set according to stopping distances. Would an increase in the maximum speed lead to undesirable dangers on our streets? Maybe the Ministry of Transport asked a researcher to analyse stopping distances depending on different velocities. The researcher then organised an experiment with a car having different velocities; the driver applied the brakes firmly at a certain position, and then the stopping distance was measured. Starting from a table and a graph, a formula emerged, allowing the researcher to analyse the problem mathematically. As such, the imaginary researcher went through the modelling cycle, from the real world to the mathematical world and back.

In the present exercise, the student has to step into the modelling cycle to move from a mathematical result to a real result (see Figure 1) by interpreting the function representations. As such, the student has to step out of the mathematical world, after being exposed to mathematical representations as well as to the context. In the first item, the function needs to be interpreted as a relation between two variables (at a velocity of 100km/h the stopping distance is 80m). For the second item, the rate of change of the one variable needs to be related to a change in the other variable, recognizing that $R'(80)$ implies units of the type 'distance per velocity' (m per km/h). The exercise does not ask for reflection on the answers [2], nor on a meta-reflection of the full modelling cycle.

METHODOLOGY

We wanted to use Bezuidenhout's research to study Dutch students in grades 11 and 12 level in the science & technology stream, which includes Advanced Mathematics. Bezuidenhout studied more than 500 university students, of which 75% gave meaningful (accurate/relevant) written responses to item 1, but only two students' written responses were meaningful in the case of item 2. The vast majority of the students raised different ideas about the meaning of the number 1.15 in the equation $R'(80)=1.15$. They interpreted it as either an acceleration or a deceleration of 1.15 km/h^2 (or 1.15 m/s^2) or as a velocity of 1.15 km/h (or 1.15 m/s).

We wanted to see whether Dutch students in senior secondary education, after being exposed to the mathematics curriculum with its abundant use of contexts, would be able to find the correct interpretations in this particular exercise. At this level of learning, students have encountered many applications and they are familiar with the derivative function, with the rate of change and they are able to determine tangents to graphs. Unlike Bezuidenhout's students, our target group had not yet entered higher education, Nevertheless, we considered the exercise suitable for them. Since we wanted to study their thinking process in more detail than only in writing, we interviewed them individually in a tasked-based setting. They were asked to think aloud, but they could also write their answers, and then describe what they were thinking while writing (stimulated recall). The interviews were videotaped, transcribed and thereafter analysed in terms of the complementarity of concepts and context in the modelling cycle. With each episode, we asked ourselves in which of the two worlds (context or mathematics) the students were, and whether they were able to bring the two together.

To minimise the effects of misconceptions through learning weaknesses, we administered the exercise to students who had been recommended by their teachers as being excellent. This would enable us to analyse the optimal level of interpretation proficiency under the current curriculum, in which interpretation does not have a prominent stance. We interviewed four students: Bas and Heiko from grade 11, and Rob and Julia from grade 12 (student’s names are pseudonyms).

RESULTS

On the first item, all four students were able to correctly interpret the stopping distance at a certain velocity, without mixing the variables. Student Bas is exemplary:

Bas: <points at the item> If he goes 100 km/h, then the stopping distance is 80m.

It is interesting to note, that Bas did not say *“if it goes..”*. Instead, he personified the subject of the exercise, a vehicle, into a male person. Possibly, the context is so familiar that the students really imagined a person driving a car. On this item, all students were able to reconcile the complementarity between the mathematical representation of the function concept (the symbols) and its interpretation in the real world. Here, the mathematics and the context did not hinder each other.

The second item, interpreting $R'(80)=1.15$ turned out to be a major hurdle. Student Rob gave a confusing answer:

Rob: But what is the distance differentiated? <sighs> I first thought... this looked like an easy exercise. Eh... if you differentiate it, you will get meters per second. 1.15 m/s is what he stops.

Obviously, Rob thought of differentiating distance to time, a familiar concept in Physics classes. Thus, he interpreted R’(80) with respect to the stopping of one vehicle. Another student, Bas followed a similar line of thought, although remaining closer to the differential as a rate of change:

Bas: The speed... eh... if he drives 80 km/h then he will need 1.15... eh... m/s... what is it? <long silence, stares at the paper, bites underlip>

Tutor: What are you thinking… ?

Bas: Yes, *R’*, so the differential of the stopping distance, eh... delta x, which is sort of a speed but then at 80km/h. 1.15 m/s... if I see it right... I don’t know whether it is m/s or km/h. It is something like transforming km/h to m/s. <long silence, stares at the paper> The 1.15... it is what he slows down.

Obviously, Bas first tried to relate the $R'(80)$ to a concept: the rate of change. He mentioned *delta x*, but then quitted and interpreted the 1.15 as being related to the slowing down. Just like Rob, he is drawn into the context of a vehicle braking. The deceleration of a car is a familiar context, and the frequent use of it elsewhere (in

Physics lessons) draws the students into another concept, namely whereby the derivative of the velocity is the acceleration (or deceleration). Thus, the context diverted these two students away from the stopping distance as a function of the velocity.

The third student, Heiko, reached a somewhat different conclusion:

Heiko: If you differentiate the distance you will get the acceleration... and…

Tutor: If you differentiate the distance… ?

Heiko: Yes, the distance, that is the stopping distance.

Tutor: Okay, and then you get the… ?

Heiko: Yes! … But eh… then you will get the time which you will need for braking. While … <knocks on the table> this is the velocity.

Tutor: This would mean that with a velocity of… ?

Heiko: 80 km, it will take 1.15 sec to stop <sounds unsure, sighs>.

Just like his fellow students, Heiko interpreted $R'(80) = 1{,}15$ in a context of a vehicle that slows down, but he contextualised the 1.15 as the time needed for braking. Heiko created an interpretation that seemed plausible in the context of a stopping vehicle. However, just like the others, he was hesitant and did not manage to correctly join the representation of the formula with the stopping distance, although in the first item they were able to interpret the function correctly. With the derivative in the second item, they moved to an associated but different concept and removed the stopping distance from the original context.

We tried to make the above students express their thoughts, but obviously they did not like their failing. Heiko enquired about an explicit function, to which the tutor obliged:

Heiko: Yes, if there was a formula given, I think it would have been easier for me.

Tutor: Okay, so if it had been something like $8v^2$... or something similar…

Heiko: <hesitates> I think then… I still would not know it.

Unlike the above three students, there was one student, Julia, who tackled the exercise, although it took her long. She first thought along a similar line as the boys, mixing concepts from Physics into her thinking:

Julia: This here, eh… <points at 2.> is the slowing down, how do I say this, the acceleration? Yes, in fact the deceleration in m/s. I think, this is simply… you should see this as a negative acceleration, an acceleration at 80 km/h …? <looks unsure>

Tutor: Can you tell me more about how I should see this.

Julia: Well, you have… the change in velocity… but it is a bit weird, because if you drive 80 km/h, then... <drops her pen to the floor in distress> ... v is still the variable, so… that 80 must be velocity, but it is weird that this figure is positive, because with a stopping distance one will always... the velocity reduces, so I thought it would be negative... I don't know why it is positive...

Tutor: Maybe I have made a mistake in typing? <laughs> No, it is positive. What is written here, is as it should be.

Julia: As it should be!

Tutor: You think it is a deceleration of 1.15m?

Julia: 1.15 m/s^2. That's what I think… No, wait a minute, the stopping distance is in meters… oh wait a moment, yes, that's what I think… Yes, in fact, I cannot think of anything else than deceleration. The stopping distance differs of course, ... how fast is one driving? It goes per speed <waves her hand>.

Julia was hesitant for two reasons. She expected a deceleration being shown by a negative sign, and she doubted the units of the deceleration from km/h to m/s. She even asked for confirmation whether the mathematics is correct. As a result, she remained attached to the mathematical world, because her interpretation (a deceleration) towards a context (a car slowing down) did not match with the mathematics (the minus sign and the units). Unlike the boys, she did not lose touch with the mathematics.

At this stage, the tutor asked her about the manual gesture:

Tutor: Right now, you were holding your pen above the paper. As if you were going to… ?

Julia: Oh, I wanted, maybe a graph, so I can see clearer… <laughs> But I don't know whether that is being required. Do you want me to spend more time on this exercise?

Tutor: Yes, yes, continue.

Julia: Well, then the R is here on this side, and here is the v <draws axes>. Then the graph could be something like this… <sketches a graph> … If you would differentiate this, then… then! … then you will get… if it would run like this…! Oh, this is the increase in the stopping distance at 80km/h! If one would drive a little faster... okay, that's it! This means, that at a velocity of 80 km/h, if... with one kilometer faster… the increase in the stopping distance is 1.15m. <looks satisfied>

Tutor: Why do you think like this suddenly?

Julia: If it is drawn, it is more logical, ehmm... the *R'* is the slope, and it indicates that… that at this moment, if you drive faster, your stopping distance increases with 1.15.

Obviously, Julia was encouraged by the tutor to make a sketch of a graph and to spend additional time on her line of thinking. The sketch of a graph kept her within the mathematical world and that triggered her to make a correct interpretation, which she even was able to explain in terms of marginal change. Thus, the heuristics of sketching a graph made her draw away from the context of a car slowing down, the brakes, the time needed to stop, the deceleration and the concepts learnt in Physics. The sketch made her stick to the function as a relation of two variables, and how a change in the one is related to a change in the other. As such, first her doubts and thereafter the sketch prevented her from returning too quickly into the real world and enabled her to reconcile the mathematical representation with the context of the stopping distance.

CONCLUSIONS AND RECOMMENDATIONS

In the current Dutch curriculum, calculus exercises are embedded into contexts. Ideally, the contexts should be starting points for constructing conceptual knowledge. However, in today's practice students are learning to delete the context in order to carry out certain mathematical techniques. As such, the current practice teaches students to separate contexts from concepts and hardly to link these.

In our exploratory study we exposed four students to an exercise, in which the mathematical concept did not ask for mathematical techniques; the concept was only present in symbols. The task was to interpret the concepts of a function and a derivative in terms of a familiar context. The exercise came from Bezuidenhout (1998) and in line with his study, the majority of our students were diverted through an interplay of the context (a car and its speed), the concept (the derivative of a distance), and prior knowledge (Physics lessons). In our study we saw how three students associated the context and the derivative symbol to only one possible concept, the deceleration. Obviously, the original concept and context hindered each other. The well-known context made these students move too quickly out of the mathematical world. However, we observed one student, Julia, who added a graphical sketch to the symbolic representation, and thus created a stronger base within the mathematical world. This enabled her to see the context through the mathematics, and by reconciling the complementarity, she made a cognitive leap forward. At that stage, concept and context fell into place.

Using the complementarity of concepts and contexts, we have treated the two as distinct but related to each other. The two can hinder each other, but the ability to reconcile concept and context is enriching. Therefore, students need exercises that ask them to simultaneously keep in mind the needs of the real world and of the mathematical world. Our study has shown that the reconciliation can take place in interpretation exercises, though it remains to be seen whether reconciliation can also take place during mathematising activities or when students complete the full modelling cycle.

NOTES

1. The answer to the final question is: a rate of inhaling of 9 litres per second. A possible question for reflection is whether that is a lot or not.

2. A possible question for reflection is whether a stopping distance of 80 metres is a lot or not.

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MEASURING PERCEIVED SELF-EFFICACY IN APPLYING MATHEMATICS

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In England a new course of post-compulsory study has been developed based on the premise that developing confidence and ability in applying and modelling with mathematics will better prepare students to elect to study courses that require relatively high levels of mathematics in their future studies in Further and Higher Education. In investigating this we have designed an instrument to measure perceived self-efficacy beliefs in applying mathematics. Here we report principles of construction of the instrument together with initial analysis which suggests that it does allow measure of perceived self-efficacy beliefs in mathematics generally and in pure and applied mathematics separately with early evidence suggesting that the new course is successfully developing students' confidence in applying mathematics.

INTRODUCTION AND BACKGROUND

The "mathematics problem" in England is deep seated: too few students are well prepared to continue their studies from schools and colleges into courses in Higher Education Institutions (HEIs) in mathematically demanding subjects. Concerns have been raised by those involved in the sector and this is reflected in national reports such as the Roberts Review (2002) which focussed on the supply of well qualified graduates in science, engineering and technology and the Smith Inquiry (2004) which investigated mathematics in the 14-19 curriculum.

Within this contextual background this paper reports one aspect of the work of an ESRC (Economic and Social Research Council) research project, 'Keeping open the door to mathematically demanding courses in Further and Higher Education'. This mixed methods project involves both case study research investigating classroom cultures and pedagogic practices and individual students' narratives of identity together with quantitative analysis of measures of value added to learning outcomes in an attempt to investigate the effectiveness of two different programmes of study. Here we report the development of an instrument designed, as part of this latter strand of the project, to measure perceived self-efficacy in applying mathematics.

Central to our research is a new qualification in the post-compulsory (post-16) sector that was specifically designed to better prepare students to continue with study of mathematically demanding subjects. This recognises that the target students will not wish to study mathematics for its own sake: high achieving students, aged 16-19, wishing to study mathematics, engineering or physical science courses at university will follow non-compulsory courses leading to the "traditional" mathematics qualification – A Level mathematics. It is the intention that the new qualification will be accessible to students who are likely to be starting from a lower base of prior attainment in mathematics, but who nonetheless wish to go on to study a course that makes considerable mathematical demands at university. This qualification, AS Use of Mathematics, is designed as equivalent to study of the first half of an A Level in the English system (although a full A Level "Use of Mathematics" is not currently available), and includes considerable preparation in the use of algebra, functions and graphs, with an option to study either "modelling with calculus" or "using and applying statistics"[1]. Due to its potential students, the new qualification, therefore, attempts to ensure that those who study it will see an immediate or potential value in mathematics within their experience of other study, work or interests. Therefore, mathematical modelling and applications are fundamental to courses leading to the AS "Use of Mathematics". Whilst students may not be explicitly taught "to model", the philosophy is such that the mathematics, as learned and practised, involves being actively engaged in aspects of mathematical modelling and making sense of commonly used mathematical models with a particular emphasis on critical awareness of how modelling assumptions affect the validity of solutions to problems.

This course contrasts with the standard or "traditional" AS / A Level route which is followed by the majority of 16-19 year-old students in preparation for further study in HEIs. In this case applications and modelling play a much less prominent role: the emphasis is on a high level of technical facility with subject content in "core" or "pure" mathematics.

Our concern, then, is to investigate practices that widen participation in the study of mathematics: consequently in evaluating the effectiveness of the different programmes, AS Mathematics (the first half of study towards a full A Level in mathematics and referred to here as AS Trad) and AS Use of Mathematics (AS UoM), one measure we are investigating is perceived self-efficacy in mathematics and in particular perceived self-efficacy in applying mathematics.

PERCEIVED SELF-EFFICACY

It is now almost thirty years since Bandura (1977) proposed the construct of perceived self-efficacy: "beliefs about one's own ability to successfully perform a given behaviour". He later situated this within a social cognitive theory of human behaviour (Bandura, 1986), before more recently developing this further within a theory of personal and collective agency (Bandura, 1997).

Perceived self-efficacy beliefs have been explored in a wide range of disciplines and settings including educational research where they have been investigated in relation to progression to further study and career choices (Lent and Hackett, 1987) and in relation to affective and motivational domains and their influence on students' performance and achievement. One's perceived self-efficacy expectations are

[1] The specifications and assessment associated with the new qualifications were designed by one of the authors (Wake) acting as consultant to the government's Qualifications and Curriculum Authority.

accepted to play a crucial role in determining one's behaviour with regard to how much effort one will expend on a given task and for how long this will be maintained. Behaviour therefore is crucially mediated by beliefs about one's capabilities. Bearing in mind that outcomes which individuals consider as successful will raise perceived self-efficacy and those that they consider as unsuccessful will lower it, we hypothesise that those students following a "Use of Maths" course will increase their perceived self-efficacy in relation to applying mathematics in particular, to mathematics in general, and this will have a positive effect on their likelihood of further study that requires mathematics. This may be particularly important in widening participation into mathematically demanding courses in Higher Education as the AS UoM course at present is often catering for those on the margins of studying mathematics in Further Education College courses.

Perhaps most important and relevant to our study are research findings that suggest that perceived self-efficacy in mathematics is more predictive of students' choices of mathematically related courses in programmes of further study than prior attainment or outcome expectations (see for example, Hackett & Betz, 1989 and Pajares & Miller, 1994). Hence, our project's need for an instrument to measure perceived self-efficacy in applying mathematics. Here we describe the underlying framework which we used in development of this instrument with particular reference to constructs relating to mathematical modelling (Blum, 2002) and "general mathematical competences" (Williams et al., 1999). Whilst drawing on these constructs from mathematics education we have also taken into account, as we shall illustrate, the important advice Bandura offers those building measures of perceived self-efficacy; namely, that they need to be clear in specificity of the tasks that respondents are asked to judge, paying particular attention to levels of task demand, strength of belief, and generality of the task.

CONCEPTUALISING APPLICATIONS OF, AND MODELLING WITH MATHEMATICS

For many years the work of the ICTMA (International Conference for the Teaching of Mathematics and its Applications) group has explored how mathematical modelling can inform teaching and learning of mathematics at all levels. An important result of the work of members of this group is the conceptualisation of mathematical modelling and how this relates to applications of mathematics. Whilst there is not room here to discuss this fully we would draw attention to some of the main features of mathematical modelling, how this relates to "applications of mathematics", and how this is usually conceptualised in implementation of mathematics curricula in schools and colleges in England.

Essential to using mathematics to model a real world situation or problem is the genesis of the activity in the real world itself. Mathematising this situation, that is simplifying and structuring it so that it can be described and analysed using mathematical ideas and constructs, leads to the mathematical model. Following analysis using mathematical knowledge, skills, techniques and understanding the outcomes and results are interpreted in terms of the original problem, being checked to determine whether or not they are valid. At this stage it may be decided that the model is adequate, or that it needs to be modified in some way, perhaps making it more sophisticated so that the results/solution to the problem are more appropriate. This can therefore be conceived of as a cyclical process with the “modeller” translating between real world and mathematical representation. Some mathematical model types are commonly found and used to describe many different situations (for example, in the sciences models of direct proportion, exponential growth and decay and inverse square laws abound) and in some instances a recognition of this allows the modeller to short-circuit some of the process and work quickly between mathematical model and real world. In the discussion document which set out the agenda for the forthcoming ICMI study of Applications and Modelling in Mathematics Education (Blum, 2002), care was taken to distinguish between use of the term “modelling” on the one hand, to describe the mathematisation as one moves from reality to mathematical model, and “application” on the other as one interprets mathematical analysis in real terms, sometimes from a given mathematical model.

In recent years in England, scant attention has been paid to the process of mathematical modelling in “traditional” courses at this level with assessment encouraging a view of problem solving / applications being something that follows learning and, if possible mastery, of “basic” techniques. As has already been suggested, the new AS Use of Mathematics attempts to bring to the fore the processes of modelling and particularly application as outlined here.

THE SELF-EFFICACY INSTRUMENT

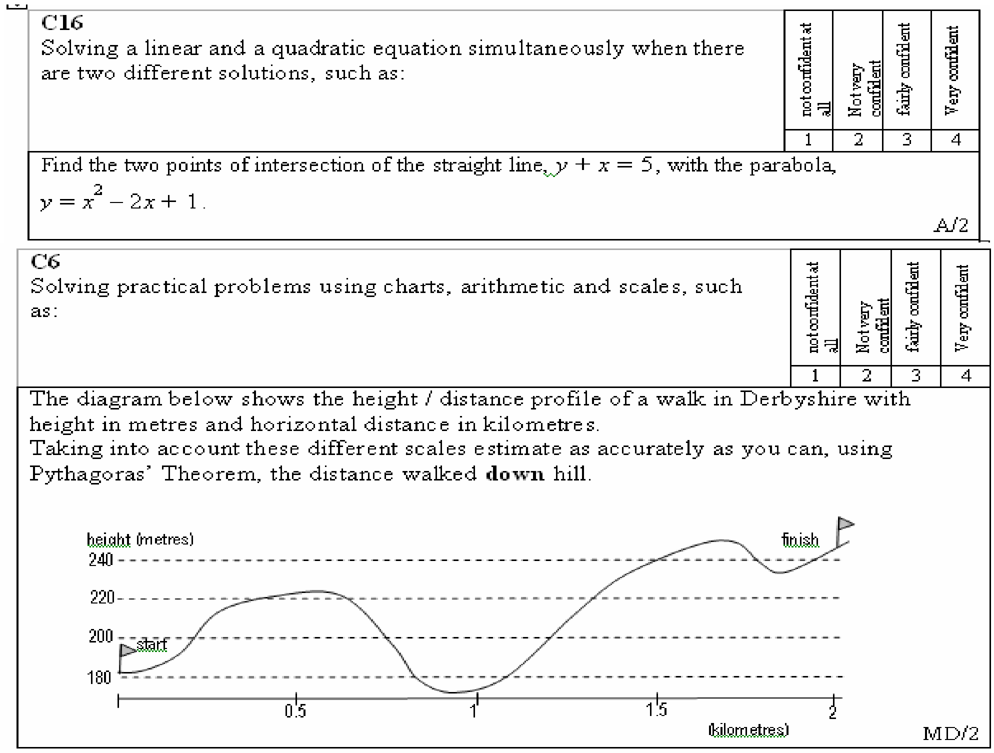
In developing an organising framework around which to build our self-efficacy instrument we turned to the construct of ‘general mathematical competence’ (for further discussion of this see for example, Williams, Wake and Jervis, 1999). This acknowledges that, as suggested above, in certain domains there are common ways of bringing mathematics together to solve problems and model situations. So mathematical modelling as practised in a range of situations by learners or workers (for example see Wake, 2007) is not a wholly open practice but is often based on common patterns of working that we have identified and briefly outline below.

In summary, a general mathematical competence is the ability to perform a synthesis of general mathematical skills across a range of situations, bringing together a coherent body of mathematical knowledge, skills and models with attention being paid to underlying assumptions and validity. Crucially then, the construct of *general mathematical competence* moves us away from thinking of mathematics as a collection of atomised mathematical skills towards consideration of it as a practice in which we bring together and use mathematics (often in common ways) to analyse situations and solve problems.

The general mathematical competences (g.m.c.s) developed were:

Costing a project: This requires the calculation of the 'cost', in monetary or other terms, of a (substantial) project or activity. Graphical / visual display of findings may be required.
Handling experimental data graphically: Developing a graphical display of experimental data requires firstly the identification of suitable data, its collection, organisation and recording prior to any scaling or processing that may be required. Following actual plotting of the raw or processed data identification and validation of mathematical functions to appropriately model the data may be necessary.
Interpreting large data sets: We increasingly have access to large sets of primary and secondary data. This g.m.c. requires initial sifting of data and identification of appropriate hypotheses, followed by the selection of the data required. Calculation of appropriate measures of location and spread and the development of appropriate visual / graphical display allow interpretation in terms of the original hypotheses.
Using mathematical diagrams: This g.m.c. requires the ability to translate reality into a diagrammatic representation (including plans or scale drawings) and vice versa.
Using models of direct proportion: The use of models of direct proportion permeates many areas of mathematical application (e.g. other g.m.c.s require the scaling of data which requires an understanding of the concept of proportionality). This g.m.c. develops numerical, graphical and algebraic understanding of this key mathematical concept requiring that one can move with ease between these different modes.
Using formulae: This g.m.c. pays attention to algebraic form and the use of algebraic formulae. It recognises that one often needs to be able to use formulae to process data and therefore requires that one is able to select appropriate data to use within any algebraic expression paying attention to units / dimensions of quantities.
Measuring: In practical work in science and technology it is important that attention is paid to measurement of data. In particular it is important that due attention is paid to the use to which the raw and processed data will be put as this will inform not only what should be measured but also the required accuracy and units with which quantities should be measured. Calibration of instruments is covered in this g.m.c.

A total of thirty items were developed with twenty four based on the seven general mathematical competences identified above and a further six being developed in "pure" mathematics areas. The items were designed at three different levels, it being the intention that, as part of the long term project, the instrument will be administered to the same student population at the start and end of a single academic year and at one point just beyond this. The different levels of items will allow for the increasing mathematical maturity of the cohort being studied. As usual in self efficacy studies, students were asked to choose the level of their confidence in solving each item (but, it was stressed, without the need for actually solving the item) using a Likert type scale. Examples of "pure" and "applied" items are given in Figure 1.



C16 Solving a linear and a quadratic equation simultaneously when there are two different solutions, such as:	not confident at all	Not very confident	fairly confident	Very confident
	1	2	3	4

Find the two points of intersection of the straight line, $y + x = 5$, with the parabola, $y = x^2 - 2x + 1$. A/2

C6 Solving practical problems using charts, arithmetic and scales, such as:	not confident at all	Not very confident	fairly confident	Very confident
	1	2	3	4

The diagram below shows the height / distance profile of a walk in Derbyshire with height in metres and horizontal distance in kilometres.
Taking into account these different scales estimate as accurately as you can, using Pythagoras' Theorem, the distance walked **down** hill.

Figure 1. Sample pure (top) and applied (bottom) items from self-efficacy instrument.

As these sample items illustrate, whilst they give a general description of the type of activity required, each ensures a greater degree of specificity by including an example of the type of task. Whilst item C16 requires application of mathematical understanding and techniques this is taken to be in the discipline of mathematics itself and is categorised as "Pure" (see next section) whereas item C6 presents a problem within a "real" context and is categorised as "Applied". Although this latter item is not explicitly presented as a modelling task its successful completion will require simplifying assumptions to be made, such as dividing the profile into a number of separate sections, using mathematics to analyse each sub-section, before making some attempt to assess the validity of the final solution.

In summary, therefore, our instrument meets Bandura's requirements, paying attention to the generality of the task, the level of its demand and strength of belief.

PILOT STUDY

The thirty self-efficacy items (ten at each level) were organised into three different versions of the questionnaire with each having link items ensuring that an item response theory (IRT[2]) model could potentially provide a good measurement model. The questionnaires were administered to a pilot sample of 341 students towards the

[2] The mathematical models of IRT calculate the probability of a correct response to an item as a function of the subject's ability, the item's difficulty and some other characteristics (depending on the relevant model)

end of their different AS courses in 23 different Further Education institutions across England.

Rasch analysis, and particularly the Rating Scale Model, was initially used to establish the validity of the instrument (Bond & Fox, 2001). The Rasch model, in its simpler form (i.e. the dichotomous model in which the responses are restricted to 1 and 0, or correct/incorrect) assigns an ability parameter to each student based on the number of his/her correct answers and a single difficulty parameter to each item, resulting from the number of students who answered that item correctly. Hence, it allows these estimates to be ordered in a common scale, using 'logit' as the measurement unit.

For this analysis we used the Rating Scale Model, which is the most appropriate for response categories in Likert type instruments that include ordered ratings such as ours ('not confident at all', 'not very confident', 'fairly confident', and 'very confident'). The Rating Scale Model (like any Rasch Model) also provides some fit statistics to indicate the fit of the data to the assumptions of the model, and particularly the dimensionality of the construct. Tests of fit aided the evaluation of the scalability of the item set and showed acceptable fit suggesting that our instrument could be used to describe the desired construct, i.e. perceived mathematical self-efficacy (Wright & Masters, 1982; Bond & Fox, 2001).

A next step in was to examine whether the items have significantly different meanings for the two groups, in which case differential item functioning (DIF) is present (as in technical guidelines suggested by Bond & Fox, 2001, p. 170-171).

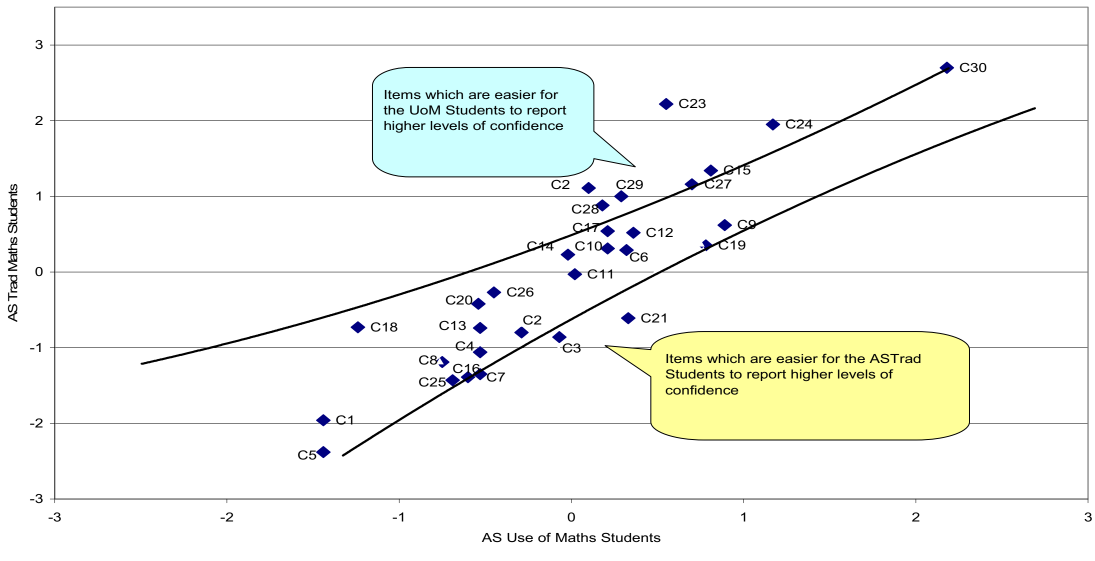


Figure 2. Item estimates (i.e. logits) for the two student groups (with the 95% confidence intervals)

The graph in Figure 2 plotted the difficulty of the items (in logits) as estimated for the two student groups separately, and the lines show the 95% confidence interval. According to the model's assumptions (and hence the 'ideal world's' scenario) it should be expected that these estimates should be invariant across the two groups, and all the items to fall inside the confidence intervals. In this case, however, it is demonstrated that certain items are outside these limits. Particularly, the items indicated at the top in Figure 2 are those that AS UoM students report to be significantly more confident in tackling than AS Trad students whilst AS Trad students report being significantly more confident than AS UoM students in tackling those at the bottom.

Given these findings, we hypothesized that there might be a latent sub-structure of the instrument that may be better modelled by a multi-dimensional model involving pure and applied dimensions. Focussing our analysis on those students with the lowest prior attainment (our target group of 70 AS UoM students and 93 AS Trad students) we performed multidimensional analysis for two different cases: (i) within-item multidimensionality (two dimensions), and (ii) between-item multidimensionality (two and three dimensions).

An instrument may, according to Wu, Adams and Wilson (1998), be considered multidimensional "*within – item*" if any of the items relates to more than one latent dimension. Within-item multidimensionality was determined by scoring the items on a 'Pure' and an 'Applied' demand on the basis of judgement about the nature of each item (i.e. items with no significant pure maths or applied maths demand, as judged by experts, were allocated to 'A' or 'P' respectively while those with some of each were allocated to both dimensions). From a different perspective *"between item multi-dimensionality*" occurs when an instrument contains several uni-dimensional sub-scales. In this case items were categorised into either two (2D model) or three (3D model) discrete categories as in Figure 3. Significantly these categorisations point to items C23 and C24, for example, as having an applied nature and C3 and C21 as having a pure nature, and the analysis points to the AS UoM students being more confident in tackling the former and the AS Trad students the latter.

<table>
<tr><th>Description</th><th>3D model</th><th>2D Model</th></tr>
<tr><td>no "real" context, may be solved using straightforward techniques</td><td rowspan="2">Pure (P)</td><td rowspan="2">Pure (P)</td></tr>
<tr><td>no "real" context (context not important to problem), requires decisions about approach to be taken</td></tr>
<tr><td>problem in "real" context - method clear</td><td rowspan="2">Applied (A)</td><td rowspan="3">Applied / modelling (A/M)</td></tr>
<tr><td>problem in "real" context requires synthesis of a range of mathematical understanding / techniques</td></tr>
<tr><td>requires assumptions / decision making in approach to solution</td><td>Modelling (M)</td></tr>
</table>

Figure 3. Sub-scale categories used to investigate "between item multi-dimensionality" of the self-efficacy instrument.

Using a ConQuest (Wu, Adams and Wilson, 1998) multi-dimensional analysis (after Briggs and Wilson, 2003) we evaluated the comparison of the unidimensional and multi-dimensional models via a chi-square test on the deviance statistic (Figure 4). As they state *"because the multidimensional approach is hierarchically related to the* *unidimensional approach, the model fit can be compared relative to the change in the deviance value, where the difference in deviance between the two models is approximately distributed as a chi-square"* (p. 95) with the difference in the number of parameters as degrees of freedom, in each case.

MODEL	Deviance	Number of parameters
Unidimensional	5670.886	33
2D –within-item	5662.499	37
2D- between - item	5723.387	35
3D- between - item	5721.527	41

Figure 4. Comparison of the Unidimensional and Multi-dimensional models

Our analysis suggests that the within-item multidimensional model (as highlighted above) has a slightly better fit to the data than the uni-dimensional model, which in turn performs slightly better than either of the between-item models. This suggests that our instrument might be successfully used to not only measure perceived self-efficacy in maths overall, but should be able to identify sub-dimensions of perceived self efficacy in pure and applied maths.

FUTURE DIRECTIONS

Our initial study has allowed us to construct and validate an instrument that we can use to measure perceived self-efficacy in mathematics, as well as in constituent dimensions of "pure" and "applied" mathematics. This will be used to track changes longitudinally in the perceived self-efficacy of students "at the margins" following AS UoM and AS Trad courses in Colleges across England. Our focus, therefore, will be students with low prior attainment who in some institutions are even excluded from study of mathematics altogether. Their perceived self-efficacy, along with other outcome measures and student disposition to study in Higher Education in general as well as disposition to study mathematically demanding courses, will be surveyed at the start of their AS Mathematics courses (UoM and "Trad") as well as towards the end of the course and into the following year in a delayed post-test.

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