WORKING GROUP 3. Building structures in mathematical knowledge 399

Xenia Xistouri

Group 3 – Structures

Milan HEJNÝ & *Graham LITTLER*
Charles University in Prague — University of Derby

GROUP LEADERS

Besides the authors of this report, Ladislav Kvasz (Slovakia) and Dvora Perez (Israel) were involved in the reviewing process as groupleaders. Unfortunately neither of these were able to come to Cyprus so Maureen Hoch (Israel) kindly accepted our request to take the responsibility of a group leader.

PARTICIPANTS AND WORKING CLIMATE

18 persons from 10 countries (CY, CZ, ES, GR, IS, NL, PL, PT, RO, and UK) participated in WG3. All of them were active during each session. In comparisson to the G3 sessions in previous CERME conferences, the climate was more collaborative and supportive especially to the younger participants. During the sessions and free time working contacts were established to develop work across countries. The group benefitted from the good climate established by the Organising Committee.

PAPERS DISCUSSED BY THE GROUP 3

1. Constructing Multiplication: Different strategies used by children
Joana Brocardo & Lurdes Serrazina & Isabel Rocha

We propose a hypothetical trajectory to multiplication and test it with second grade pupils, which put problems in several levels. We emphasised the aspects related directly to the learning of multiplication using multiplicative contexts, the nature of tasks and the role of teachers.

Questions for Discussion

- When can we say that a given task is a multiplicative situation or a division one?
- What are the main differences between multiplicative and division structures?
- What are the difficulties by proposing open tasks?

2. The scheme A + B = C as the basis for Arithmetic structure
Milan Hejný, Jana Slezáková

- Fill the blank boxes with numbers in such a way that both the sum of three adjacent numbers in a row and in a column is always 6.

3				
		1		
2				1

Questions for Discussion

Why do nearly all solvers use Trial and Error methods?
Do you know how many doors there are in your house/flat?
You should know since you have the scheme of your house/flat in your mind.
Can you give similar usage of scheme from mathematical area?

3. Recognising an algebraic structure
Maureen Hoch, Tommy Dreyfus

- We tried to answer these questions by analysing and comparing the learning paths of two students. The students' structure sense did improve: they learned to recognise and use the structures in complex forms.
- We asked the group to discuss why this ability did not always withstand the test of time.

Questions for Discussion
- What is algebraic structure at high school level?
- What is structural about a2 – b2 or a2 + 2ab + b2?
- How does a student learn to recognise structure?
- Can structure sense be taught?
- Structure sense was defined in earlier papers and is briefly summarised here.

4. Creating a mental image of dice – the game blackjack
Antonín Jančařík

- The experience of the author from workshop and seminars with teacher-education students, selected the game 'dice blackjack' as an example of more important games with perfect information.
- In the experiments analyse the process of finding a winning strategy and creating mental structure of the game.
- The results show that the understanding of the game comes at one specific point when the situations become more complicated.

Questions for Discussion
- Is it possible to find such a point in a different situation?

5. Classification, Manipulation and Communication – work with pupils and student-teachers
Darina Jirotková, Graham Littler

- Levels of *tactile manipulation* have been defined - these are linked to the ability of pupils to classify solids and to the individual pupil's communicative ability.
- Links between classification, manipulation and communication in 9 to 11 year old pupils when asked to divide a group of solids into two groups were found.
- It was found that communicating about what the tactile perceptions were as the pupils manipulated the solids helped to build the structures. Three types of structure were developed.
- It goes on to describe what the researchers found when they investigated student teachers' knowledge of and competence in building structures in 3D geometrical structures.

- By **structure** we mean the linking of information gained about a solid through the tactile perception so that a solid or an attribute found in several solids can be described uniquely.

Questions for Discussion

- Are tactile manipulation, communication and classifying 3D solids linked in any way?
- Is there difference between 3D geometrical knowledge structures of student-teachers' and 11 year-old pupil's?
- What attributes of solids are structure making?

6. Investigating the processing structures of student's inductive reasoning in mathematics

Eleni Papageorgiou & Constantinos Christou

- The study is aimed to empirically test a theoretical model formulated to identify and classify students' processing structures when they solve inductive reasoning mathematics problems. We constructed a mathematical test in order to assess the components of inductive reasoning of sixth Graders. The data suggested that inductive mathematics reasoning is a process consisting of three factors: the "similarity", the "differences" and the "similarity and differences" factors.
- The proposed model provides a framework of students' thinking while solving various formats of inductive mathematics problems, and a prototype for further investigation of the components of inductive reasoning.

Questions for Discussion

- Find the common feature of the numbers: 4, 16, 8, 32, 20, 100.
- Underline the number that does not fit in with the others: 3, 9, 15, 30, 81, 5.
- Complete with the right number 1, 5, 13, 29, ….
- Find the number that disturbs the sequence 1, 1, 2, 3, 5, 7, 13, 21.

7. Empirical hierarchy of pupils' attainment of measurement in early primary years.

Alexandra Petridou, Maria Pampaka, Constantia Hadjidemetriou, Julian Williams, Lawrence Wo

This study describes a developmental 'map' of performance in the context of measurement in the early primary school years (ages 5-7). This study uses data from diagnostic, age-standardized tests from a sample of 5120 pupils in England. The map was constructed using Rasch measurement methodology and specifically the Partial Credit model. This model enables to describe typical misunderstandings and errors alongside a constructed hypothetical learning trajectory. We interpret the scale from the analysis as a hierarchy of five levels of measurement performance. We then compare this empirical hierarchy with the one described in the English National Curriculum for mathematics.

Questions for Discussion

- Describe the strategies used by 5-7 year old pupils when measuring a toothbrush.
- The paper is written from a professional psychological point of view. Would the conclusions change if a mathematical analysis was used?

8. How students from primary school discover regularity

Marta Pytlak

The research experiments asked pupils to determine the sequence of values based on the perimeter and interior area of squares. The perimeter (8, 16, 24, 32,…) gives a linear sequence while the area (1, 4, 9, 16, …) gives a quadratic one. Comparing the first few elements of these two sequences a pupil, she/he hypothesises that the perimeter will always exceed the area. In the paper different ways of approaching this problem are discussed.

Questions for Discussion

- What are the ways of discovering the regularity by 5th Grade pupils?
- In what ways can a teacher stimulate the pupil's process of thinking on the abstract level?
- How can the interaction between pupils influence the development of structure of thinking in the process of discovering regularities?

9. Students' 'reflections on activity- effect relationships' in solving word problems

Ana I. Roig & Salvador Llenares

- This study focus on how the mechanism *"reflection on activity-effect relationships"* (Simon et al., 2004) is involved in the students' problem-solving processes.
- A five-question test paper was prepared and all the participants were interviewed. The results show that if the students don't remember a mathematical tool that allows them to solve the problem, the process of recognising the underlying structure of the situations can be carried out by means of the *"reflection on activity-effect relationships"* mechanism.
- Different components of the mechanism were identified and the comparison of activity-effect relationships seems to be a determining factor in the characteristics of the solving processes. If the comparison remains at the projection phase, the underlying structure of the situation is not (or is partially) recognized. If the comparison is done at the reflection phase, the student achieves a complete structural understanding of the situation.

Questions for Discussion

- What is the least size of a square floor which can be made using tiles 33 cm × 30 cm?

- What is the least size of a square floor which can be made using tiles m cm × n cm?

10. Childrens' perceptions of infinity – could they be structured?

Mihaela Singer, Cristian Voica

- The paper is focused on explaining what kinds of structures are activated when dealing with the intuition of infinity in school context. To get a better view of the students' insights about infinity we used a variety of questions, covering the following categories: vocabulary, intuitive representations; how does intuition work? how does one prove the infinity of a given set? how does one compare infinite sets?
- Starting with the sample data, we made a qualitative analysis concerning the following aspects: Children's everyday meaning for infinity; Perceptions and arguments in understanding infinity; Children's arguments in comparing infinite sets, and we come back to describe arguments for infinity starting from the structures point of view.
- We can conclude that some of the young children have a structured representation about the infinite sets. This is happening as soon as they learn about the set of natural numbers, in primary grades. At the age of 10-11, when students are learning about the decimal numbers, they are able to identify structures helpful in arguing about the infinity of some sets or giving hints for the cardinal equivalency. Also, we see that, when the students' arguments are consistent, they seem to be based on connections between algebraic and geometrical thinking.

Questions for Discussion

- Could pattern regularity be important at University level? If regularity as an idea is not grasped until university age, what can we do?
- What roles does structure play in young children's everyday activities (building with blocks, learning to count, comparing quantities)? How do these roles compare and how may they be related to each other?

11. Students' thinking about fundamental real number properties

Eustathios Giannakoulias, Alkeos Souyoul, Theodossios Zachariades

- We focus on the difficulties concerning the identification of rational and irrational numbers, the importance of the decimal and fraction representation in this process, as well as the real numbers density. The data reported was collected by questionnaires administered to first year students who studied mathematics and had mathematics as a major subject in school. Data analysis provided us a classification of the students into four groups. We make an attempt to characterise each group's common answers as certain thinking strategies (mental structures). Some groups consider rational only those that have a finite number of (non zero) decimal digits in their decimal representation.

- Many students in one group did not consider irrational numbers as real numbers. Problems across all of the groups were found concerning the dense structure of the real numbers. This paper concludes that several difficulties seem to remain after school graduation, while the students have developed interesting thinking strategies.

Questions for Discussion

- Compare the following pairs of numbers a) 2.999… and 3, b) 1.888… and 1.9.
- Decide whether the following numbers are rational or irrational: 3.146 and √2.
- Which of the two above questions would you think university students found easiest?

12. The role of spatial configurations in early numeracy problems

Fenna van Nes & Jan de Lange

- The mathematics Education and Neurosciences (MENS) project is a unique initiative to integrate research from the field of mathematics education with research from the fields of cognitive psychology and neuroscience in order to come to a better understanding of how the early talents of young children (aged four to six years) can best be cultivated for supporting the development of mathematical abilities in formal schooling.
- The present paper investigates relationships between the development of early spatial thinking skills and emerging number sense. We examined how a child is able to distil a structure form a spatial arrangement of objects (i.e. domino dot configurations, finger counting images) and how the structure is applied to determine and compare quantities.
- Fifteen four-year olds, fifteen five-year olds, and fifteen six-year olds were interviewed as they performed a series of number sense and spatial tasks. In the preliminary qualitative analyses, an association was found between a child's ability to apply spatial structures to numerical tasks and their level of mathematical achievement. This association has triggered questions for subsequent research into the role of structure in the development of spatial and numerical thinking.

Questions for Discussion

- What roles does structure play in young children's everyday activities (building with blocks, learning to count, comparing quantities)?
- How do these roles compare and how may they be related to each other?

13. Students' ability in solving line symmetry tasks

Xenia Xistouri

- The aim of this study is to propose and evaluate a model of 4th, 5th and 6th grade students' structure of knowledge in line symmetry. The model used is the Taxonomy of Structure of the Observed Learning Outcome (SOLO). The model describes the structure of students' aptitude to respond correctly to tasks of line symmetry, and thus it can be used by teachers to enhance students' learning.

Questions for Discussion

- Can a hierarchy model for the structure of a concept be valid in every regular concept.
- How useful can a model for the concept structure be for the classroom teacher?
- Is there a more appropriate methodology for studying structures?

CONSTRUCTING MULTIPLICATION: DIFFERENT STRATEGIES USED BY PUPILS

Joana Brocardo **Lurdes Serrazina** **Isabel Rocha**

ESE de Setúbal *ESE Lisboa* *ESE Leiria*

Abstract

The Project "Number sense development: curricular demands and perspectives" aims to study the development of number sense in elementary school (5 to 12 years old). This paper presents a discussion based on one of the six case studies developed by the project. We will focuses on the strategies used by 7-years old pupils when solving multiplication problems, namely on the awareness of existence of different strategies and the inclination to utilize an efficient representation or method.

Introduction

Number sense has been considered one of the most important components of elementary mathematics curriculum. The development of personal strategies of calculation and its implications to solve problems in real situations are recommended by both international literature (Fuson, 2003) and Portuguese curricular documents.

The term number sense has been used by several researchers to mean a group of numerical competencies that, nowadays, are considered very important to develop with students. For us, the meaning of this term, adopted by McIntsoh, Reys and Reys (1992) includes all the main points. They consider that number sense comprehends:

1. Knowledge and facility with numbers, witch includes multiple representations of numbers, recognizing the relative and absolute magnitudes of numbers, composing and decomposing numbers and selecting and using benchmarks.
2. Knowledge and facility with operations, witch includes the understanding of the effects of operations on numbers, the understanding and the use of the operations properties and their relationships.
3. Applying knowledge of and facility with numbers and operations to computational settings, witch includes the understanding to make connections between the context of a situation and the computation procedures, requiring knowledge of multiple computational strategies.

Most countries have emphasized during the last twenty years, the development of number sense together with the development of strategies and computation procedures and their flexible applications both in real practice contexts and in new ones. In NCTM Standards, understanding number and operations, developing number sense and gaining fluency in arithmetic computation form the core of mathematics education for the elementary grades (NCTM, 2000). The Project *National Numeracy Strategy* also points out these ideas when states that England changed the way in which mathematics is taught in many schools, with a new emphasis on mental mathematics, with new teaching approaches to help children develop a repertoire of computational skills involving work on mental calculations and strategies. (Askew and Ebbutt, 2000).

Instead, Portuguese elementary school tradition emphasizes arithmetic algorithms and that way of facing number sense is far away from it. Some studies reflected on the effect of teaching written algorithms on the development of children's mental strategies and number sense (Clarke, 2004). Encourage students to use only one method to solve problems limits their capacity to use flexible and creative thinking. In counterpoint, Clarke shows the benefits of developing concepts and strategies for mental computation prior to more formal written computation.

A closely relation between research and practice is seen as an important element to help changing teacher's practice. It is very important to work with teachers, to develop new materials with them, to refine these materials according to instruction experiments results in a process that leads to construct a conjectured local instruction theory (Gravemeijer, 1997).

The project *Number sense development: curricular demands and perspectives*

The project *Number sense development: curricular demands and perspectives* aims to study the development of number sense in elementary school (5-12 years old). The team Project is composed by classroom teachers and researchers who work collaboratively in all phases of research. We developed related sequence of 3 or 4 tasks – *task chain* - and implemented them in a particular classroom, covering different grades from kindergarten (5 years old) to 5th grade (11 years). Each task chain was developed as an hypothetical learning trajectory in the sense used by Simon (1995).

Therefore, one of the components of the project *Number sense development: curricular demands and perspectives,* is to develop chains of learning activities that supports the development of number sense. Those chains has been thinking with the support of Simon`s model. For Simon, the teacher in order to plan his teaching, has to make decisions about contents and about the learning tasks. So, in this context, it is introduced by this author, the concept of *hypothetical learning trajectory*, a cycle of learning assigned with some tasks constructed by the teacher, attending to the mathematical ideas and processes that he intends to develop in the students.

The learning trajectory is hypothetical because is thought as experimental and because is not possible to be sure that it will be a real and efficient way of learning. Of course, it is possible to make previsions because the teacher may anticipate the approaches, the discussions and solutions that may be stimulated by the potential of the tasks.

It was in this context, that the task chain experimented in the case study presented in this paper, was developed and implemented.

In addition the project has three other objectives:

- To understand the main difficulties faced by pupils when they develop number sense;
- To study the curricular integration of these activities and the demands they pose to teachers;
- To identify professional practices that facilitates number sense development.

We developed six qualitative case studies. Each case study analyzed the implementation in a particular classroom of a related sequence of 3 or 4 tasks – *task chain* - and covered different grades from kindergarten (5 years) to 5th grade (11 years).

Methodology

We will present a discussion based on the analysis of one of the case studies developed by the project and implemented in a second grade classroom (20 seven years old children). These pupils had already worked addition and subtraction with numbers up to 100.

The focus of the *task chain* experimented in this case, is a learning trajectory to develop the multiplication concept. It is composed by four tasks that were presented in the last semester of school year. This work was developed along four weeks, one task per week, explored every Monday morning.

Each Monday morning the classroom teacher organized a brief oral introduction to each task. He mainly tried to capture pupils' attention to the context presented in each task. He also proposed a given period of time to explore the task in small groups. After that he organized a whole group discussion where all groups had the opportunity to share different approaches to the problems posed.

This *task chain* focus the construction of a learning trajectory, which starts from the knowledge related to additive computation to develop the multiplication concept. More precisely it deals with relation between some table products and the understanding of specific properties of multiplication. This learning trajectory was also foreseen to introduce the double number line model and to enhance the concept of multiplication relating it with the rectangular model.

Classes where tasks were developed were videotaped by one of the members of the research team, who also took some written field notes from her class observations. After that all videotapes were watched and more meaningful episodes transcribed and completed with field notes.

Data analysis was made task by task, first a description how task was presented to pupils, following how it was explored and discussed in classroom. It also includes the final discussion when it was not embedded with the exploration. It ends with a synthesis of each task centred in the processes used by pupils according to categories about number sense, adapted from Mcintosh, Reys and Reys (1992).

After the analysis of each task of the chain, the hypothetical learning trajectory and the learning goals for each pupil, were confronted with the process followed by each one in order to try to establish a learning trajectory followed by these classroom pupils.

In this paper we focuses our analysis in the category - *Applying knowledge of and facility with numbers and operations to computational settings* - which includes: (1) understanding the relationship between problem context and the necessary computation, (2) awareness that multiple strategies exist and (3) inclination to utilize an efficient representation and/or method.

Understanding the relationship between problem context and the necessary computation

One of the conclusions of a previous study carried out by Brocardo, Serrazina, Kraemer (2003) shows a strong tendency of Portuguese pupils to a mechanical use of algorithms: they read the question and they ask themselves which computation should they do. They do not analyse the meaning of question and the answer they propose.

In this case was not observed this kind of procedure. Pupils tended to analyse the context and to relate it with the computation they performed. This tendency seems to be related with the teacher's concern to enhance pupils' interpretation of the proposed problem, as he did when he proposed the first task:

> Teacher: And now you can read what is presented in the task and, first of all we are going to interpret it, ok? Is there any word you can not understand?

At the beginning , although pupils had a concern to analyse the context, many times that analysis tended to be more important than the mathematical work related with the problem that was been discussed. For instance, in the first task, the discussion of the meaning of the word aperitif drove pupils to talk about meals (main course, desert, aperitif) and about the sort of aperitifs that they had already tasted.

In the following tasks this aspect was less relevant. The pupils did not take so much time to talk about task lateral aspects. However, they always made some comments to the task context. We will show in the next section the strategies they used.

Awareness that multiple strategies exist

Our data show a progressive awareness of different strategies. Pupils strategies repertoire was enriched along the work developed with this task chain.

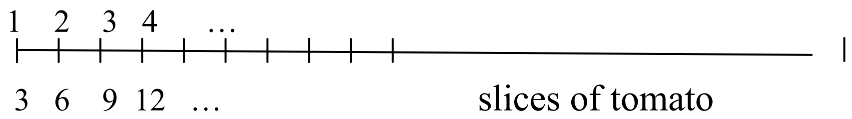
In the first task pupils had to calculate the number of sausages' slices. Most of them used additive strategies:

> 1 sausage - 2 slices
>
> 2 sausages - 4 slices
>
> 3 sausages - 6 slices

The additive strategy was also used in a more sophisticated way: two groups related the questions like this:

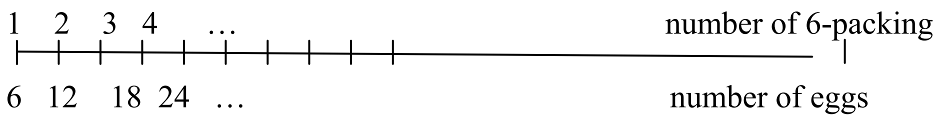
> If I need 24 slices of cheese then I have to add 24 to know the number of sausages.
>
> To know the number of slices of tomatoes I have to add 24 to the number of sausages.

Some pupils had difficulties in organizing their thought. So, the teacher decided to introduce the double number line:

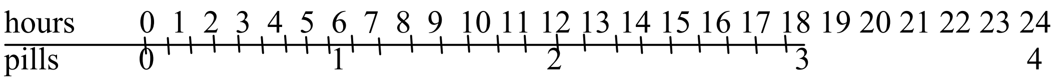


This strategy was easily understood by the pupils that begun to use it to solve some of the questions posed in the next tasks.

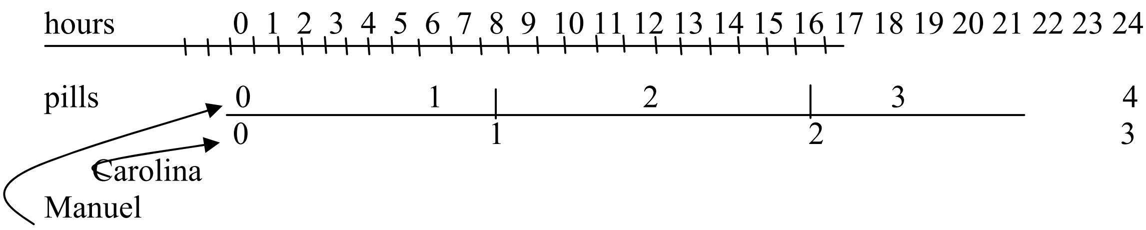
In the second task most of the pupils used this strategy, for instance, to find the number of boxes they need to pack 100 eggs:



This strategy was also used to support the explanations that teacher had to provide to some of the pupils. For instance, in the 3rd task, as some of the pupils had difficulties to express how they could explain that Manuel took 4 pills in one day, the teacher used the representation:



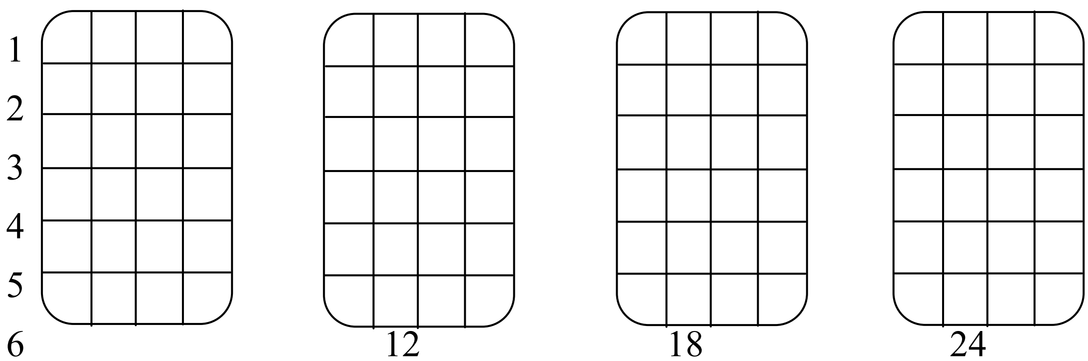
The double number line was also the support that teacher used to compare the number of pills that Carolina and Manuel took in one day.



The concept of multiplication and its relation with the rectangular model was progressively developed. In the last task, when they were asked to propose different possibilities to pack 30 Chiclets, most pupils used the concept of multiplication relating it with the rectangular model and begun to present several solutions:

- 1x30
- 2x 15
- 3x 10
- 5x 6

The use of certain strategies seemed to be related with the context. For instance, the drawing of the blisters supported the strategies used by Francisco and his peer:



Francisco: I already knew that Manuel took 4 pills per day and Carolina took 3 and then I did 4 pill boxes …. 4 blisters and then I did.

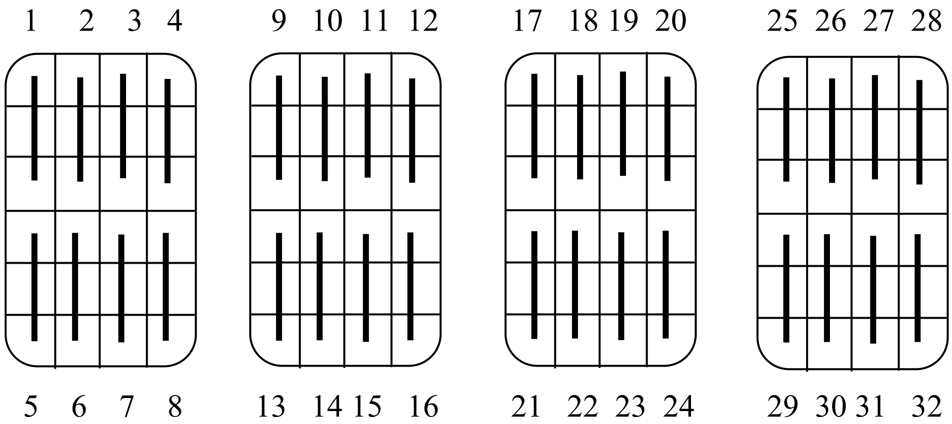
Teacher: In one row.

Francisco: In one row… aah… one day, and the other row it was the other day … (…)

Francisco: Because the rows have 4 pills and it was 24, he needed 24 days to take all the pills.

(…)

Francisco: And then I did the same thing …but Carolina took 3 pills per day and I did this:



Inclination to utilize an efficient representation and/or method

Besides the progressive awareness of different strategies, an aspect which was emphasized by teacher was the use of more efficient strategies.

Data shows that this is a more difficult level for most of the pupils. Many of them can persist to use a long process – like jumping one by one in the double number line.

During the two first tasks only one group seemed to be interested in thinking in the "most efficient process". However the reflection proposed by the teacher when whole class analyses the different strategies used by the small groups, seemed to be an important way of facilitating the inclination to use a more sophisticated strategy.

An example of this was the discussion that Tiago and Rodrigo strategy facilitated:

To calculate the number of days that Manuel and Carolina needed to take all pills they wrote on the blackboard this:

Manuel	Carolina
1 day 24 h	1 day 24 h
6 days one blister	8 days one blister

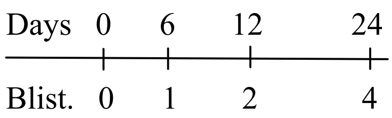
12 days two blisters 16 days two blisters

24 days four blisters 32 days four blisters

The teacher helping whole class to understand what Tiago and Rodrigo presented commented:

> Not to jump one by one …you may do large jumps …for 6 days how many pills? … 24 …24 with 24 is … right! Can you see? Now you have to do another jump. 6 days more is ? … So, more 24 … do bigger jumps.

and used this representation.



Concluding remarks

We tried to illustrate that along the exploration of these 4 tasks, pupils develop a clear and powerful understanding of multiplication. They begun, in first task, to use additive strategies that corresponded to an activity in the task setting: interpretation and solutions depend on understanding of how to act in the setting (Gravemeijer, 2005). During discussion the teacher facilitated the change of ideas and strategies among pupils. He also had the initiative to introduce different approaches that facilitated the development of more powerful strategies.

In the second task most of the pupils already used models of the explored situation as an approach to the task.

In the fourth task pupils clearly used different models and they could relate multiplication with the rectangular model.

According to Dolk and Fosnot (2001), counting one by one is not multiply. The development of new strategies as doubling and the use and understanding of properties of multiplication facilitate the growing capacity of children thinking in terms of number relations and enhance the number sense. In this sense we can say that these pupils developed number sense: they could use different models; they could begin to use the ones that were more "powerful"; and they could relate multiplication with rectangular model. However not all students were able to reach the same flexibility and level of understanding. For some of them successive addition and jumping by ones continued to be strategies they preferred (or were able) to use. The context seems to have an important role in this process.

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STUDENTS' THINKING ABOUT FUNDAMENTAL REAL NUMBERS PROPERTIES

Eustathios Giannakoulias, Alkeos Souyoul, Theodossios Zachariades

Department of Mathematics, University of Athens

This paper presents a part of a research concerning school graduate students' difficulties in their understanding of real numbers and fundamental calculus concepts. Particularly, we focus on the difficulties concerning the identification of rational and irrational numbers, on the importance of the decimal and fraction representation in this process, as well as on the real numbers density. Based on these difficulties, the data analysis suggested a classification of the students into four groups with certain characteristics. Several difficulties seem to persist even after school graduation, while the students have developed interesting thinking strategies.

INTRODUCTION

There are numerus pieces of research that confirm several cognition problems about the real numbers (Zazkis and Sirotic 2004; Moseley, 2005). Particularly, many studies show that students face difficulties in identifying rational and irrational numbers. The distinction between the different categories of numbers remains fuzzy and strongly dependent on their semiotic representations (O' Connor, 2001; Munyazikwiye, 1995). The order and density of real numbers also cause cognitive problems (Merenluoto and Lehtinen 2006; Vamvakoussi and Vosniadou 2006). Most of the studies about the above mentioned difficulties concern elementary or junior high school students (ages 6-15). In this paper we examine school graduate students' comprehension of the structure and the representations of real numbers. Using a methodology, which is based on statistics, we have divided the whole set of students into four groups and we compare the structure of understanding among the groups. One of the main aims of the paper is to compare different levels-degrees of understandings of real numbers.

THEORETICAL BACKGROUND

Real numbers appear in school mathematics education through a process of enrichment of the set of natural numbers. The set of natural numbers expands to the set of the integers in order to include negative numbers. The integers extend to the set of rational numbers so as to provide ratios of integers. Finally irrational numbers join the set of rational numbers and construct the set of real numbers. Each of the sets mentioned above appears normally in a context of a necessary expansion of each set in order to solve problems that the subset cannot interfere with. Every bigger set preserves some of the properties of the subset (but not all) and it has its own new properties. Research on numbers education shows that the process described above hides a number of problems in students' understanding. It has been shown that students' knowledge of real numbers is often highly compartmentalized, and not linked to their broader mathematical knowledge (Moseley, 2005).

Some cognitive problems concerning the number concept arise from the fact that in school mathematics numbers are not (and cannot be) defined in a formal way. Pupils in school instead of a definition for real numbers, have some concept image, in the sense of Tall and Vinner (1981), acting as a definition. An essential component of the fundamental change from elementary to advanced mathematical thinking is described schematically in the following:

Concept image → Definition

Definition → Concept image

Sometimes the formal definition of a concept comes in a later step of the didactical process, after the students have already been familiar with the concept in an intuitive/informal context. In this case the concept image determines the formal definition. On the other hand, in formal mathematics the definition is used to prove the properties of the mathematical concept which it defines. In this case the definition determines the concept. This reversal is an epistemological obstacle which can cause great difficulty (Pinto, Tall 1996).

The number concept image in school mathematics involves multiple representations for numbers such as, points on what is called the "real line", decimals, fractions and some other numbers –the irrationals– that cannot be expressed as fractions. Problems involving the ability to move between different representations of the real numbers are discussed in Zazkis, Sirotic (2004) and Pinto, Tall (1996). In a conceptual change framework (Vosniadou, 1994; Vosniadou and Verschaffel 2004) students form synthetic models when they face problems of rational numbers. The natural numbers' discrete structure usually acts as a barrier when students have to cope with the rational numbers dense structure (Vamvakoussi and Vosniadou, 2004). The counter-intuitive nature of incommensurability and density, seems to cause some of the problems in real numbers' understanding (Fischbein et al. 1995). Incommensurability and density in the real numbers set are considered to have poor intuitive representations. This results to a counter-intuitive nature for the irrational numbers.

Understanding of the real numbers structure is a presupposed knowledge for university mathematics. Students should be familiar with the real numbers in order to face the fundamental calculus concepts. Most of the studies in mathematics education, about numbers understanding, concern primary or junior high school students (6-15 years old). The study presented in this paper focuses on two research questions:

- Do some of the above mentioned problems rersist after school graduation? In particular, we focus on the distinction between the elements of the basic subsets of the real numbers, on the role that the fraction and decimal representation plays in this distinction, as well as on the dense structure of the real numbers set.

- Do school graduates have certain thinking structures about the real numbers and in what extend do these appear?

METHODOLOGY

Data reported in this paper were collected by questionnaires administered to 215 first year students who studied mathematics and had mathematics as a major subject in school. The tests were administered during the students' Calculus course early in their first semester. They had not yet been taught in a university level, the structure of real numbers. So, it is assumed that they answered the questionnaire using their knowledge from school. The questionnaire is part of a larger diagnostic test that we have devised, in order to identify problems that first year mathematics students face in the fundamental calculus concepts.

Students' fully correct responses were marked with 1 and the incorrect responses with 0. The quantitative data analysis was made with the use of latent class analysis (LCA) with categorical variables (Barholomew et al. 2002, Kline, 1998). This analysis, which is part of mixture growth analysis, is a statistical method for finding subtypes of related cases (latent classes) from multivariate data. The results of LCA were used to classify cases to their most likely latent class. That is, given a sample of subsets measured on several variables, one wishes to know if there is a small number of basic groups into which cases fall. The statistical software used for the analysis was Muthen & Muthen Mplus, which is appropriate for discrete variables. More information on the statistical method used can be found in Bartholomew et al. (2002), Muthén (2001), and Muthén & Muthén (2006).

QUESTIONNAIRE

The questionnaire was divided into four parts. The questions are displayed providing also the percentage of correct answers in the parentheses. The first part consisted of four questions asking the students to distinguish the basic subsets of real numbers.

A1. Write a natural number (99.1%)

A2. Write an integer number that is not natural. (96.3%)

A3. Write a rational number that is not integer.(97.2%)

A4. Write an irrational number. (92.6%)

Questions in the second part are related to the order and the density of the real numbers.

Compare the following pairs of numbers.

B1.1	0.999…	0.999	(80,9%)
B1.2	1.888…	1.9	(95,8%)
B1.3	2.999…	3	(10,7%)

In each of the following pairs of numbers write a number lying between them (if such number exists). If there is no such number, write "there is not".

B2.1	0.1	0.11	(71,2%)
B2.2	1.888…	1.9	(58,1%)
B2.3	2.999…	3	(90,2%)
B2.4	1/3	2/3	(86,5%)

B3. Is there any rational number q being greater than 3/5, having the property: 'there is no number between q and 3/5'? If there is such a number, write it. If it does not exist, write 'there is no such number'. (65,5%)

B4. Can you find two real numbers such that there is no other number between them? If you can find a couple of numbers with this property, write the numbers. If you believe that there is not such a couple, write 'there are no such numbers'. (59,5%)

In the third part students have to characterize the following statements as 'true or false'. In this part of the questionnaire the students have to identify five different numbers as real, rational, irrational. For the numbers √2 and 2/3 which are not given in decimal representation the students have to answer three more questions about their decimal representation.

C1.1	√2 is a real number.	(86%)
C1.2	√2 is a rational number.	(95,8%)
C1.3	√2 is an irrational number.	(95,3%)
C1.4	√2 has a decimal representation with infinite decimal digits.	(78,1%)
C1.5	√2 has a decimal representation with finite decimal digits.	(80,9%)
C1.6	√2 does not have a decimal representation.	(84,7%)
C2.1	3.46 is a real number	(96,7%)
C2.2	3,46 is a rational number..	(85,1%)
C2.3	3,46 is an irrational number.	(85,1%)
C3.1	0.78634… is a real number.	(90,2%)
C3.2	0.78634… is a rational number.	(79,1%)
C3.3	0.78634… is an irrational number.	(78,1%)
C4.1	0.777… is a real number.	(89,8%)
C4.2	0.777… is a rational number.	(30,2%)
C4.3	0.777… is an irrational number.	(32,6%)
C5.1	2/3 is a real number.	(95,3%)
C5.2	2/3 is a rational number.	(75,8%)

C5.3 2/3 is an irrational number. (74,9%)

C5.4 2/3 has a decimal representation with infinite decimal digits. (79,5%)

C5.5 2/3 has a decimal representation with finite decimal digits. (76,3%)

C5.6 2/3 does not have a decimal representation. (93,5%)

The fourth part has two general questions about the decimal representation.

D1 Every real number has a decimal representation.(63,7%)

D2 Every number having a decimal representation is real. (67,9%)

RESULTS

In the modelling process we used a method of successive steps. That is, we tested the model under the assumption that there are two (BIC 6406.705), three (BIC: 6385.987), four (BIC: 6355.086) and five (BIC: 6477.895) groups of subjects. The best fitting model with the smallest BIC was the one involving four groups. The clarity of the classification was indicated by the Entropy summary measure which had its maximum value for the models tested. The average latent class probabilities for the groups are 0.964, 0.985, 0.992 and 0.967 respectively, which enable us to conclude that the four classes are quite distinct, thus indicating that each class has its own characteristics.

We should note that there are many questions in which students have a high percentage of success while there are some questions that ask similar things (for example whether a certain number is rational or irrational). This results to extremely highly correlated variables. In general, the use of such variables should be avoided in LCA as they can result to more classes having no real meaning rather than explain these high correlations. In the present analysis this did not happen and we have not excluded these questions for the following reasons: We do not want to lose some valuable information. Some students think that the rational and irrational numbers are not distinct sets, some other that a real number can be neither rational nor irrational, or finally that a number can have no decimal representation. Furthermore, in our study, LCA was used as an exploration tool and it provided us with a very interesting categorisation which enabled us to focus on certain similarities. We then went back to the individual questionnaires; we examined the similarities provided by the analysis more closely and confirmed their existence. Such a categorisation was not achieved by omitting some of the questions.

Table 1 displays a summary of the results for the class analysis. Each of the columns represents a group and each of the rows represents a question. The number of each cell displays the probability that a student in a particular group answers correctly in the corresponding question. We have to note that this is not equal to the fraction of the students in the group that have answered correctly the corresponding question although we will treat them as such. This happens because the model class for the latent class patterns is based on estimated posterior probabilities and does not result to integer counts for the groups. For example, based on the most likely latent class membership, the first group consists of 63 students, while the model class count for this group is 62.269. Low performance (0-50%) is displayed in grey, average performance (50%-75%) in normal and high performance (75%-100%) in bold.

We first present a description of the main characteristics of each group. The first group consists of 63 students and has the highest performance among the groups. They answer correctly in most of the questions of the first part, something that does not happen for the second part where a problem with questions B3 and B4, related to the density, is observed. Question B1.3 is the most difficult question in the whole questionnaire. In this question the first group has significantly higher score than the other groups, but it still remains very low. Most of the students in the first group identify as real all the numbers of the third part (C1.1, C2.1, C3.1, C4.1 and C5.1). They recognise that whenever a number can be turned into fraction of integers it is a rational number (C5.2 and C5.3). They also know that whenever a number is given in decimal representation it can be turned into fraction if the decimal part has a recurring pattern in its decimal part (C2.2, C2.3, C3.2, C3.3, C4.2 and C4.3). The lowest performance is observed in questions C3.2 and C3.3 where 9 of the 20 students who answered this question wrong, gave the response that they do not know whether 0.78634… is rational or irrational as there is not enough information provided in order to decide. The didactical contract suggests that if there was some recurring pattern, it should have shown before

	group	group	group	group
A1	**1**	**0,986**	**1**	**0,972**
A2	**1**	**1**	**0,978**	**0,805**
A3	**1**	**1**	**0,955**	**0,889**
A4	**0,966**	**0,976**	**0,912**	**0,771**
B1.	**0,781**	**0,916**	0,733	0,74
B1.	**0,9**	**0,976**	**0,978**	**1**
B1.	0,244	0,067	0,067	0
B2.	**0,807**	0,738	0,671	0,544
B2.	0,603	0,638	0,551	0,47
B2.	**0,889**	**0,918**	**0,979**	**0,798**
B2.	**0,878**	**0,898**	**0,841**	**0,808**
B3	0,694	0,697	0,65	0,515
B4	0,7	0,59	0,56	0,47
C1.	**0,935**	**0,961**	**0,801**	0,604
C1.	**1**	**1**	**1**	0,749
C1.	**1**	**1**	**1**	0,722
C1.	**0,827**	**0,825**	**0,934**	0,426
C1.	**0,914**	**0,852**	**0,868**	0,47
C1.	**0,796**	**0,916**	**0,893**	0,735
C2.	**1**	**1**	**0,935**	**0,887**
C2.	**0,903**	**0,878**	**0,87**	0,684
C2.	**0,903**	**0,878**	**0,848**	0,712
C3.	**0,983**	**1**	**0,846**	0,637
C3.	0,716	**1**	**0,974**	0,272
C3.	0,684	**1**	**0,974**	0,272
C4.	**0,983**	**1**	**0,846**	0,609
C4.	**0,848**	0,019	0	0,301
C4.	**0,91**	0,007	0,023	0,328
C5.	**1**	**1**	**0,89**	**0,859**
C5.	**0,984**	**1**	0	**0,83**
C5.	**0,984**	**1**	0	**0,774**
C5.	**0,871**	0,707	**0,978**	0,613
C5.	**0,84**	0,679	**0,916**	0,605
C5.	**0,933**	**0,93**	**0,956**	**0,923**
D1	0,615	0,574	**0,8**	0,599
D2	**0,85**	**0,781**	0,516	0,383

Table 1

the dots "…". They however support that there could be a recursion in the digits that are not shown on the paper. Questions of the fourth part were posed in a theoretical way and in general they seem to confuse many of the students, even those of the first group. There is a subgroup of about 40% of the students in this group who support that real numbers which do not have a decimal representation exist.

The second group consists of 72 students. In the first two parts they have a performance similar to the first group. They too have difficulties with the questions regarding density. Students of the second group also identify as real the numbers of the third part (C1.1, C2.1, C3.1, C4.1 and C5.1). However they have a problem in rational-irrational identification. Although they recognise as rational the numbers that can be turned into fraction (C5.2 and C5.3), they additionally use another incorrect criterion when they have to decide whether a given number is rational or irrational from its decimal representation. For the students of this group a number is rational if it has a finite number of decimal digits different from 0. So, $\sqrt{2}$, 0.78634… and 0.777… are all irrational numbers as they have infinite decimal digits (C1.2, C1.3, C1.4, C1.5, C2.2, C2.3, C3.2, C3.3, C4.2, and C4.3). An interesting point is that about 30% of these students do not even make the division 2/3 in order to answer questions C5.4 and C5.5. They know that 2/3 is rational so they conclude that it has finite decimal digits. The rest of them consider 2/3 as rational although they can see that it has infinite decimal digits. This means that **the fraction criterion is stronger than the infinite digit criterion**. As we will see, this is the major difference from the third group. The performance of Group 2 in the fourth part is slightly poorer that that of group 1, where there were also many students who supported that real numbers which do not have a decimal representation exist.

The third group counts 45 students. In part A students do not have any particular problem. In part B, students answer in the same way with the students of the second group, facing difficulties with the density questions. Most of the students in this group identified the third part numbers as real (C1.1, C2.1, C3.1, C4.1 and C5.1). They identified as rational, numbers those that have finite decimal digits different than 0 (C2.2 and C2.3). They also identified as irrational numbers, those that have infinite decimal digits different than 0, without checking recurrence (C1.2, C1.3, C1.4, C1.5, C3.2, C3.3, C4.2 and C4.3). Finally they concluded that 2/3 equals 0.666… (C5.4, C5.5 and C5.6) and therefore 2/3 is an irrational number (C5.2 and C5.3). Students in this group believe that rational numbers can be written with finite decimal digits, while the decimal representation obscures the fraction representation. The difference from group 2 lies in the fact that in the third group **the digit criterion dominates the fraction criterion**. Students in the third group have the highest score in D1 and this is fully compatible with their answers in questions of the third part, as decimal representation plays a major role in their concept image for real numbers. On the contrary, half of these students believe that there are numbers having decimal representation which are not real numbers. (D2).

The fourth group has 35 students who in general have lower performance than the students in the other groups. This can be seen even in the easiest questions contained in part A. This group also has lower performance in most of the questions of part B. A high percent of the students of the fourth group have a different view of real numbers. They identify as real only the numbers which have a finite number of decimal digits different than 0 (C2.1, C5.1, C5.4 and C5.5). Numbers which have infinite decimal part are identified as not real (C1.1, C3.1 and C4.1). We can also remark that they identify the set of rational numbers with the set of real numbers (questions 2 and 3 in C1, C2, C3, C4 and C5). This is compatible to the group's low percentage of success in question A4. Students' answers in question C5 also show that 2/3 is considered rational with finite decimal digits. This resembles the second group's view, as the fraction criterion also prevails over the digit criterion. The fourth group has low scores in the fourth part. Their percentage of success in D2 is very low and this agrees with their answers in the third part of the questionnaire, as they do not identify irrational numbers as real (although they have a decimal representation). Students in the fourth group are also uncertain about whether real numbers having no decimal representation exist (D1).

DISCUSSION

The identification of rational-irrational numbers appears to be difficult for the majority of the study subjects. Only students of the first group check for recurring digits when they are asked to identify a number in decimal representation. Although all of the students have been taught a general way to convert a rational from decimal to fraction representation, this knowledge (if such exists) remains unconnected to the real numbers structure.

There is also an interesting remark for the whole sample. A 45% of all the students fall in a contradiction by giving the following combination of answers:

- 2.999… is less than 3 (an expected answer for question B1.3). So 2.999… is different than 3.
- In question B2.3 they answer correctly that there is no number between 2.999… and 3.
- Finally, in question B4 they also answer correctly that there is no pair of numbers having no number between them, instead of giving 2.999… and 3.

This contradiction is regardless of the students' distribution into each of the groups. The percent of students falling into contradiction for each group are 44.4%, 47.2%, 45.1% and 34.3% respectively.

Another interesting point comes from the identification criteria priority. Students in groups 2 and 3 use the finite digits criterion in order to identify rational numbers. In group 2 the fraction criterion dominates over the digits criterion while in group 3 the fraction representation is not adequate to guide the identification. A high percentage of the students in group 4 consider as real numbers only those with finite decimal digits. This leads them to exclude irrational numbers from the real numbers structure. Throughout school mathematics, there is no formal definition for the real numbers set. Taking into account Greek school textbooks, school graduates' concept image for the real numbers is expected to be the set of numbers in decimal representation. Questions D1 and D2 show that there is a general difficulty with decimal representation of real numbers. Only 39% of the sample has answered correctly in both D1 and D2. Question B3 is closely connected to the density of the rational numbers. Question B4 tests students' knowledge on real numbers density. A 42.3% of the students have answered to both of these questions correctly. Density is highly related to the understanding of the decimal representation's structure. However, only 18% of the students have answered correctly to all of the questions B3, B4, D1 and D2. The 48.7% of these students is in the first group, 30.8% in the second, 15.4% in the third and 5.1% in the fourth. This shows that students from the first group have a better understanding of the real numbers structure.

Concluding, we argue that several problems remain in graduate students concerning the rational-irrational numbers identification as well as the real numbers density. The importance of the decimal or fraction representation in the process of identification provided a classification of students into groups with several unique characteristics. On the other hand some other problems on real numbers structure appear unrelated to this classification.

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SCHEMA A ± B = C
AS THE BASIS OF ARITHMETIC STRUCTURE

Milan Hejný, Jana Slezáková

Charles University in Prague, Faculty of Education

Abstract: *An image of a triad of numbers, one of which equals the sum total of the two remaining ones, is regarded as the basic schema of arithmetic thinking. This schema is studied through several semantic and structural models. More thorough examination is given to key semantic models that only consist of operators of both types – operator of change and operator of comparison.*

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Key words: schema, semantics vs. structure, number as a status, address, operator of change, operator of comparison, additive triplet, concept-process-procept, language of arrows.

INTRODUCTION

The paper presents some results of the ongoing research done by the above authors and D. Jirotková. The research is aimed at creating educationally effective environments through which a child can penetrate into the realm of arithmetic. It is an attempt to offer teachers an alternative to the traditional educational strategy based on routine drill and memorising on the part of the pupils.

The theoretical frame of the research is based on procept theory (Gray & Tall, 1994) and the theory of generic model (Hejný & Kratochvílová, 2005).

The key concept of the research is *Additive Triplet Schema*. The main objective of this article is to describe and analyze the above-mentioned mental construct.

SCHEMA

When someone asks you about the number of doors or carpets in your flat or house, you will probably not be able to give an immediate answer. However, in a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects in question. Both of the required pieces of information and many other data about your dwelling are embedded in your consciousness, as a part of the schema of your flat. We use schemas to recognize not only our dwellings, but also our village or town, our relatives, interpersonal relationships at our workplace, etc.

Specialized literature gives various connotations of the term "schemata". The following quote by R. J. Gerrig provides a rather loose definition that serves our purposes. "Theorists have coined the term *schemata* to refer to the memory structures that incorporate clusters of information relevant to comprehension … . A primary insight to schema theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units." (Gerrig, 1991, pp. 244-245)

ADDITIVE TRIPLET

By the term "additive triplet" we mean an orderly triplet of numbers (A, B, C – not necessarily in that order), one of which equals the sum total of the two remaining ones. The relation can be written as follows: A±B=C. We focus primarily on preschool children and early learners, which is why the term "number" shall herewith stand for a non-negative integer, smaller than 100. Negative numbers and fractions will enter the picture at a later stage. The term "additive triplet" was inspired by the idea of an additive family, as introduced by Repáš and Černek's textbooks (Repáš et al, 1997).

The standard model of an additive triplet is expressed in Arabic numerals (e.g., 2, 3, 5). The child will then become acquainted with the additive triplet via many other representations, and the aggregate set of all these representations helps create the additive triplet schema within the child's consciousness.

Note. In this study, we only explore the *additive triplet schema*, we shall abbreviate this term to *schema*.

FRAME OF ANALYSIS

Schemata are explored in a multiple of layers. We attribute the most significance to the layer that classifies schemata into two main categories: the *semantic* schemata – which are rooted in the everyday experience of the pupil, and *structural* schemata – which are not rooted in such a way, being only confined to the realm of mathematical terms, signs and ideas. The boundary between these two categories is blurry.

The second layer focuses on the *process – concept* polarity, in terms of the procept theory formulated by E. Gray and D. Tall (1994). With every schema, we try to establish the degree to which it contains concepts or processes, sometimes even trying to recognise the shape of the particular procept.

The third layer is devoted to *language*. We examine the way in which numbers and operations are represented. Numbers are represented by sets of objects (apples, fingers, matches, …), sets of pictures/images, dots, commas, or arrows. They can also be represented by ephemeral vehicles, such as words, handclaps, steps or bell chimes. And, last but not least, they can also be expressed by Arabic numerals (we are not considering Roman numerals or other systems for the moment). The key pupil activity is *translation* – the process of transforming the situation from one language to another.

Other layers like problem-solving strategies or adjustability of the level of difficulty of the problem are not considered in this article.

THE LAYER OF SEMANTIC MODELS

The typology of semantic models of additive triplets is based on the typology of semantic anchoring of numbers (Hejný & Stehlíková, 1999). We distinguish between four basic semantic types of numbers: *status* (number, magnitude), *address* (in terms of place or time; the temporal address can be either linear or cyclical), the *operator of change* and *the operator of comparison*. These are abbreviated as follows: status = St, address = Ad, operator of change = Ch, operator of comparison = Co. The symbol O will stand for the operator, when there is no need to specify its particular type.

The above-mentioned quadruplet of semantic anchors of numbers can be subdivided into eight basic semantic types of additive triplets:

1. St + St = St	2. St – St = St	3. St ± Co = St	4. Ad ± Co = Ad
5. St ± Ch = St	6. Ad ± Ch = Ad	7. $^{+}$Co ± Co = Co	8. $^{+}$Ch ± Ch = Ch.

Three of these types are illustrated as follows: .
4. (Ad ± Co = Ad) Jan is 8 years old. Rita is 1 year older/younger. Rita is 9/7 years old.
6. (Ad ± Ch = Ad) Cid used to live on the 5^{th} floor. He moved 2 floors up/down. Now he lives on $7^{th}/3^{rd}$ floor;
8. ($^{+}$Ch ± Ch = Ch) The number of bus-passengers increased by 7 persons at the first stop. At the second stop it increased/decreased by 5 persons. At these two stops the number increased by 12/2 bus-passengers.

Key semantic model. The key semantic model, whose mastery is the decisive step towards understanding the schema, can be written as $^{\pm}$O ± O = O. Many years of experience, substantiated by the experimental research of Ruppeldtová (2003), clearly indicate that problems that use only operators are among the most demanding problems for first- to fourth-grade pupils.

This is caused by the differing perceptions of the statuses and addresses on the one hand, and the operators on the other. The status and address are both *closed* data. Information such as “there are 5 chairs around the table” does not generate any further questions concerning numbers. The operator is, by contrast, an example of *open* data. The information “there are two chairs fewer” provokes the question as to what was the original number of chairs and how many chairs are there now. These two numbers are *virtually present* in the operator of change. The accuracy of the above thesis is confirmed by the behaviour of pupils who are assigned such operator problems. When given such a problem, they keep asking for virtual data and for explanations on to how to deal them. These pupils clearly have not had enough experience with numerical situations that feature solely the operator of change. That is why the current situation might be improved by incorporating operator problems already in first-grade primary school curricula. In order to achieve this goal, we elaborated a “walking” environment (see below) which we are currently testing in several classes.

THE LAYER OF STRUCTURAL MODELS

A triplet of numbers (A, B, C) may be inserted into a non-semantic context. The *carriers* of numbers can be words, body movements or a set of symbols that we choose to call *scaffolding*. By placing three numbers in their assigned spots on the scaffolding we have created a *concrete model.* If we place only two numbers in their appropriate spots, we have obtained a problem that can be formulated as “Find the third number”. The standard scaffolding is $\square + \square = \square$; the concrete model is $\boxed{2} + \boxed{3} = \boxed{5}$, and the problems are e.g. $2 + 3 = \square$ or $\square + 3 = 5$.

The standard model is generally known. We shall touch on five models which we find interesting in terms of both research and education. The first one will be *carried* by words and body movements, while the remaining four will have the graphic scaffold as their carrier. The second and the third model are also associated with movements.

1. **Walk**. The teacher (and later one of the pupils) gives a command and another pupil(s) walks according to it. Sample commands: 1. Three steps forward, go! 2. Two steps, then one step, forward, go! 3. Three steps forward, then two steps backwards, then one step forward, go! After this warm-up stage, the additive triplet is introduced by the following scene: Two pupils, A and B, are standing side by side. Pupil A receives the following command: Three steps forward, then two steps forward, go! Pupil B receives the command: Five steps forward, go! Both pupils, A and B, eventually end up standing side by side again. The entire scene is accompanied by words and body movements, and can be classified as a walk representation of the additive triplet (2,3,5). We shall henceforth write this performance as W(2,3,5). The problem originates by concealing one of the three numbers on offer. The given situation therefore leads to three problems: W(?,3,5), W(2,?,5), W(2,3,?). The concealed number has been replaced by a question mark. Backward steps are symbolized by negative numbers.

2. Footprints. By recording the walk in symbolic terms, we have created yet another model. A number of experiments have been performed with the sole aim of finding appropriate symbols for the 'footprint'. In the end, arrow symbols were chosen as the most appropriate for children of 6 to 8 years of age. One such model is illustrated by figure 1a. We shall henceforth write this model as F(2,3,5). A substantial difference between models W(2,3,5) and F(2,3,5) resides in the fact that the first one is ephemeral, while the second one is permanent. Words and steps will fade away, but Figure 1a will remain.

Both of the models bear the strongest resemblance to the semantic model $^{\pm}Ch \pm Ch = Ch$. In the course of our work within these models, we in fact employ negative numbers without having to use the minus sign or to explain anything. This aspect is quite different from other symbolic models.

By introducing the arrow-language to the environment 'walk' the environment of footprints is created. As soon as these two environments merge in pupil's mind his/her understanding of the walk + footprint model of triple reaches the procept level.

The wide range of variously oriented arrows hereby creates as many as eighteen types of footprints task. Three of these are illustrated by Figures 1b, 1c and 1d, which we write as F(3,-2,?), F(-1,?,3) and F(2,?,-6).

→→	→→→
→→→→→	

Figure 1a

→→→	←←

Figure 1b

←	
→→→	

Figure 1c

→→	
←←←←←←	

Figure 1d

3. Staircase. Similar to Walk model tasks in Staircase model are also carried by words and body movements. On the floor of a class there is a number line as a stage for staircase performances. Sample tasks: 1. Stay at 5; four steps forward, go! What will be your terminal? 2. Stay at 9; four steps backward, go! What will be your terminal? 3. Stay at 5; your terminal should be 9; how many steps you have to do? Later on these tasks will be written as shown on Figure 2. Figure 2a shows a concrete type of staircase. We shall henceforth write this illustration as S(5,4,9). Tasks 2 and 3 described above can be written by Figure 2b and 2c or in the shorthand by S(9,-4,?) and S(5,?,9).

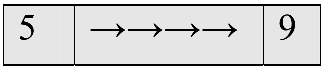

Figure 2a

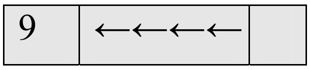

Figure 2b

Figure 2c

This model bears the strongest resemblance to the semantic model A ± Ch = A. The middle number, represented by the set of arrows, is a carrier of the dynamic element, while the numbers on the outside are the carriers of the static elements of the triplet.

4. Triplet, or 3-plet. Figure 3a shows an example of a concrete triplet. We shall henceforth write this illustration as T(6;1,5). When we have deleted all three numbers from the model, we will be left with the scaffolding of the triplet. Figures 3b and 3c show two problems within the context of a triplet. The first one is written as T(?;1,5), the second one as T(6;1,?).

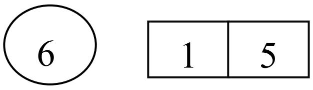

Figure 3a

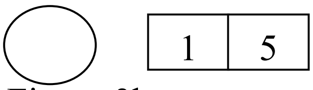

Figure 3b

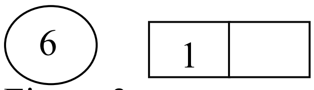

Figure 3c

5. Fourplet, or 4-plet and 3-string. There are two important issues of extensions of triplet. The first extension introduces the term 4-plet (quadruplet) – a group of four numbers, the first of which equals the sum total of the remaining three (see Figure 4). In the same way, we could introduce 5-plet, 6-plet, …, *n*-plet, but we are not going to do it here. The second extension introduces the term string. It can generally be said that any *n*-plet can be transformed into an (*n-1*)-string whose length is *m* (*m* > *n*). For example, the 4-plet in Figure 4 can be transformed into a 3-string whose length is 9 (see Figure 5). The sum total of any 3 adjacent numbers within this quadruplet equals 7.

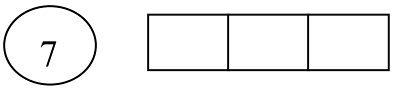

Figure 4

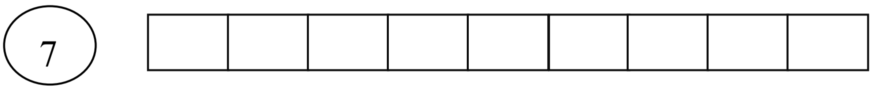

Figure 5

METHODOLOGY

There are four different ways of putting together a database that has been generated by researching the additive schema build-up mechanism in various types of school environment, with pupils aged 6 to 8. The database can be built up by means of:

1. episodic communication, imparted by the teachers who have familiarised themselves with the new approaches by attending various seminars;
2. systematic record-taking done by four teachers with whom we have cooperated for a long time;
3. "in vitro" experiments, carried out by the authors;
4. testing various types of induced environment at seminars attended by future teachers.

In this article we list one episode of the 1st type, two recorded situations of the 2nd type, one experiment of the 3rd type and several comments of the 4th type. The systematic record-taking activities of the two collaborating teachers were carried out in two 1st grade classes. The first one – we may designate it as "class A", consists of 18 pupils, while the second one – we may designate it as "class B", consists of 22 pupils. The pupils are 6 to 7 years old.

EXPERIMENTS

Walk. Experimental teaching indicated that the command "Two steps forward, go!" can be carried out in three different manners: 1. without putting one's legs together (the performing pupil ends up with his/her legs apart), 2. putting one's legs together (which is not classified as a step), 3. one-by-one ("step" as employed here stands for two body movements – the step forward and putting the other leg in line with the first one). In the beginning we assumed that there was just one way of carrying out the order, i.e., by putting the legs together as the final movement. However, the pupil experiments demonstrated that the varying perceptions of the walk commands need to be thoroughly analyzed. The following example illustrates typical class behaviour.

<u>Illustration 1.</u> Pupils in class A were quite correct in counting the steps that the teacher, or the performing pupil, had taken. When the teacher had made five steps, a female pupil named Hana said, "Our teacher took six steps." Dan disagreed, claiming that she had only taken five steps. The experiment was repeated, with the children counting aloud. A part of the class finished the counting with the number five, but another part said six, when the teacher put her legs together in her final movement. There was a dispute between the two parts of the class, one part defending the "five step" thesis, while the rest stood behind the "six steps" proposition. Dan said, "But the movement that you count as six is in fact not a step, because it does not bring the teacher's body any further. "Well, yes", Eva conceded. The pupils accepted Dan's argument and from there onwards they used the word "step" to stand for the entire movement, including the act of putting the legs together. Despite this, some weaker pupils tended to count the incomplete movement as a step, when there was a larger number of steps. This mistake would occur even half a year afterwards.

<u>Illustration 2.</u> In class B, a teacher introduced the process of walking by always putting the legs together. She was surprised to see two girls walk in a "one-by-one" manner. When she was telling us about it, she described the girls as very poor

mathematicians. We had the same experience with university students – primary pre-service teachers. One of these students also used the "one-by-one" walk. The teacher asked the students to evaluate this method of walking and especially weaker students supported this method, arguing that this method was more certain than walking by putting the legs together.

Commentary. A. From a strictly mathematical viewpoint the only correct method of walking is "one-by-one". The formulation 1 + 1 + 1 = 3 consists of the three ones, which represent the same object. If you put both legs together at the end of the walk, your final step is different from the two previous ones. That is why this method is not quite correct in mathematical terms. Moreover, when we do not put the feet together at the end, the situation is not quite clear, because we actually step backwards as well. This happens for example when we step according to the relation 3 - 2 = 1. However, there is a more relevant educational viewpoint, which we shall explore in the following sections.

B. The walk where the legs are eventually not put together has three basic deficiencies: 1. The walking process remains incomplete. 2. The command "Two steps, another two steps, forward, go!" consists of two disparate parts. The first pair of steps starts from a basic position, while the second one starts from a forward straddle. 3. After the introduction of the addresses (staircase) it is not quite clear which of the addresses the walking pupil is actually standing on (he or she has one foot on the *n* address, while the other one is on the $n+1$ address).

C. Deficiencies of the walk where the legs are eventually put together. 1. Some children count the final movement that brings the two legs together as a regular step (see Illustration 1). 2. Those children who do not classify this body movement as a step are able to see that the last step differs from the first step and from the steps in between. 3. Model W(2,3,5) is de facto not a representation of equality 2 + 3 = 5, because the movement that represents the process of adding up a 2 + 3 sum in fact resides in two movements of putting legs together, while the movement that represents the result 5 only puts the legs together once.

D. Educational application. 1. We introduce the walking activity by a one-by-one method. 2. If some pupils shift towards the more economical method putting the legs together, we should tolerate this new convention, but stop short of codifying it. This also corresponds with the use of an abacus, when one child counts to five by shifting every single ball, whereas a more advanced pupil separates the whole group of five balls.

E. There is reliable evidence that the Walk environment generates strong motivation. After one seminar for teachers, where we had been presenting our experience with walking in steps, one teacher wrote us the following comment, "Today I tried the walking game with children and they were absolutely taken in. This afternoon I put sticky notes on the floor to indicate the length of individual steps and am very much looking forward to tomorrow, as we continue this activity. Even today, the kids were falling over backwards to be asked to perform. They actually made a waiting list as to who will be the first to perform tomorrow. The walking method is an excellent idea."

F. The strong motivation of Walk model indicates that the pupils feel the need for an operator perception of numbers. In the case of first and second-graders in primary schools, this numerical model seems to belong to Vygotsky's zone of proximal development. This also indicates that the "walking environment" is capable of effectively fulfilling the educational objective.

Illustration 3. Second-grade pupils were assigned a problem concerning a 3-string whose length was 4 (see Figure 6). The assignment was: Fill the blank boxes with numbers in such a way that the sum total of three adjacent numbers should always equal 7. We were surprised to find that the pupils have considerable difficulties comprehending the notion of "three adjacent numbers". This indicated pupils should practise the correct application of the term for a longer period of time.

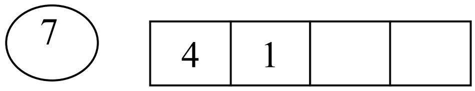

Figure 6

Commentary. G. Recently we found out that the idiom in question can be mediate to pupils by tasks of the following type: Find three adjacent letters in the word 'father'. How many solutions of this task you can find?

Illustration 4. Primary pre-service teachers did not seem to find it difficult to solve problems such as the one shown by the previous illustration. It did not take them very long to discover a pattern that permeates the string: periodical repetition of three numbers (e.g. 4, 1, 2, 4, 1, 2, 4, 1, …). Figure 7 shows the way in which Jana solved the problem with a 3-string whose length was 8. The numbers 8, 2, 6 had been pre-printed and the remaining ones were filled in by Jana. She began the problem-solving procedure by considering the number in the last box. She said the number could not be larger than 8 – 6 = 2. She then proceeded by trial and error, gradually trying out all the three potential numbers, 0, 1 and 2. The last number eventually proved to be the correct solution. An important aspect of this problem-solving process resides in the fact that in

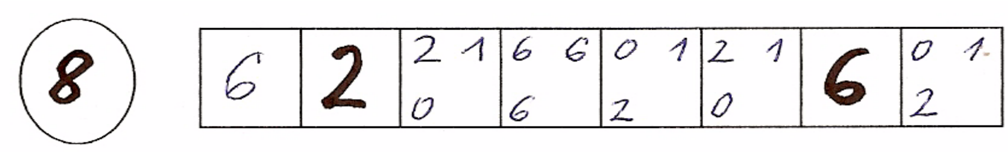

Figure 7

order to arrive at the correct solution, the girl did not use the overall pattern of these strings that she had discovered earlier.

Commentary. H. It is interesting to observe that in spit of the knowledge of the periodicity of each 3-string Jana did not apply this knowledge when solving the task. Why? Since the schema of the quadruple is not yet created in girl's mind.

Illustration 5. Two second graders, Fin and Tom, top maths pupils in their class, were assigned several problems with a 3-plet whose length was 4. The solution of one such problem is shown by Figure 8a. Then they were given the next problem (see Figure 8b).

They spent some time searching for a solution and we finally ended up by asking them to do it as their homework. A week later, we discussed the problem with Tom. He said. "My mum said it didn't make any sense." Even so, he agreed to make one more attempt at finding the solution, at the request of the researcher. He tried to switch the respective positions 2 and 3, but the result ended in yet another disappointment. "It won't do", he concluded. The researcher then took a card with number 2 on it and put it on the left desk, then put a card with an inscribed number "3" on the right desk and handed Tom two blank cards. She said, "Write a single number on each card, so that the sum of these two numbers, added to the number on the left, would make a total of 9, and that these numbers, added to the number on the right, would make a total of 9." After four unsuccessful attempts, Tom said in a disappointed tone, "I don't know. I cannot cope."

(9)

2	3	4	2

Figure 8a

(9)

2			3

Figure 8b

Commentary. I. Illustration 3 shows that the process of building up a scheme of triads inserted in graphic contexts may encounter didactic problems outside the realm of arithmetic. This discovery is currently being explored in more depth by further experiments.

J. By successfully solving problems such as the 3-string whose length is *m*, a pupil gradually grasps the issue; this happens in several stages:

1. Comprehending the condition "the sum total of each three adjacent numbers equals number *x*".
2. Discovering the fact that if $m = 4$, then the numbers on the outside of the string are identical.
3. Discovering the pattern (the periodicity of numeric triplets) for larger *m*.
4. Discovering a way to use the pattern to solve problems, in which there is a wider span between the two pre-assigned numbers in the string. These stages function as guidelines for our current experiments.

K. The surprisingly groping approach on the part of Tom shows that a pupil of his age may find it very difficult to automatize a schema. Tom does not realize that the two numbers that he writes on the blank card could be replaced by a single number, i.e., their sum total. If he knew that, he would quickly find out that if a sum total were added to a figure of two, it would produce a figure different from the same sum total added to a figure of three. This means that Tom does not see the possibility of writing the number 7 to stand for 6 and 1. He has not internalized the additive triplet schema yet. We think that the explored phenomenon can be used for diagnosing the quality of a schema as a base element of arithmetic structure.

CONCLUSIONS

This article describes several results of ongoing research aimed at triggering and embedding arithmetic structures in a pupil's consciousness. It points out the key role of additive schema, which can only be established if the pupil has had rich experience with various types of generic models. The paper describes the kind of educational environment that we think would get rid of the difficulties the pupils tend to have when trying to solve operator problems. And finally, the paper includes several observations that will be the subject of our future research.

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RECOGNISING AN ALGEBRAIC STRUCTURE

Maureen Hoch and Tommy Dreyfus

Tel Aviv University

This paper follows two students through a sequence of tasks and observes how they acquire the ability to recognise the structure $a^2 - b^2$ and apply the substitution principle: if a parameter is replaced by a product or a sum the structure remains the same.

INTRODUCTION

What is algebraic structure at high school level? What is structural about $a^2 - b^2$? How does a student learn to recognise structure? Can structure sense be taught? In this paper we try to answer these questions, by analysing and comparing the learning paths of two students. We talk about relearning since the structure $a^2 - b^2$ is one that these students have met earlier. The information on these two students is part of a larger body of data obtained by observing ten 11th grade students (age 16) learning or relearning four algebraic structures during three-session individual teaching interviews designed to improve structure sense.

Structure sense

The term "structure sense" was coined by Linchevski and Livneh (1999). Subsequently the idea was developed and refined by Hoch and Dreyfus (2004, 2005, 2006) who arrived at the following definition.

A student is said to display structure sense for high school algebra if s/he can:

- Recognise a familiar structure in its simplest form.
- Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
- Choose appropriate manipulations to make best use of a structure.

See Hoch and Dreyfus (2006) for full definition and examples. See also Novotná, Stehlíková, and Hoch (2006) who adapted structure sense and defined it for university algebra. An important feature of structure sense is the substitution principle, which states that if a parameter is replaced by a compound term (product or sum), or if a compound term is replaced by a parameter, the structure remains the same. In this study we look at two aspects of structure sense for high school algebra – recognising a familiar structure in simple or complex forms, and applying the substitution principle.

Algebraic structure

The term "structure" is widely used and most people feel no need to explain what they mean by it. In different contexts the term structure can mean different things to different people (see for example Dreyfus and Eisenberg, 1996; Hoch and Dreyfus, 2004; Stehlíková, 2004).

The term "algebraic structure" is usually used in abstract algebra and may be understood to consist of a set closed under one or more operations, satisfying some axioms. In this paper we are concerned with algebraic structures met in high school. Hoch (2003) discussed and analysed structure in high school algebra, considering grammatical form (Esty, 1992), analogies to numerical structure (Linchevski and Livneh, 1999) and hierarchies (Sfard and Linchevski, 1994) culminating in a description of algebraic structure in terms of shape and order.

In this research we took a similar approach, relating to any algebraic expression or equation as possessing structure, which has external components such as shape and appearance, and internal components determined by relationships and connections between quantities, operations and other structures. Five structures were examined: $a^2 - b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax + b = 0$; and $ax^2 + bx + c = 0$. Hoch and Dreyfus (2005, 2006) identified students' difficulties with these structures. Dreyfus and Hoch (2004) looked at the structure of equations. In this paper we look at the structure of expressions, specifically the expression $a^2 - b^2$.

The structure $a^2 - b^2$

How does this view of algebraic structure apply to $a^2 - b^2$? In appearance, it is an expression with two terms, each of which is a perfect square, connected by a minus sign. The internal structure is that of a quadratic expression which can be factored. Thus the general formula, $a^2 - b^2$, is equivalent to $(a - b)(a + b)$. Students in Israel first meet $a^2 - b^2$ in 8^{th} grade as a special case of $(a + b)(c + d)$, the extended distributive law, and again in 9^{th} grade when they learn to factor special quadratic expressions. An assumption of this research was that students at advanced and intermediate levels are familiar with this expression and proficient in working with it in its simplest form. Whether this assumption was justified will be seen later.

METHODOLOGY

The first step in designing a teaching unit to facilitate the improvement of structure sense was to develop a sequence of appropriate tasks. These tasks were designed with a certain age and ability of students in mind. The process of selecting the subjects included a pre-test, which was administered to two whole classes. The intervention comprised a series of three one-on-one teaching interviews, so called because they were planned as a careful mixture of clinical interview with tutorial session. These sessions took place a few days apart from each other. A post-test was administered individually a few days after the third session. A brief post-mortem discussion took place with the student after the immediate post-test. Several months later a delayed post-test was administered.

Task design

The tasks included sorting, comparing, factoring expressions, solving equations, and creating new examples, with the aim of encouraging the student to learn to look for and recognise the five structures mentioned above in their simplest form, and in more complex forms where the compound term is a product or a sum.

The first task is to sort several algebraic entities including $49 - y^2$ and $x^2 - 16$, each printed on a separate card, and presented altogether in random order, into groups of similar items. One aim here is for the student to differentiate between equations and expressions. Subsequently the student is asked to sub-divide the two groups and characterise the resulting five structures: describe common properties and find names and formulae.

We will now describe only the tasks relating to the structure $a^2 - b^2$, which can be divided into four groups as shown in Table 1. In the teaching unit these tasks were given in the order described, but were mixed with similar tasks involving the other structures. The expressions are also printed on cards, but are presented to the student one at a time, in the given order. The expressions are presented at first in simplest form (G1 in Table 1), then with compound terms which are products (G2) and then with compound terms which are sums (G3). In G1 the student is asked to suggest a name and formula for the structure. In G2 and G3 the student is asked to factor and to name the structure. If the student is unable to factor, then s/he is asked which structure the expression possesses, and encouraged to use the structure to help to find the factors.

G	Expressions	Tasks	Structure sense
1	$49 - y^2$ $x^2 - 16$	Create similar expressions Describe common features Factor Find a name and formula	Recognise a familiar structure in its simplest form
2	$25x^2 - 36y^4$ $16x^2y^2 - 49z^6$	Factor Recognise and name	Substitution principle - product
3	$(x+8)^2 - (x+7)^2$ $x^2 - (x+1)^4$ $(x+3)^4 - (x-3)^4$	Factor Recognise and name	Substitution principle - sum
4	$a^2 - b^2$ Some of above	Name the structure Create similar expressions	All of above

Table 1: Tasks involving $a^2 - b^2$

Another task is for the student to describe the structure, given by its general formula, in words and make up expressions similar to those shown (G4). The student is asked to create expressions that s/he considers might be difficult for a friend to recognise. The idea here is that the need to explain a structure in words causes the student to reflect on it more carefully, and that the act of creating more examples deepens his or her personal relationship with the structure.

The tasks were designed to encourage students to verbalise in order to make explicit what was previously only implicit - the structures and their properties.

Student subjects

The tasks were aimed at developing structure sense in 11^{th}-grade intermediate to advanced level students. The focus group consisted of 10 students chosen on the basis of their low structure sense score on the pre-test and their willingness to participate in the research.

Here we report on two of these students, chosen for the fact that neither of them displayed any knowledge of the structure $a^2 – b^2$ on the pre-test. Anne studied in the advanced class, Brian in the intermediate class.

Pre-test and post-tests

A pre-test of twelve items was designed, based on the four structures $a^2 – b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax^2 + bx + c = 0$ and on three types of structure sense: simple recognition and dealing with compound terms which are products or sums. For the immediate and delayed post-tests the same items were used with different numbers and /or letters.

Teaching interviews

One-to-one teaching interviews were decided on, as being the most promising method of encouraging the students to verbalise about how they worked on the tasks. The researcher sat with each of the 10 students individually, for three sessions of approximately 45 minutes each, over a period of up to two weeks. The interviews were audio taped and a post-test was administered in a fourth individual session. Throughout the sessions the student was encouraged to verbalise about what s/he was thinking and doing, with emphasis placed on the correct naming of each algebraic entity and structure.

The interviews were designed to proceed as follows. Initially the student was instructed to “talk aloud about what you are thinking and try to explain your decisions and actions. Feel free to write down anything you want on the blank paper.” The researcher presented the student with a task and verbal instructions, e.g. “factor”. Depending on how the student completed the task, the researcher then asked a series of pre-arranged questions, designed to lead the student to the desired conclusion. Sometimes the researcher had to ad-lib, relying on her experience as a teacher. Once the desired conclusion was reached, the next task was presented, and so on. At the end of each session the student was asked some self-reflective questions – what s/he felt s/he had learned and how it might help him/her.

RESULTS AND DISCUSSION

Pre-test and post-tests

Tables 2 and 3 summarise the results of the pre-test and post-tests on the three items involving the structure $a^2 - b^2$.

Pre-test	Immediate post-test	Delayed post-test
$81 - x^2$ = (9 – x)(x – 9)	$25 - y^2$ **correct**	$81 - y^2$ **correct**
$x^4 - 36y^2z^2$ $=(x+1)^2(x-1)^2 - 36yz(yz)$	$49x^2y^2 - z^4$ $=(7xy - z^2)^2$ then **correct**	$25x^4 - 16y^2z^2$ **correct**
$(x-3)^4 - (x+3)^4$ $=(x-3)^2(x-1)^2 - (x+3)^2(x+1)^2$	$(x+2)^4 - (x-2)^4$ $=[(x+2)^2 - (x-2)^2]^2$ then $(x+2)^2(x-2)^2 - (x-2)^2(x+2)^2$	$(x+5)^4 - (x-7)^4$ **correct**

Table 2: Anne's performance in pre-test and post- tests

In the pre-test, neither Anne nor Brian was able to factor any of the expressions correctly. This was surprising in the light of the assumption mentioned earlier that these students would have well-based knowledge of the formula.

Pre-test	Immediate post-test	Delayed post-test
$81 - x^2$ $\pm 9 = x$	$25 - y^2$ **correct**	$81 - y^2$ **correct**
$x^4 - 36y^2z^2$ blank	$49x^2y^2 - z^4$ **correct**	$25x^4 - 16y^2z^2$ $=(5 - x^2)(5 + x^2)$ then $(5x)^4 - (4yz)^2$
$(x-3)^4 - (x+3)^4$ $x^4 - 12x + 81 - x^4 + 12x - 81 = 0$ $0 = 0$	$(x+2)^4 - (x-2)^4$ **correct**	$(x+5)^4 - (x-7)^4$ blank

Table 3: Brian's performance in pre-test and post-tests

We can see that on the pre-test Anne clearly attempted to factor and was aware that raising to the fourth power is equivalent to repeated squaring, but otherwise seemed to be very confused. Brian, on the other hand, converted two of the expressions to equations by adding "= 0". One of these equations he solved correctly and in the other he opened brackets, seemingly according to an over-generalisation of the formula $(a - b)^2 = a^2 - 2ab + b^2$.

In the post-test, immediately after the three-session intervention, Brian factored all of the expressions correctly but Anne only succeeded with $25 - y^2$, and, after some prompting, $49x^2y^2 - z^4$. She recognised $(x+2)^4 - (x-2)^4$ as being difference of squares but was unable to factor it. From this we might surmise that the intervention had succeeded in considerably improving Brian's structure sense, at least concerning this particular structure, and to a lesser extent had led to an improvement in Anne's structure sense.

However, in the delayed post-test, several months later, Anne factored all of the expressions correctly while Brian only succeeded with $81 - y^2$.

Clearly both students have learned something about the structure and about the substitution principle. In Brian's case the learning appears to be immediate but short-lived. In Anne's case, the learning seems to be deeper, not appearing immediately, but somehow absorbed and emerging later. Will an examination of the teaching interviews help to explain this? Perhaps Brian had more to learn and thus might have been expected to retain less. Brian's immediate success and later failure is less surprising than Anne's initial failure and later success.

Overall Anne was the stronger student and retained most of what she learned. Her improvement in structure sense does not seem to be dependent on a particular structure. She maintained the same overall score between the two post-tests, due to the fact that in the immediate post-test she succeeded in applying the substitution principle to certain structures, and in the delayed post–test she succeeded in others. Brian's results are more in keeping with the results of the rest of the group, and with what would normally be expected – immediate improvement, only part of which is sustained over time.

Teaching interviews

Let us examine how Anne and Brian dealt with the expressions $49 - y^2$ and $x^2 - 16$. When asked to create two similar expressions, Anne wrote $x^2 - 25$ and $x^2 - 64$. It appears that Anne understood this structure but her comment suggests otherwise:

Interviewer: What do these expressions have in common?

Anne: That you solve them by taking the square root.

After a short discussion about what constitutes an expression, and what one might do with an expression Anne was still unable to overcome the visual impact of these square numbers:

Interviewer: Why did you connect these two together? [Points to $49 - y^2$ and $x^2 - 16$]

Anne: Ah, they just seemed to me to be simpler?

Interviewer: What do you mean simple?

Anne: Well simply you just take the square root. Raise it to … Oh no, it's impossible.

Anne seemed to be relating only to the appearance, with no conception of any inner structure. Although she did not actually state it, there is a hint here that she was thinking of turning the expression into an equation, which she knew how to solve. When asked to factor she suggested extracting a common factor, but when urged to write it down she immediately factored $x^2 - 16$ correctly. This is interesting since she could she not factor $81 - x^2$ in the pre-test. Did she suddenly remember the formula? Perhaps she just made a lucky guess, as suggested by …

Interviewer: You factored correctly here. [Points to $(x-4)(x+4)$] Do you remember from which formula that came?

Anne: Em. a plus b squared? [Writes $(a+b)^2$]

Anne was once more asked to look at the expression and was able to describe it as two squares with a minus sign, but unable to come up with a name, which the interviewer provided. But later …

Interviewer: Okay. So now let's try to find a formula for the difference of squares.

Anne: Em … a … em

Interviewer: Let's find a formula with a and b which will

Anne: a squared minus b squared.

Where did this formula appear from? Did Anne have a store of formulae in her head, which she produced on a trial and error basis? Or had she been constructing it while going through the earlier process of description?

Brian's learning process was somewhat slower. He sorted the two expressions $49 - y^2$ and $x^2 - 16$ into the same group. He was aware that they have something in common, but at this stage he did not know how to articulate it. When asked to create two similar expressions, he produced $x^2 + 81$ and $y^2 + 36$. It would appear that he noticed only the squares but not the minus sign.

Interviewer: Can you explain to me how you got to these two expressions?

Brian: These two have a root, [Points to $49 - y^2$ and $x^2 - 16$] that's 16 and 49.

Interviewer: What does that mean – they have a root?

Brian: It's possible to extract the root of 16 and also of 49.

Brian's description was similar to Anne's, referring to the square numbers as having roots. The interviewer asked if $x^2 - 7$ belongs to this group. Brian was not sure, and tried to describe numbers like 16, 49, 36 and 81 saying that "they have real roots" and using other terms like “nice” roots and “realistic” roots until eventually he reached “whole”. He concluded that $x^2 - 7$ does not belong to this group. After some discussion of factoring Brian recognised the importance of the minus and was able to create new examples of this structure. But still he was unable to describe it or talk about square numbers. He even called the expression an equation. We have seen this so often with students that we wondered whether it was just a careless way of talking. However we see also from Brian’s confusion in the pre-test, and from Anne’s reference to solving an expression that this is not just a question of semantics but a deeper conceptual problem. Eventually Brian too arrived at the formula, and a name for the structure, but with a great deal of prompting.

Next the expressions $25x^2 - 36y^4$ and $16x^2y^2 - 49z^6$ were introduced. From the students’ reaction to them we see whether they can recognise the structure in more complex form and apply the substitution principle when the compound term is a product.

Anne quickly identified $25x^2 - 36y^4$ as having structure $a^2 - b^2$, “These are both squares”. However, when asked to factor, she wrote $(5x^2 - 6y^2)^2$. She had to be coaxed to remember how to factor $a^2 - b^2$ and from there she had no difficulty in correctly factoring $25x^2 - 36y^4$. She also identified $16x^2y^2 - 49z^6$ immediately as $a^2 - b^2$ but hesitated over the factoring. She wrote $(4xy)^2 - (7z^3)^2$ and when asked if that is the factoring was unable to answer, despite having shown earlier with other expressions that she knew what factoring means. Again, after light prompting, she factored correctly. This is interesting as it shows that Anne had no difficulty recognising the structure and applying the substitution principle, but that she did have a problem with the factoring.

Brian could not factor or identify $25x^2 - 36y^4$ so it was put aside. After working with other structures he was presented with $16x^2y^2 - 49z^6$. He said “Eh ... they both have roots” and wrote $(4xy)^2 - (7z^3)^2$. When asked if this is the factoring he hesitated. He identified that the expression has the structure $a^2 - b^2$, but like Anne he was unsure how to factor $a^2 - b^2$. Once the formula was established he contradicted himself, saying that the formula could not be applied in this case because “there’s no root”. After extensive prompting he managed to factor correctly. On returning to $25x^2 - 36y^4$ he identified the structure, but wrote $(5x - 6y^2)$before factoring it correctly.

Later the expressions $(x + 8)^2 - (x - 7)^2$, $x^2 - (x + 1)^4$ and $(x + 3)^4 - (x - 3)^4$ were introduced, necessitating recognition of the structure in more complex form and application of the substitution principle when the compound term is a sum.

When asked to factor $(x + 8)^2 - (x - 7)^2$ Anne’s first response was “impossible” but then she identified the structure and with only slight coaxing, and one small error, she succeeded in factoring. She identified $x^2 - (x + 1)^4$ correctly but again factored as

$\left(x-(x+1)^2\right)^2$ before realising her mistake and factoring correctly. But finally success – she factored $(x + 3)^4 – (x - 3)^4$ flawlessly. Brian looked at $(x + 8)^2 – (x - 7)^2$ as the sum of two expressions with structure $(a + b)^2$. When asked to look at the whole expression he immediately recognised the structure and factored it correctly. He identified $x^2 – (x + 1)^4$ correctly and with only minimal help succeeded in factoring. And like Anne he factored $(x + 3)^4 – (x - 3)^4$ flawlessly.

Finally the students were asked to describe the structure $a^2 – b^2$, and create expressions similar to those shown in Table 1. Anne's description was "Difference of squares. At the two edges there's squares. And they are … squares". She produced these expressions: $144x^2-64z^6y^4$ and $(x+20)^8-(x-16)^8$. Brian simply named it "Difference of squares" and wrote $\left(144x^2y^2\right)^2-\left(81z^4y^6\right)^2$ and $\left(289+x^4\right)^2-\left(z^2y^2+256\right)^2$. It is interesting to notice that Brian used only square numbers or expressions for the parameters a and b in his examples. It is not clear whether he used them, as he said, "To make it more difficult" or whether he believed they are necessary.

CONCLUSION

Both students clearly learned to identify and factor the expression $a^2 – b^2$, and to apply the substitution principle. They displayed similar difficulties with articulating the features of the structure, and with applying the simple formula but both overcame these difficulties. The fact that they could both easily factor the expression $(x + 3)^4 – (x - 3)^4$, not only in the teaching interview but also on a post-test, is impressive as factoring this has been shown to be extremely difficult for students with similar background (Hoch and Dreyfus 2005).

Although both were unable to factor any of the three expressions in the pre-test, Anne displayed a better understanding of factoring, and seemed to pick up on the structure faster, although it took her longer to learn the basic formula $a^2 – b^2 = (a – b)(a + b)$. Brian seemed to be slower at recognising structure, and if we look at how he built the new expressions in the last task we might suspect that he had not quite grasped it. This might have been expected as Brian learns in an intermediate stream while Anne learns in an advanced stream. However, the results of the post-tests are surprising. Brian excelled in the immediate post-test, while Anne excelled in the delayed post-test. Brian's result in the delayed post-test is disappointing but a certain loss of knowledge or skills over time was only to be expected. What is surprising is Anne's performance in the two post-tests. Is it possible that the experience of not succeeding in the immediate post-test, and a short discussion of her mistakes which followed that post-test, constituted another learning experience and succeeded in consolidating the structure in Anne's mind?

We have dealt here with only two students and one structure. It will be interesting to compare these stories with those of all 10 students and all four structures.

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CREATING A MENTAL IMAGE OF DICE BLACKJACK GAME

Antonín Jančařík

Charles University in Prague, Faculty of Education

The structure of students' mental image of the Dice BlackJack game is studied and some cognitive and meta-cognitive phenomena of this structure are presented and briefly described. The motivation of this experiment is based on the experience from experiments and workshops with future mathematics teachers.

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INTRODUCTION

In his research, the author of this article – a mathematician – is concerned with alternative methods of acquainting students, future mathematics teachers, with "the world of mathematics". To this aim the author uses non-traditional means – above all, games and puzzles. From his seminars and workshops for students and teachers, the author the means that seem to be appropriate as they illustrate in a practical way some of the key concepts and methods. One of them is the Dice Blackjack game. This game is usually used by the author as an example of a more complicated mathematical game at the stage when students have formed an image of what a winning strategy might be, and once they have learnt basic methods of combinatorial theories through simple games, such as the NIM game.

The rules of the game are very simple. There are two players. The first player throws a die to begin, after that there is no more die throwing; both players take turns in flipping (turning) the die. The values on the top side of the die are added up (for both players). The player who goes over the total sum of 21 loses.

The game may progress in the following way: the first player gets 6, the other flips the die to 5 (total 11), the first flips to 6 again (total 17) and the second flips to 4 (total 21), and then the first player flips to 1 and loses, because the total is greater than 21. The last move would see the player lose regardless of the number he flips to, because the sum must be greater than 21 at this point. However, the player could have chosen in his first turn to flip to 1 (making the total 12) and thus gain a possibility of victory with no regard to the move of his adversary (for an analysis of this situation, see below).

THEORETICAL BACKGROUND

Using games in mathematics education is not a new idea. The author believes that games provide a unique opportunity for integrating the cognitive, affective and social aspects of learning. They can be used to introduce new ideas and lay the foundations for processes and thinking strategies (Booker, 2004). Games are a suitable motivating tool (Ein-Ya, 2005) and at the same time a foundation for communication among students (Cañizares, 2003). The use of games in statistical education (Teles, 2006) could be used as an interesting comparable example. Next, let us consider the mathematical background of games.

Combinatorial game theory is a relatively new discipline. It dates back to 1902 when an analysis of the game NIM was published (Bouton, 1902). The discipline underwent a tremendous development in the second half of the 20th century. The research of that period is tied to people like Guy, Berlekamp, and Conway (see Berlekamp, Conway and Guy, 2001), with especially the latter's results having had great influence even on fundamental fields of mathematics (Conway, 2001).

The basic tool of the Combinatorial game theory used for searching for winning strategies is the fact that an evaluation of one's position in the game is possible without looking at the whole game but only at the current state of the game. Since there is, apart from the initial die throw, no other influence of probability, the result of the game is, at the point when the thrown die stops, already determined, in other words, the winner is known (assuming no one makes any mistakes). Therefore, there must be a given valid method for each player to win, or *the winning strategy*.

Finding the winning strategy is the same as finding critical positions. A critical position is a position in which the current player can only win with the help of his/her foe's mistake, and a winning position is a position in which the player will win regardless of his opponent's moves provided he/she makes no mistakes. A more accurate definition: A critical position is such that all following moves lead to a winning position, a winning position is such that at least one move leads to a critical position.

In the Dice Blackjack game, a position is a set of two numbers, one is the total sum of the game, the second is the number on top of the die from the last move. The game given in the Introduction has the following positions: (6, 6), (11, 5), (17, 6), (21, 4).

To evaluate a position, we only need to know the evaluation of those positions, that we can attain with one move. Therefore we begin the game evaluation from those positions that can be decided easily. In the case of Dice Blackjack, such a position occurs when the total sum is equal to twenty one. In that case, the player to move must lose whatever number may be on top of the die. The position (21, x) is therefore critical for all x values. Next, we can evaluate positions with the total sum of 20. Obviously, if 1 can be played (1 is not on the top or bottom side of the die), the position is won. Let us assume the use of a standard die with the opposite side sum of 7. We have four winning positions, (20, 2-5), and two critical positions, (20, 1) and (20, 6), because in their case the die flip must breach the total sum of 21. A similar method can be used to evaluate all other positions (Table 1).

	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
1	C	C	W	W	W	W	W	W	W	C	W	W	W	W	W	W	W	W	C	W	W	W
2	C	W	W	W	W	C	W	W	W	C	W	W	W	W	C	W	W	W	C	W	W	W
3	C	W	W	C	C	W	W	W	C	C	W	W	C	C	W	W	W	C	C	W	W	C
4	C	W	W	C	C	W	W	W	C	C	W	W	C	C	W	W	W	C	C	W	W	C
5	C	W	W	W	W	C	W	W	W	C	W	W	W	W	C	W	W	W	C	W	W	W
6	C	C	W	W	W	W	W	W	W	C	W	W	W	W	W	W	W	W	C	W	W	W

Table 1: Evaluation of all game positions

The positions marked in colour can be attained in the initial die throw. Since only two of them are critical, the second player has the advantage, having a probability of 2:1 to make his/her first move at a winning position. The basis of the winning strategy for the player who is to move at a winning position is to select a move each time so that a critical position is attained.

METHODOLOGY

The experiment presented here was embedded in term-long work with a group of students within a game theory seminar. *Its aim was to describe the phases of the creation of a structure of a mental image of the Dice BlackJack game and to distinguish basic phenomena that are important in the process of this structure creation.*

Two students, year 4 future mathematics teachers took part in the experiment described below, both were male. They already had some experience with games; they had dealt with some types of NIM games. Thus they knew what a winning strategy was and had some idea how to look for it.

The experiment went as follows. In the first 2 minutes, the students were reminded what a winning strategy was (based on the NIM 1-2-3 game). Next, the rules of Dice Blackjack were explained. The students were asked to comment on their thought processes and to communicate with each other while looking for a solution. Below we will use the names Peter and Paul for our two students and Ex to mark the speech of the experimenter (author).

The whole experiment took about 45 minutes. The author was present and intervened several times. He answered questions about the game, asked the students to write the results on a blackboard and suggested minor improvements of the written record. In one case he influenced the experiment by alerting the students to a mistake using a question, and thus guided them to remove the mistake. Once the experiment was done, a short interview was held.

The experiment was started with a try-out game which was not, however, finished. It stopped at the point when any one player was about to reach the sum of 21. At this stage, the students asked the experimenter to further specify the rules of the game.

PAUL: Can I just flip it or can I also turn to the bottom side?

After the initial game, the students started discussing certain situations that they tried to complete. They simulated a situation, when there is already a certain sum and they tried to finish the game. This was repeated twice. The students started at sum 9 and finished the game, next they started at 14 while at the same time looking for the winning strategy. Since the situation was complicated (there were many possibilities), the students decided to discuss the options systematically. The following dialogue (Dialogue 1) describes this spontaneously arisen strategy change:

PETER: Well, let's start at 21, depending on what's on the up-side.
PAUL: If it's 21, then you've always won. If it's 20 and a six or 1 on top, you've won. … If it's anything else, then you've lost.
Ex: Could you put these results down?

While writing, the students decided which positions to mark as winning and which as losing (Dialogue 2):

PAUL (writing): If it's sum 20 and 2,3,4,5 on top… (hesitates).
PETER: This way it means that it's the … situations when …
PAUL: The player to move loses.
PETER: Right, because you'd get …
PAUL: If you've 2, 3, 4, 5 then the player to move wins (writing).
PAUL: With 21 numbers one through six lose (edits the record).
PAUL: 19, if it's 1,3,4,6, then you win.
Ex: Why is it won in this case?
PAUL: Because the player would play the two.
PETER (examining the die): The sum is 19, you are on the move, on the top is 1, you play a 2, 3 is on top, that's the same.
PAUL (writing): The other numbers, one, six, that's lost.
PETER: You sure it's lost?

Since the students' record was very untidy, the experimenter suggested another form of record. The original record had one group of winning positions and another group of losing positions for each line. The new record was much like the table above. The students accepted the suggestion; they used 0 and 1 to mark the losing and winning positions, respectively. They copied their results into the table and began discussing the position (19,2) (Dialogue 3).

PETER: So when there's a two on top?
PAUL: Then you play 1 and you get to (points to the zero in table cell (20,1), thinks and goes to a table to try out the position using a die).
PETER (setting the die): Nineteen and two on top.
PAUL: That means it's won for the first player. (Goes back to the blackboard, writes).

At sum 18, the positions with numbers 1, 2, 5, 6 were immediately marked as winning. Positions with 3 and 4 were discussed using all possibilities. With sum 17, again the positions that win with one move were marked first. Next, an analysis of all possibilities was not done but the losing positions were found in the table and it was determined whether these positions can be attained. At sum 16, the same method was used, however only the attainable positions were discussed (Dialogue 4):

PAUL: You get to 17 through 1, you get 18 through 2 …

At sum 15, again the one-turn winning positions were marked first and then the table results were discussed diagonally (16,1), (17,2),… The students discovered that the position (18,3) was marked with a zero.

PAUL: You get to sixteen with 1, to seventeen with 2, and to (moves the finger in the table) uh-oh.

PETER: (Looking at the die) Well, you can get there.

PAUL: That means it's winning.

PETER: Oh, yeah.

Paul records the results in the table and circles them. Then he moves on to line 14.

PAUL: You can't add a seven.

He is thinking the situation over.

PAUL: If you get to 15 with a one, then you've lost. (Shows the zeros in the table) You need to get here, here and here. So if you can play two, then you've won.

This thesis was then doubted and both students spent a long time finding the connections between the winning and the possible moves. Most hypotheses were accompanied by pointing to the table; the students commented on them very little. The following dialogue followed after approximately five minutes (Dialogue 5):

PAUL: If you can play one or two, then it's these numbers (points to the table), if five or six, then it's these numbers (points to the table again). So this diagonal direction is important (fills in all results).

Similarly, all other lines of the table were filled in, with increasing speed, up to the line zero. After filling the table, the students hesitated what to do next. Upon the experimenter's request they first marked 6 lines as an answer, after one throw of the die they specified that the only relevant positions were (1,1) through (6,6) and they answered that the game favored the second player. Finally, the students played one game at their own request using the correct strategy based on the table.

The experiment was recorded with a camcorder and evaluated by standard methods of Grounded Theory (Strauss, Corbin, 1998). The evaluation was based on an analysis of the video recording, on an interview with the students after the experiment (the students' self-evaluation) and on the experimenter's previous experience. Discussions and experiment analyses done with the help of other experts were also a valuable asset and they were used as a source of further knowledge and data to be used in the analysis. The theoretical basis for the interpretation of the structure of mental image creation analysis results was the theory of the generic model (Hejný, 2003 and Hejný, Kratochvílová, 2005).

At this stage of introduction to game theory, the students only know some isolated models – examples of mathematical games. They do not possess the knowledge of an abstract model of a "mathematical game" and must therefore look for solutions step by step, while of course using the methods that proved useful in their studies of NIM; for example, they begin the analysis starting at the end (see Dialogue 1).

The aim of the experiment was to observe the students' methods of finding a winning strategy and to characterize a method of creating a mental image of the game. Since this paper is limited in length, the following text is concerned mainly with the analysis of the cognitive and meta-cognitive phenomena directly connected with the structure of the game. The communication phenomena are not analyzed deeply. Communication obstacles in student communication should be discussed in a separate analysis.

Next, some cognitive phenomena arising from the analysis will be given.

CONCEPTUAL PHENOMENA

Understanding the rules: Understanding the rules is a necessary prerequisite to further work. It depends mainly on the experience with other types of games. One of its important aspects is the understanding of the initial position, the possible moves, the roles of the individual players and the rules of finishing the game – understanding especially when the game finishes and what determines the winner.

Understanding the winning game strategy: A winning strategy is a very abstract notion. Its understanding is carried out on several levels.

- The first, lowest level is the ability to apply a winning strategy – to understand that a given method guarantees victory and to be able to apply that method.
- The second level is shown in the ability to modify the strategy based on a change in the input parameters.
- The third level is the ability to actively discover the winning strategy in a new situation; this is the level of understanding that the analyzed experiment stresses.
- The final level is working with a winning strategy at an abstract model level. Here we can also talk about, for example, finding the sufficient and necessary conditions for the game to have a winning strategy – for the game to be determined.

The students in the above experiment worked with the notion of winning strategy on the third level, they were individually able to discover winning strategies in simpler game types and to modify them subsequently.

Understanding the determination factor of the game: In the case of Dice Blackjack, we need to separate the influence of the probability of the initial dice throw from the determined rest of the game. The initial dice throw is probable, but the rest of the game is determined and it is therefore possible to discover its winning strategies. The evaluation of the game is based on the evaluation of all initial throws and their respective probabilities. The main manifestation of understanding the determination factor of the game is (1) coming to terms with the fact that the game does have a winning strategy and (2) the decision to look for it, since a winning strategy can only be discovered in determined games.

In order to find a winning strategy, it is absolutely necessary for the students to understand the concept of critical position. In relation to the determination factor of the game, it is important to understand the fact that every position is either critical or winning. Looking for a winning strategy can thus be transformed into the problem of evaluating all positions. The understanding of the notion of critical position can be observed in the students' decisions about the individual positions – the isolated game models. A manifestation of understanding the critical position notion and mastering the rules was observed in the fact that the students did not need to finish the first trial game as soon as it became "clear" who is to win. A manifestation of the understanding of the existence of the need to evaluate all positions was seen in the switch to a systematic evaluation of all game positions (Dialogue 1 and 2).

PROCEDURAL PHENOMENA

The procedural phenomena played a major role in forming the mental structure of the game. The first one concerns gaining **an insight into the situation by playing trial games**. During the trial games, the non-clarities of the game rules become apparent. If there are any doubts, the game is stopped and questions are asked. In this, key concepts like move, victory and loss are also explained. Students come to know the influence of the basic properties of the die on the game – the player to move can only select four (not six) options. The sum of two consecutive moves cannot be seven. An important phenomenon is the **recognition of a lost game** before it is finished. This recognition comes from the students' experience as players. In the trial game, the first notions of strategy are formed. Students assume that the strategy is a complicated one and therefore they do not try to analyze all the possible procedures of the game, but rather to look for a solution in shortened games.

Another important procedural phenomenon is the **study of the procedure of shortened games**. Shortened games give us an opportunity to study the regularities of the game. An important discovery is the fact that a position is an arranged pair, the total sum and the state of the die. Our experience from the above seminar shows that this discovery is usually made at the position (19,2). In shortened games, students also discover that the evaluation of a position is based on the subsequent positions. At this moment, they discover that the winning strategy can be formulated by a systematic evaluation of all possible game positions. Based on their experience with isolated models, the students move on to a systematic analysis of all game positions.

The forth phenomenon is the sophisticated **evaluation of all game positions**. While moving to a systematic evaluation of the positions, it was found that a more sophisticated situation in some cases forced the students to verify the matter on a practical level (Dialogue 3 – the students came back to the table and simulated the analyzed situation by throwing a die).

The discovery that the winning move can even be a move which does not directly achieve a victory is very important. During the course of the experiment, such situations were marked by students by circling the number in the table. The students first evaluate the position based on an evaluation of all further possible procedures of the game. They gradually discover that the evaluation of a position can only be determined once some of the subsequent positions are evaluated (Dialogue 4). In time, the students discover that the evaluation is determined by attainable positions, and they can find these positions (Dialogue 5).

The key situation in this process is the situation (14,X). At this moment, there are no more points of support, that is the points that definitely win a move. All positions must be evaluated based on the preceding results only. For this reason, this situation is crucial for the formation of the mental image of the game. If this position is handled successfully, the solving strategy advances by a considerable stretch. The students already have a sufficient insight into the whole situation and they can fragment the entire process without worrying about getting lost. At this point, there is a move from isolated models to a generic model of the game. An important fact is that the students stop to differentiate between the situations where one and where more moves are required to win. At the same time, they can apply an evaluation algorithm smoothly and use it to determine a complete evaluation of game positions. They can also modify the algorithm for variations of the game (for example, a game using a different end sum).

Finally, we will briefly mention five meta-cognitive phenomena.

META-COGNITIVE PHENOMENA

Selecting a solver strategy (starting at the end): When choosing a strategy, there was an apparent influence of the experience with the NIM game, especially the discovery that shorter game solutions can be applied to more complicated games.

Creating and noting isolated game models: Since the given game is overly complicated, the students switched to analyzing simpler situations. The isolated models of the easier situations were crucial for obtaining an insight into the situation.

Searching for a generic model: An important point in switching from isolated models was the decision to analyze all the positions systematically, including a systemization of recording the results.

Method of describing the winning strategy: While choosing a method of recording results, the students again derive from their previous experience. Records in a table make further procedures much easier. The algorithm of the winning strategy derives from a correct interpretation of the table.

Checking results: With the students participating in the experiment, a need for checking the results by playing games according to the winning strategy and thus confirm the correctness of the results was apparent.

CONCLUSION

The author carried out several similar experiments as the one above and they brought a result which was very surprising for the author. Contrary to his expectation, the students who took part in the experiment were able to obtain a winning strategy on their own with only marginal interferences by the teacher *faster* than other students for whom the author employed the transmission way of work, that is he guided them step by step. Not only was the time taken for obtaining the winning strategy shorter but the understanding of the game was deeper at the same time. This fact needs to be further studied in the same context of games and winning strategy search.

In the analysis of the experiment, it became apparent that there was *a key position* in the game, namely (14,x). This position was crucial for understanding the game and finding a winning strategy. The key position (14,X) can be characterized as follows:

- This position is more difficult to grasp than the previous ones and plays a major role in a model of the game.
- This position greatly differs from the preceding positions and the mastering of it is decisive for the mastering of the whole game.

It can be assumed that the key position (14,X) will play a similarly crucial role during the process of creating a mental image of this game. The author's experience from seminars with students suggests that this position often played an important role in the process of understanding the winning strategy of this game. More experiments are planned for testing this hypothesis.

If the important role of a key position is confirmed, it can be assumed that the key positions can be found in other situations as well and that their mastering will play a similarly crucial role. Searching for and characterizing the crucial positions while creating generic and abstract models will be the subject of further research.

From the point of view of a teacher when working with students, it's very important to focus on the understanding and, if possible, on the unassisted derivation of solutions in case of these key positions.

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CLASSIFICATION, MANIPULATION AND COMMUNICATION: WORK WITH PUPILS AND STUDENT TEACHERS

Darina Jirotková, Graham H. Littler
Charles University in Prague, CZ; University of Derby, UK

Abstract. *Pupils' tactile manipulation of 3D solids is closely linked to their geometrical understanding. The links found between manipulation and communication when pupils are solving classification tasks are described in the first part of the paper. The second part opens new research working with future teachers to determine their abilities to create structures within the family of 3D solids. No significant knowledge difference in the ability to build geometrical structures between 10/11 year old pupils and student teachers was found.*

Key words: tactile perception, manipulation, classification, communication

INTRODUCTION AND AIMS

In our previous research aimed at learning about pupils understanding of geometrical solids using games and non-standard tasks[1], we found that pupils' ways of tactile manipulation of solids could be classified into three levels: global, random and systematic (Littler & Jirotková, 2004). This paper reports a further development of the research, the aim of which was to see whether, by asking the pupils to communicate their thoughts as they carried out classification tasks, there were links between manipulation and communication. We found these actions were related and reflected the pupils' geometrical insight and understanding. The results from the research from the pupils' work prompted us to work with student teachers to see whether or not their knowledge of 3D geometry mirrored that of the pupils. Our long term research has shown that many pupils have only a limited knowledge of both two and three dimensional shapes (Jirotková & Littler, 2002, 2003), and by comparison of syllabuses, text-books and teachers' comments in several countries we have found that little time is given to geometry in primary schools. Moreover, the dominant part of this time is given to transmissive methods of teaching. Where three-dimensional shapes are dealt with in primary schools, the teachers generally concentrate on terminology rather than concept building processes.

Fujita and Jones (2002) state that the teaching and learning of geometry is a major problem in mathematics education and cite Villani (1998) and the Royal Society (2001) reports which consider that good teaching methods are needed to be developed together with appropriate activities and resources if there is to be an improvement in pupils' knowledge and understanding of the subject. Much of current research focuses on the difficulties pupils have understanding geometrical theory particularly at the secondary level, and neglects the relevance

[1] By non-standard tasks we mean tasks which cannot normally be found in text-books in CZ and UK.

of the practical nature of the subject. We believe that this problem has its root in the primary school where the pupils are not given tasks which will deepen their understanding of objects and their relationships.

We have found that by solving tasks using tactile perception only, pupils are able to create a mental picture on the basis of the information delivered to the brain by tactile perception and then describe this picture. To be able to do this successfully, it requires many experiences of handling solids and describing what they perceive using appropriate language before the solid becomes a 'personality' for them (Vopenka, 1989). As a simile a wine-taster would have to taste many wines a considerable number of times building up a suitable descriptive vocabulary before s/he was able to undertake a 'blind tasting' in which the task was to classify and describe the wines by region, grape and date. Pupils take on a similar role when they are asked to describe a solid by tactile sense only. They have to recognise the attributes which characterise the solid and then find suitable words to describe these attributes. Previous work has shown that the tasks we set the pupils help to develop their structural approach to 3D solids, identifying three kinds of geometrical structures – single solid, cross solid and web structures (Jirotková & Littler, 2005)[2].

FRAMEWORK AND LITERATURE

In her thesis, Jirotková (2001) guided by M. Hejný considered some of the mental processes, mechanisms, in solving 3D geometrical tasks. The defining of these processes came from the analysis of pupils' responses to the tasks. The mechanism most applicable to the research task is the *Mechanism of Tactile Classification* (MTC).

The mechanism of tactile classification is our construct of the pupil's mental process by which s/he divides a group of solids tactilely into two groups such that at least one of the groups has solids which all have a common attribute. We identified three types of this mechanism:

MTC1. The pupil makes a mental note of the first tactile perception, which is associated with certain geometrical phenomenon recalled from his/her long-term memory. This phenomenon becomes the criterion for classification. This process of classification is started by tactile perception and is completed by recall of certain geometrical concepts from the long-term memory. It could be graphically expressed by the diagram below:

MTC2. The decision regarding the choice of criterion takes place before any tactile perception occurs and is based on the pupils' knowledge of solids, which is immediately recalled from long-term spatial memory.

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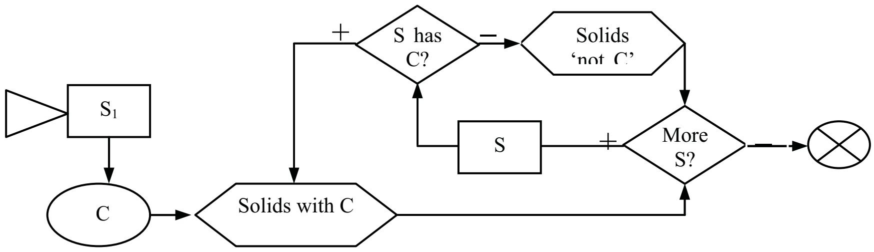
[2] '*single-solid structures'* are the structures of the attributes within a solid itself , *'web structures'* are developed when several solids are linked in pairs or small groups by some attribute but not to each other and *'cross-solid structures'* are those structures which linked several solids to each other by each having the same attribute.



S_1 - the first solid chosen. C –an attribute is recognised and used as the criterion for selection. S - a solid is taken.

MTC3. The pupil sets up his/her criterion, A, on the basis of their initial tactile perception of the solids. After making the first selection, he/she tactilely perceives another solid, which matches the first criterion but has another dominant attribute. If the next solid also has this other dominant attribute then this attribute takes over as the criterion, B, by which the solids are selected but still put in then original group.

If the decision for classification is based on the question: 'has the solid got the attribute A or not?', then we call this *complementary* classification. One subgroup is characterised by property A, the remainder of the set, the complement, is characterised by not A. If the decision for classification is based on the question: 'has the solid got attribute A or B?', then we call this *attributal* classification – each subgroup within the given set is defined by a particular criterion. Often the pupils' method of classification is not clear.

Our earlier work with 11 year-olds (Littler & Jirotková, 2004) in which we analysed video recordings of the pupils whilst they were undertaking the task enabled us to distinguish three different processes of tactile manipulation which we call the *Type of Tactile Manipulation*.

The type of tactile manipulation is the process by which a pupil manipulates a solid to determine its 'shape' and transfers this perception into a mental image of the solid. We delineated three types of manipulation: global, random and systematic. These manipulation processes mirror closely the first three levels of insight into three-dimensional geometry given in van Hiele's theory (1986) which would apply to most primary age pupils and are juxtapositioned below.

Global. The pupil holds the solid in (both) hands getting a perception of the whole solid (a gestalt), the solid is rotated in the hands but there is no attempt to focus on any attribute.

Level 1. *The solid is seen as a whole and usually as a specific shape. The properties/attributes of the solid play no part on the recognition of the shape.*

Random. The pupil usually holds the solid in one or both hands and uses their fingers to feel certain attributes. There is no order to the checking of attributes. A vertex might be touched and this followed by a fac, rather than follow through to see how many vertices a particular face might have.

Level 2A. *The shape is identified by a single attribute rather than the whole shape.*
Systematic. The pupil holds the solid in one or both hands and feels systematically for attributes with fingers. In this process the pupil checks all the faces say, then checks how many edges each face has or how many vertices and so built up a composite mental image of the solid.

Level 2B. *The solid is recognised by several of its properties/attributes, which are seen as independent of each other.*
Level 3. *The attributes are considered logically and the relationship between them recognised.*
(Van Hiele's level 2 was split into two parts by Pegg (1997))

If the pupil's understanding of 3D solids is at the Van Hiele level 1 then their manipulation will be global and they are likely to describe using everyday language. A better mental image might be achieved if the solid was familiar to them but they are likely to give the name only, such as cube and not describe its attributes. Pupils with Van Hiele level 2 understanding use random manipulation and we believe the mental image generated is not clearly defined. Those pupils who apply systematic manipulation and build up their mental picture of the solid sequentially are at Van Hiele's level 2B or 3 of 3D geometrical understanding. They sometimes feel the solid completely before gaining information about the attributes of the solid in a logical sequence. This enables them to get a clear mental image of the solid, even though they may not have met the particular solid before. It also helps them to keep the attributes in their short-term memory, which enables them to recognise cross-solid structures (Jirotková & Littler, 2005) when classifying their solids. It can be seen that as a pupil moves from general to random and systematic manipulation, they move from a general descriptive recognition of a solid to an analytical one (Rønning, 2004). Using tactile perception only it is much more difficult than is the case when visual perception is allowed since such things as similarity, equality of lengths can be seen, faces of solids recognised as two-dimensional shapes and pattern recognised (Callingham, 2004).

The new aspect of this research is the requirement of the pupils to 'commentate' on what they are thinking and doing as they go through the process of classifying the thirteen solids through tactile manipulation. The importance of communication in the diagnosis and education process (Hejný, 2003; Sfard 2002) was an aspect of an earlier paper (Jirotková & Littler, 2003). We were aware that most pupils' mathematical vocabulary comprised everyday language, specific words only used in mathematics and words which have different meanings in mathematics and general language usage. We were interested to see whether the language the pupils used was related to their tactile manipulation and therefore to their classification of the solids. Their mental image has to be described and the quality of the image together with the pupil's ability to communicate and use appropriate language will determine whether or not a second person can recognise the solid so described.

THE EXPERIMENT WITH PUPILS

Methodology

Over the last two years the authors have used the same task in the United Kingdom and the Czech Republic with 11 year pupils (grade 5), 10 year pupils (grade 4) and an 8 and a 6 year old pupil. The basic task has remained the same during this period, namely to divide a set of thirteen solids into two groups with at least all the solids in one group having a common attribute. In the last year we have asked the pupils to give a commentary on their thinking as they manipulated the solids. These commentaries have been transcribed and viewed against the relevant video to see what the pupil was actually doing as s/he spoke and to determine what connection there is between these two processes.

The solids used (of a suitable size for their hands) in the task were: 1. Cube; 2. Square based prism; 3. Rectangular prism; 4. Triangular prism; 5. Non-convex pentagonal prism; 6. Hexagonal prism; 7. Tetrahedron; 8. Square-based pyramid; 9. Truncated rectangular-based pyramid; 10. Non-convex pentagonal pyramid; 11. Cylinder; 12. Cone; 13. Truncated cone.

The solids were chosen to present the pupils with some that they would recognise from everyday life or school; cube, cuboids, cylinder and cone, some which they might have met; pyramid, tetrahedron, hexagonal prism and right-angled triangular prism, and those which we thought might be new to them: non-convex pentagonal prism and pyramid, truncated cone and pyramid.

These solids were either hidden behind a screen through which the pupils put their arms or the pupils were blindfolded. All the experiments were video recorded. These were transcribed and the actions of the pupil as seen on the video were recorded in written form. The transcriptions and the description of the pupils' actions were used in the analysis for this paper. The task was undertaken individually by each pupil in a quiet room, the researcher explaining what the task was, which included the pupil giving a commentary on what s/he was thinking as they made their decisions. At no time from the pupil entering the room to completion of the task, were they able to see the solids.

Results and discussion

We have initially divided the results and discussion into the three different aspects of our research - classification, communication and manipulation and hope to show the linkages found between these aspects.

(i) Classification. The first solid the pupils picked up had a great impact on the criterion they used for selection. For those pupils at the MTC1 stage, this first solid gave them a strong characteristic, which they were able to sustain throughout the task. For instance a pupil felt the rectangular based prism first and used the criterion: *These solids have only got quadrilateral faces. The other group may have quadrilateral faces but they have other faces as well.* His complementary classification comprised the cube, cuboids and truncated prism.

Some pupils process was at MTC3 stage, one of the 10 year old girls was using the phenomena of 'four-sideness' to begin with (audio-tape), but when she handled the cone she construed this as *irregular* and from then on put those solids which she thought of as regular with her four-sided group. These included the square-based pyramid, tetrahedron and the cylinder. It was noted that her interpretation of 'irregular' varied from meaning *I have not met this solid before* to saying what she thought was irregular about the solid – *this is a cone too but it hasn't got a bottom*(truncated cone, apex down). From the analysis of the commentaries there was no indication of pupils using MTC2.

(ii) Communication. Our analysis of the commentaries showed a difference between the geometrical language used by the 11 year old pupils and those of 10 years. The 11 year-olds clearly indicated that when they used a 2D geometrical term it was being applied to a face, as in the example above. When they referred to the solid they generally used 3D terms. However there was geometrical confusion in the minds of two girls who used 'square or rectangular *base*' for their criterion. If the square based pyramid was laid on a triangular face they did not want to put this solid in their original selection, since it was now a triangular–based solid.

The 10 year-old pupils used a much wider selection of language: some geometrical 2D and 3D terms were used correctly (triangle, rectangle, square, hexagon, circle, cube, cuboid, cylinder, cone, pyramid); there were geometrical terms with distorted meaning (pentagon – miscount of sides on hexagonal face, polygon for polyhedron, side for edge); some terms were the pupils' construct (circular triangle for cone, four-cornered triangle for trapezium); every-day words were common (points, flatness, curves, tent for triangular prism, diamond for hexagon); and some made use of descriptions ("bottom missing" and "cut-off at the end" for truncated cone, "stereotype solids" for cube, cuboid, part missing, piece taken out of it, piece bitten from it for non-convex pentagonal prism or pyramid). These pupils did not use the word face but from the video it could be seen that when they used phrases such as *like a triangle* they were feeling a triangular face.

For these younger pupils, quadrilaterals were either squares, rectangles or 'four-sided'. Similarly it was the minority of this age-group who used cuboid and cone. The girl who described the cone as a *circular triangle* was rotating the cone by its vertex with one hand whilst the other felt the curved surface. Two pupils used everyday language most of the time, one girl only using two geometrical terms during the task, circle and square. She used the word *points* six times and *flatness* four times. Pointedness seemed to be the dominant attribute for her when describing her tactile perception and yet her final description of the two groups was *this one(*group) *with curves and this one with flatness.* It was interesting to hear another pupil of Afro-Caribbean origin describe the tactile perception of the sides of the hexagon to be *like a diamond,* which could be because the women in that society wear many rings each having multi-facetted 'gems'. Many pupils found the non-convex pentagonal solids confusing, *it's funny* was one comment and there were several descriptions of 'a piece missing'.

(iii) Manipulation. The first point we noted was that only one boy (11 years) from all those tested took the trouble to feel all the solids carefully, to get a mental picture of the solids, before starting to solve the task. He counted the number of edges, faces and vertices as he traced the outline of each solid with his finger and only when he had completed this did he attempted to divide the solids into two groups. In our most recent experiments the pupils (10 years) either laid their hands over the tops of all the solids without feeling individual ones and/or took the first solid their hand encountered and began to analyse it. This pupil showed that his manipulative skills were most advanced, his commentary indicated he had clear within-solid and cross-solid structures developed and that his level of communication was high and mirrored his manipulative level.

The majority of the 11 year-old pupils and some of 10 years were at the systematic level of manipulation. As reported earlier the communication of the 11 year-olds showed great mathematical sophistication and this paralleled their manipulative skills. Many felt the solid as a whole and then held it with one hand whilst investigating the dominant perception with the other, for instance with square-based pyramid, they usually felt the square face first and then the triangular faces, then counted the number of edges before making their decision. In general, the solids were picked up separately but occasionally a pupil picked up more than one solid for comparison. Very often the 10 year-old pupils used only 2D language in their commentary which reflected the way they manipulated with the solids, usually at the random level. It was noticeable that they made a decision about a solid and did not refer to it again, whereas the 11 year olds kept re-feeling the solids about which they had made their decision, to compare their attributes with the one being investigated. Three of the 10 year old pupils were at the random manipulation level, their feeling of the attributes having no order. Their communication was generally in everyday or descriptive terms. The solids easily recognised by these pupils were the cube, cuboids, cylinder and square based pyramid since the tactile perception of them was quickly matched to an image in the long-term memory. These shapes were personalities for the pupils in Vopenka's terms.

(iv) General comments. It was clear that there was a link between the level of sophistication of the language used and the level of manipulative skill of the pupil. The level of manipulation indicated the soundness of the grasp of both the within solid structure and the cross solid structure. This was shown by those pupils who manipulated the solids well and were able to communicate about the common attribute especially. Our task helped the pupils to look for the attributes of the solids hence building up the within solid structure and finding the common attribute in a number of solids enabled the pupils to have experience of noting a particular attribute in several solids and then giving this as the 'generic' attribute for a group of solids (Hejný, 2005). Even the younger pupils formed the within solid structure quickly by working through the task.

Following our analysis of these experiments we considered whether or not the lack of geometrical structure, which we had found in many pupils, was remedied as they progressed through secondary school. We thought that the primary school student teachers in our universities might give us research data on this.

EXPERIMENT WITH STUDENT TEACHERS

Methodology

We worked with 28 students in the Czech Republic who were at the beginning of a geometry course in the second year of their study and 32 students in England undertaking the one-year post-graduate certificate of education course. We first gave them a questionnaire in which we asked them what they considered most important in geometry. The overwhelming response was that they had to be precise and careful, particularly when drawing constructions; they needed to be able to draw well; they had to remember theorems and formulae.

With each group, we discussed 2D and 3D shapes and they had the opportunity to handle the thirteen solids we had used with the pupils. Several sets of the solids were placed around the room so that the students could use them if they wished. Finally we presented them with a worksheet, which was a picture of the thirteen solids.

The task we gave them was to use the worksheet to establish as many links/relationships/common attributes as they could between the various solids, joining them by different coloured lines and writing down the relationship implied by the links in a key. When this was completed they were then asked to choose two of the solids and to list those attributes which were the same and those which were different.

Results

(i) Classification: There were very few students who used a point on the paper to show a many to one relationship, or for instance, showing the transitivity between the six solids which all possessed the attribute 'had at least one face a square'. The majority of them drew a single line between each solid to link them. The attributes used to describe the relationships were faces, non-convex solid and truncated solid, both of the last two phrases had been given in the previous discussion. There were no relationships given which one could classify as invisible feature such as ‘needs two measures to calculate its volume’. The cube was described by seventeen different words or phrases. The question of whether the square was a member of the family of rectangles was answered clearly by the results. Most students considered the square as a separate shape so that the cube was not included in the relationship ‘Have all rectangular faces’. This last point was more clearly defined in the results of the Czech students where the square is not considered a special case of a rectangle. The truncated pyramid and cone were often linked in the relation ‘is a pyramid’, and only one student included the tetrahedron in those who used the relation ‘has triangles’.

The student teachers found the second task difficult since their answers showed a lack of analytical approach to the comparison of the two solids. The choice of solids was interesting, some choosing very different solids in the hope of defining differences easily and others choosing similar solids to determine similarities easily. Both types found the counter analysis difficult. One student who chose the square based prism and the truncated pyramid wrote that the attributes which were the same were ‘6 sides, 8 corners and 12 edges’; those that were different were ‘the truncated pyramid is smaller at one end and the long sides taper inwards’.

(ii) Communication: During our discussions before setting the task, many of the students commented that they did not recognise many of the solids and so could not give them names. They exhibited all the problems we had encountered with pupils such as often using words from everyday language to describe the solids and their attributes. In addition they used different words when speaking about the same attribute indicating that they were not sure of the correct meanings of the words. The analysis of the students’ responses to our tasks showed that their vocabulary ranged from everyday language such as ‘Have pointed tops’ and ‘Have flat tops’; incorrect mathematical vocabulary such as using the word ‘side’ when they were speaking about a face of a solid; and correct mathematical terminology – ‘Are truncated solids’ and ‘Has 12 vertices’.

Conclusions

Until the pupils worked with ‘our tasks’ many of them had not consider solids analytically and so were unable to build up the geometrical knowledge structures listed earlier in the paper. Their knowledge was fragmental and their vocabulary limited to common solids and 2D shapes. The work which we did with student teachers would suggest that nothing done in secondary geometrical syllabuses helps develop these structures. We found the same use/misuse of mathematical language in the two different age-groups and the student teachers in general did not show they had firm within, or cross solid structures in their minds.

We believe that primary school geometry should focus on the concept building processes in the widest possible meaning. Therefore we recommend that our tasks could be used to advantage with primary school pupils and with student-teachers in their mathematics pedagogical courses. We would repeat that we feel the benefit of using tactile perception in many of these tasks is that it enables the pupils/students to build up their geometrical structures more quickly.

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INVESTIGATING THE PROCESSING STRUCTURES OF STUDENTS' INDUCTIVE REASONING IN MATHEMATICS

Eleni Papageorgiou & Constantinos Christou

Department of Education, University of Cyprus

The study aimed to empirically test a theoretical model formulated to identify and classify students' processing structures when they solve inductive reasoning mathematics problems. Based on previous theory (Klauer, 1999), we constructed a mathematical test in order to assess the components of inductive reasoning of sixth Graders. The data suggested that inductive mathematics reasoning is a process consisting of three factors: the "similarity", the "differences" and the "similarity and differences" factors. Each of these factors involves the integration of other subordinate procedures. The proposed model provides a framework of students' thinking while solving various formats of inductive mathematics problems, and a prototype for further investigation of the components of inductive reasoning.

INTRODUCTION

The purpose of this research is the validation of a new model, which describes the components of students' mathematics inductive reasoning. The new model is based on Klauer's definition of inductive reasoning (Klauer, 1999). Although Klauer's definition refers to general psychological tasks of inductive reasoning, the proposed model provides an integral approach to mathematics inductive reasoning tasks.

Reasoning, in general, involves inferences that are drawn from principles and from evidence, whereby the individual either infers new conclusions or evaluates proposed conclusions from what is already known (Johnson-Laird & Byrne, 1993). Inductive reasoning is the process of reasoning from specific premises or observations to reach a general conclusion or overall rule. It usually refers to the given instances and does therefore reach conclusions that are not necessarily valid for all possible instances (Klauer, 1999). Thus, the inductive reasoner can only use probable conclusions to predict further instances (Sternberg, 1999). In this paper we refer to the mathematical inductive reasoning of students of the primary education when they solve various formats of mathematics inductive tasks.

The process of drawing inductive conclusions about general laws in mathematics starts with single observations, which are combined with the strength of previous observations in order to arrive to a conclusion. However, the derived conclusion is not necessarily accurate or logically valid. Nevertheless, in many cases the inductive inferences are valid and provide an important basis for the understanding of regularities in mathematics. Regularities as well as uniformities are the basis for the generation of concepts and categories, which play a fundamental role in our every–day life (Klauer & Phye, 1994). Thus, the focus of our study is on the inductive reasoning as it is required in primary mathematics problems and in most intelligence tests (e.g. analogies, classifications, series completion problems, matrices etc.). The student's task is to discover the pattern of relations or attributes among several elements given in a problem.

Specifically, the study aimed (a) on the integration of the processes students use when they solve various mathematics problem types into one common classification scheme prescribed by a model, and (b) on the development of an instructional program, which specifies the thinking skills typical of inductive reasoning. This paper presents the results of the first purpose of the study and delineates the verification of the inductive reasoning model.

THEORETICAL BACKGROUND

The process of inductive reasoning has been a topic of considerable interest in mathematics education, and is one of the most important goals of the curriculum of mathematics (Serra, 1989, NCTM, 2000). Indeed, inductive reasoning plays a critical role in mathematics and in problem solving situations (Koedinger & Anderson, 1998). However, in primary school, the stimulation of thinking skills is not pursued explicitly. It is usually assumed that these skills develop as a by-product of the teaching of content as defined in traditional curricula for different subjects (Hamers, De Koning, & Sijtsma, 1998). As a result, most students cannot comprehend the basic concepts of mathematics and have a lot of difficulties in solving problems. To remedy this limitation, a range of programs for training of thinking has been developed (Klauer, 1999; Hamers & Overtoom, 1997). For example, Klauer (1988) developed a training program that can be used to both teach and assess the students' inductive reasoning and problem solving abilities. Klauer's program was based on an analytic definition of inductive reasoning. Specifically, Klauer defined inductive reasoning as the systematic and analytic comparison of objects aiming at discovering similarities and/or differences between attributes or relations (Klauer, 1999). This definition results in the identification of six classes of inductive reasoning problems according to the cognitive processes required for their solution: the generalization, the discrimination, the cross-classification, the recognizing relationships, the differentiating relationships and the system construction problems.

The generalization problems are characterized by the need to find similarity of attributes for different objects in order to make up a group, while the discrimination problems are related with noting differences among objects with respect to attributes, in order to identify the object that does not fit in with the others. The cross-classification problems are characterized by a classification scheme, in which at least two attributes must be considered simultaneously. All combinations, which are possible, will occur: similarity in both features, dissimilarity in both features, similarity of one of the features with differences in the second feature, and vice versa. The solution strategy of the cross-classification problems requires a determination of both common and different attributes. The recognizing relationships problems' subset includes analogies and sequences' problems and involves the identification of similarity among relationships. Specifically, analogy problems require the determination of a specific relationship between a given pair of objects. Then the solution strategy consists of mapping the relationship onto an incomplete pair in order to establish a new pair of objects that exhibit the same kind of relationship. Similarly, the solution strategy of sequences' problems is related to the sequential check on the relationships that exist between the pairs of objects in a given series, in order to complete the series with a missing object, or the recognition of a pattern among objects, in order for them to be arranged sequentially. The differentiating relationships problems are concerning sequences that include an object which disturbs the sequence. This class of problems involves the recognition of differences in relations, and therefore they require the sequential check of the relationships existed between consecutive objects in a given series in order to identify and exclude or correct the object that disturbs the sequence. Inductive problems that are included in system construction subset consist of series of objects arranged in a nxm matrix ($n \geq 2$). One of the cells of the matrix is empty, usually the right down, and the solution strategy is related with the determination of at least two relationships that must be recognized and applied in order to generate the correct content of the empty cell (Klauer & Phye, 1994).

It is obvious that the above six classes of inductive reasoning problems are interrelated, since all of them can be solved by a core strategy of inductive reasoning, namely the process of comparing. Klauer (1999) developed an analytic and a heuristic strategy, both of which share the core process of comparing. The analytical strategy compares objects with respect to their common attributes or relations. After evaluating all objects regarding the similarities and dissimilarities of all attributes or relations, the problem solver will discover the rule, and consequently the solution. The strategy assumes that the problem solver is able to recognize all attributes or relations inherent in the problem. Specifically, the generalization and the recognizing relationships problems' solution is considered to be based on finding similarities among attributes or relations, while the solution of the discrimination and the differentiating relationships problems is considered to be based on finding differences among attributes or relations. In the same way, the cross-classification and the system construction problems' solution is based on finding similarities and differences among attributes or relations (Klauer, 1999). Mastery cross-classification and system construction is considered to be the final stage of inductive reasoning.

Even though many research projects were developed to validate this classification system proposed by Klauer (Roth-van der Werf, Resing, & Slenders, 2002; Klauer, Willmes, & Phye, 2002; Hamers, et al., 1998; Klauer, 1992, 1996; De Koning, Hamers, Sijtsma, & Vermeer, 2002), all of them have been developed in a general content domain. Accordingly, the literature does not provide any coherent picture of the reasoning processes required for the solution of the problems included in specific academic subject areas. Considering the importance of inductive reasoning in mathematics education, there is a need for a framework of cognitive processes that can be used in solving mathematics inductive tasks, in order to foster children's inductive reasoning ability in mathematics. The model proposed in the present study adapted Klauer's classification scheme specifically for the content of mathematics. Its focal point is on the structures of the problems and the corresponded cognitive processes that are necessary to solve them.

THE MATHEMATICS INDUCTIVE PROBLEMS

One of the main purposes of the present study was to develop a model that encompasses the whole spectrum of inductive reasoning processing structures. Thus, the proposed model presupposes that the whole set of inductive mathematics problems is consisted of problems of classification, analogy, series and matrix (Van de Vijver, 2002) and/or similar varieties and combinations of these problem-formats. The proposed model was hypothesized that the set of inductive mathematics tasks can be classified into two main subsets, according to their reasoning structure. That is, classification tasks which are dealing with scanning attributes, and analogy, series and matrices tasks, which are dealing with scanning relationships. Classification tasks required detecting similarity and/or differences among attributes, while analogy, series and matrices tasks required detecting similarity and/or differences among relationships. Thus, it was hypothesized that three kinds of cognitive procedures are required for the solution of the tasks of the two subsets, i.e., detecting similarity or differences or both (Klauer, 1999).

In order to specify the cognitive processes required for the solution of the problems included in each subset and therefore to classify the problems according to their processing structures, it was hypothesized that detecting similarity strategy is applied on problems required one to note an attribute common to each number or geometric figure included in the problems in order to form a class (generalization); on problems required the comparison of properties of numbers or geometric figures of a given set, in order to distinguish the attribute have in common some of the given numbers or figures (generalization); on series problems required the sequential check on the relationships that exist between the pairs of figures or numbers in a given series, in order to complete the series with a missing figure or number (recognizing relationships); and on analogy problems required the determination of a specific relationship between a given pair of numbers or figures and the application of that relationship onto an incomplete pair in order to establish a new pair of numbers or figures that exhibit the same kind of relationship (recognizing relationships). Detecting differences was hypothesized that it is applied on the problem format required the identification of the number or geometric figure that does not fit in with the others with respect to attributes (discrimination) and on number or geometric figure series problems that required the exclusion or the correction of the member that does not fit in with the others with respect to the recognized relationship, in order to define a correct series (differentiating relationships). The procedure consisted of detecting similarity and difference was hypothesized that it is applied on problem formats required the completion of a specific classification scheme represented by a 2x2 matrix, where at least two attributes of numbers or geometric figures included in a specific cell must be considered simultaneously (cross-classification), and on matrices where at least two relationships is to be verified (system construction).

METHOD

Participants

Participants were 139 Grade six students (71 females, 68 males), from seven existing classes at elementary schools in an urban district of Cyprus. The school sample is representative of a broad spectrum of socioeconomic backgrounds.

Instrument

Inductive mathematics reasoning was determined using a test that involved all six problem formats. Specifically, each student completed a 40-minute written test, which contained 25 inductive reasoning mathematics problems. The test included 12 classification problems, which are dealing with grouping objects with respect to attributes, and 13 problems that are dealing with the seriating of objects on the basis of their relationships. Five of the problems required detecting similarity of attributes, two problems required detecting similarity of relationships, four problems required detecting differences in attributes, five problems required detecting differences in relationships, three problems required detecting similarity and differences in attributes, and six problems required detecting similarity and differences in relationships. Examples of all the problem formats used in the test are shown in Table 1.

Data Analysis

Inductive mathematics reasoning was determined using a test that involved all six The assessment of the proposed model was based on a confirmatory factor analysis, which is part of a more general class of approaches called structural equation modeling. EQS computer software (Bentler, 1995) was used to test for model fitting. In order to evaluate model fit, three fit indices were computed: The chi-square to its degrees of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). These indices recognized that the following needed to hold true in order to support model fit (Marcoulides & Schumacker, 1996): The observed values for x^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be close to or lower than .08.

Types of problems	Class of problems	Tasks	Cognitive operation required
Classification tasks	Generalization	Find the common feature of the numbers: 4, 16, 8, 32, 20, 100	Similarity of attributes
	Discrimination	Underline the number that does not fit in with the others: 3, 5, 9, 15, 30, 81 Justify.	Differences in attributes
	Cross Classification	Write the number 24 in the appropriate cell. Justify. [6 18 12 36 | 16 22 4 8] [15 9 21 3 | 7 5 25 31]	Similarity and difference in attributes
Seriation tasks	Recognizing Relationships	Complete with the right number. 1 5 13 29 ………	Similarity of relationships
	Differentiating Relationships	Find the number that disturbs the sequence: 1 1 2 3 5 7 Justify.	Differences in relationships
	System Construction	Complete the cell with the appropriate number. [8 | 4 | 2] [24 | 12 | 6] [72 | 36 |]	Similarity and difference in relationships

Table 1: Examples of the six types of problems included in the test

RESULTS

In this study we proposed an a-priori model consisted of six first-order factors, three second-order factors, and one third-order factor. The first-order factors represented the structures of the cognitive processes required for the solution of inductive mathematics problems: Similarity of attributes (F1), similarity of relationships (F2), differences in attributes (F3), differences in relationships (F4), similarity and differences in attributes (F5), and similarity and differences in relationships (F6). The similarity of attributes (F1) and the differences in relationships (F4) factors were each measured by five tasks. The differences in attributes factor (F3) was measured by four tasks, the similarity of relationships factor (F2) was measured by two tasks, while the similarity and differences in attributes (F5) and similarity and differences in relationships factors (F6) were each measured by three and six tasks respectively.

The above six factors were hypothesized to construct three second-order factors: the "similarity in attributes or in relationships" factor (F7), the "differences in attributes or in relationships" factor (F8), and the "similarity and differences in attributes or in relationships" factor (F9). The second-order factors were hypothesized to represent the inductive reasoning procedures and are postulated to account for any correlation or covariance between the first-order factors. Finally, the F7, F8 and F9 factors were hypothesized to construct a third-order factor "inductive reasoning strategy" (F10) that was assumed to account for any correlation or covariance between the second-order factors.

Figure 1 presents the structural equation model with the latent variables and their indicators. The descriptive-fit measures indicated support for the hypothesized first, second and third order latent factors (χ^2/df=1.29, CFI=0.912, and RMSEA=0.046). The fit of the model was very good and the values of the estimates were high in all the cases. It is clear that the three-level architecture accurately captures the data. Specifically, the analysis showed that each of the tasks used in measuring inductive reasoning in mathematics loaded adequately on each of the six cognitive processes (F1-F6), as shown in Figure 1. This finding indicates that similarity of attributes, differences in attributes, similarity and differences in attributes, similarity of relationships, differences in relationships and similarity and differences in relationships can represent six distinct functions of students' thinking in solving inductive mathematics problems. These six factors were regressed on three second-order factors: the "detecting similarity", the "detecting differences" and the "detecting similarity and differences" factors, which in turn were regressed on a third-order factor which concerned with the inductive reasoning strategy. Therefore, the three-level model, as it is presented in Figure 1, is consistent with the theory.

DISCUSSION

Inductive reasoning considered as one of the most important goals of mathematics education, because of its fundamental role to the learning and performance in mathematics and in problem solving situations (Serra, 1989, NCTM, 2000). Even though research demonstrated the importance of inductive reasoning in mathematics and in problem solving, the literature does not provide any framework of the types of cognitive processes used for the solution of inductive mathematics problems. Hence, the goal of this study was to formulate and validate a theoretical model of cognitive processes used in various types of inductive mathematics problems. The design of the model was based on Klauer's classification scheme of the structures of the cognitive processes required for the solution of inductive reasoning problems in a general content domain (Klauer & Phye, 1994; Klauer, 1999). Thus, this model

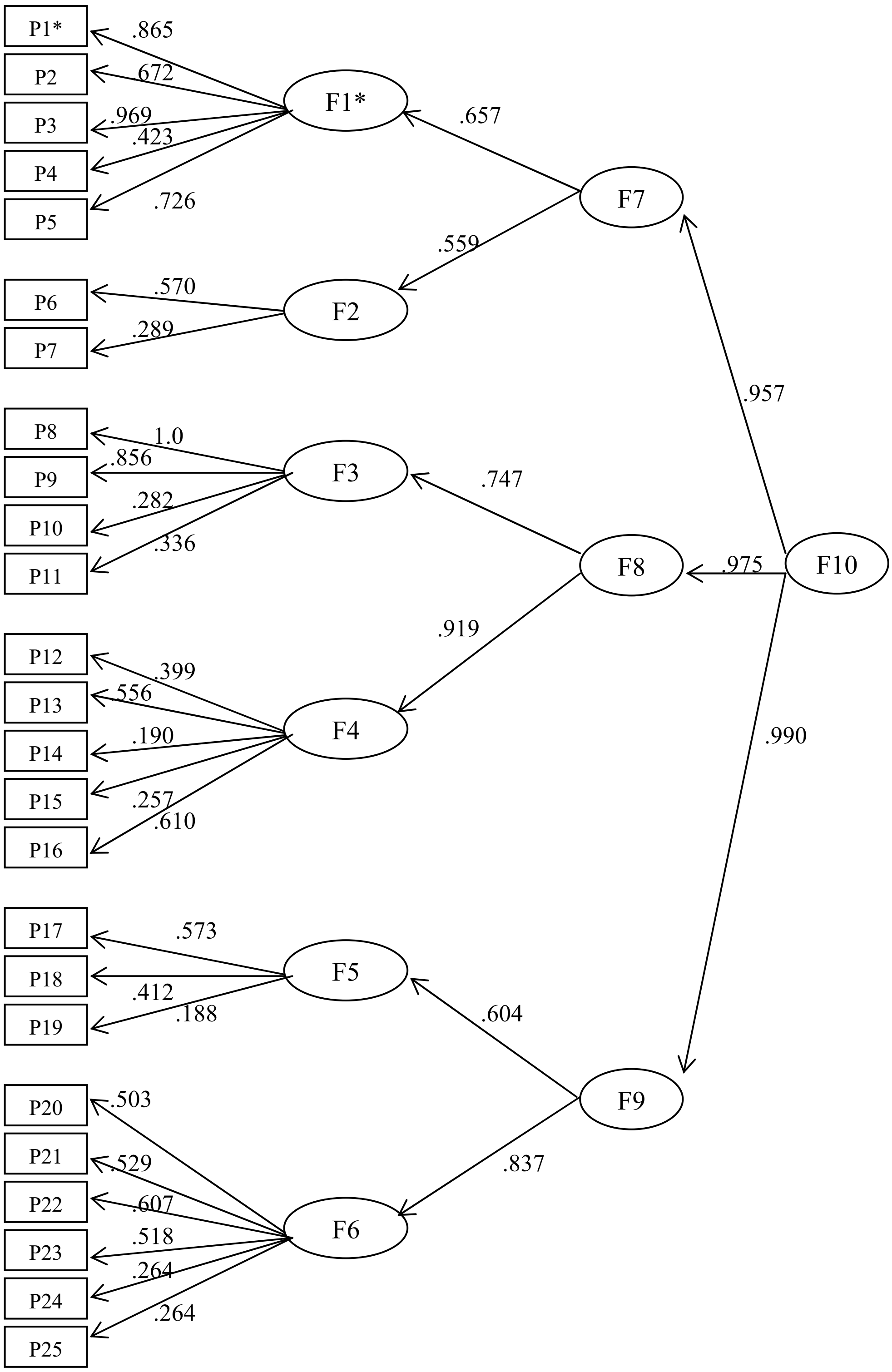

Figure 2: A structural model of the inductive mathematics problems

* P1-P25 refer to the problems assigned to students, F1=Similarity of attributes, F2=Similarity of relationships, F3= Differences in attributes, F4=Differences in relationships, F5=Similarity and differences in attributes, F6=Similarity and differences in relationships, F7="Detecting similarity" process, F8="Detecting differences" process, F9="Detecting similarity and differences" process, F10=inductive reasoning strategy.

extended the literature in a way that similar processes are recognized and can be used for the solution of mathematics problems of inductive reasoning.

The proposed model proved to be consistent with the data, leading to the conclusion that six distinct cognitive processes aiming at detecting similarity and/or differences in attributes or relationships are used for the solution of inductive mathematics problems dealing with attributes or relationships. Specifically, three types of strategies consisted of detecting similarity of attributes, differences in attributes, or

both required for the solution of classification tasks. Classification tasks include problems that required one to form a class by selecting objects which have a common attribute or to determine the common attribute having the objects of a defined set; problems required one to identify the object that does not fit in with the others in a defined set; and classification problems represented by 2x2, 2x3 and 3x3 matrices. Similarly, three types of strategies consisted of detecting similarity of relationships, differences in relationships, or both, are used for the solution of analogy problems, sequences' problems and matrices.

Taking into consideration that inductive reasoning ability improves the learning of mathematics, this model offers teachers a framework of students' thinking while solving various formats of inductive mathematics problems; it can be used as a tool in teachers' instruction. For the research, the proposed model could be useful as a prototype for further investigation of the processes used older students while solving tasks of specific mathematical areas, such as finding a function.

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EMPIRICAL HIERACHY OF PUPILS' ATTAINMENT OF MEASUREMENT IN EARLY PRIMARY SCHOOL YEARS

Alexandra Petridou, Maria Pampaka, Julian Williams, Lawrence Wo
University of Manchester

Constantia Hadjidemetriou
Intercollege

This study describes a developmental 'map' of performance in the context of measurement in the early primary school years (ages 5-7). This study uses data from diagnostic, age-standardized tests from a sample of 5120 pupils in England. The map was constructed using Rasch measurement methodology and specifically the Partial Credit model. This model enables to describe typical misunderstandings and errors alongside a constructed hypothetical learning trajectory. We interpret the scale from the analysis as a hierarchy of five levels of measurement performance. We then compare this empirical hierarchy with the one described in the English National Curriculum for mathematics. Finally, we discuss educational implications of our findings.

INTRODUCTION AND BACKGROUND

Measurement is one of those topics in mathematics which can be regarded as a prime candidate for the *maths is useful* description (Hart, 1981, p.21). This is because through measurement we apply mathematics in many aspects of our personal and professional lives. Therefore, measuring skills are considered to be life skills. Education stakeholders have long recognised the necessity and usefulness of measurement. The English National Curriculum for mathematics outlines not only objectives for developing numerical concepts but also objectives for developing measuring concepts. In fact, objectives that refer to measurement are included among the key objectives [1] for each yearly teaching programme for ages 5-11 in England. In Table 1 we present the key objectives that are relevant to the early years of primary education.

The purpose of this study is to construct a performance hierarchy in the context of measurement and specifically in relation to three aspects of measurement (length, capacity and mass) across early primary school years (ages 5-7). According to Hart (1981) the word 'hierarchy' is used in a number of ways when applied to how and in what order children learn mathematics. In this study the word hierarchy is used to describe a staged hierarchy of attainment or a hypothetical developmental scale based on empirical evidence.

Table 1: Key Objectives related to measurement as defined in the English National Curriculum

National Curriculum Key Stages	Year groups	Key Objectives
KS1	R	-Use language such as more or less, greater or smaller, heavier or lighter, to compare two numbers or quantities.
	Y1	-Compare two lengths, masses or capacities by direct comparison. -Suggest suitable standard or uniform non-standard units and measuring equipment to estimate, then measure, a length, mass or capacity.
	Y2	-Estimate, measure and compare lengths, masses and capacities, using standard units; suggest suitable units and equipment for such measurements. -Read a simple scale to the nearest labelled division, including using a ruler to draw and measure lines to the nearest centimetre.

According to Copeland (1984), the investigation of the ways through which children come to measure is particularly interesting because "the operations involved in measurement are so concrete that they have their roots in perceptual activity (visual estimates of size, etc.) and at the same time so complex that they are not fully elaborated until sometime between the ages of 8 and 11" (p. 254). This means that children's understanding about measures depends on concepts which are not completely developed until sometime towards the end of primary education. This has special implications for this study as the purpose is not just to present increasing levels of measurement understanding and capability but also to present associated errors and misconceptions that accompany this development. Errors and misconceptions provide not only insights into the growth of understanding measures but also useful diagnostic and formative information (i.e. information which can be used as guidance to bridge the gaps between current knowledge, understanding, or skill and the desired goal). Research evidence shows that high quality formative assessment does have a powerful impact on student learning (Black & William, 1998; Williams & Ryan, 2000).

Research into hierarchies began with Gagne and his co-workers (Gagne & Paradise, 1961). Since then a considerable amount of research in hierarchies (in different mathematical topics) has been carried out over the years, both empirical and theoretical (characteristic examples are Piaget's stage theory on the development of measurement in children and Van Hiele's levels of development in Geometry). The idea behind these research efforts was to identify and describe children's mathematical development and then to link it to the cognitive level of the child. This study is closely related to the work of those researchers in mathematics education who aimed to describe children's mathematical development through assessment data (Gagatsis, Kyriakides & Panaoura, 2001; Callingham & Watson, 2004; Doig, Williams, Wo & Pampaka, 2006; Davis, Pampaka, Williams & Wo, 2006). These studies combine the idea of competency with the notion of development in order to provide a process that both acknowledges what students can do but also indicates to teachers the next step in the learning process (Callingham & Griffin, 2005).

There are researchers however who oppose to this approach and suggest that the construction of learning hierarchies through assessment data might be flawed on educational and technical grounds (see Noss, Goldstein & Hoyles, 1989). We accept and acknowledge some of these hierarchies' limitations (for example, the adoption of the notion of context-free assessment, lack of accountability for differentiation of topic exposure among schools, invariant sequence for all children). Nevertheless these hierarchical constructs are useful in describing generalities especially in England where the National Curriculum imposes a specific structure to the teaching of mathematics (Davis et al., 2006). Such schemes can also enrich diagnostic assessment and help teachers organise their teaching based on this information.

This study aims to construct a developmental scale of pupils' measurement attainment in the early primary school years by using data from a large-scale assessment project, the Mathematics Assessment for Learning and Teaching (MaLT) at the University of Manchester with Hodder Murray. The MaLT project developed age-standardised diagnostic assessment materials for pupils aged 5-14 to be used for both formative and summative purposes. Test development was based on: (i) assessment research literature so that important strategies and conceptions were included and assessed by the tests and (ii) the National Curriculum for Mathematics in England so that the tests would be in agreement with the key objectives and difficulty levels described for each year group in the National Curriculum. The standardization of the tests involved a nationally representative sample of pupils. A total of approximately 14000 pupils aged 5-14 from 120 schools in England and Wales participated in this project. Validation analyses took place at both the pretest and main test stages and included test and item-fit, subgroup DIF etc. For the purposes of the current study only pupils' responses to measurement items included in the MaLT diagnostic assessment tools were used in order to construct a map of typical development. Specifically, we report only the test items that are targeted for pupils aged from 5 to 7 (i.e. Reception to Year 2 or Key Stage 1 as it is called in England). We have decided not to include items from the rest of the papers (i.e. for ages 8-11 or Key Stage 2 according to National Curriculum) as vertical equating becomes less reliable and problematic with changes in curriculum content. However measurement items from MaLT8 (i.e. for eight year olds) were also used in the analysis in order to create a link with items of Key Stage 2 (ages 8-11) items as we intend to develop a similar map for measurement concepts for ages 8-11, in the near future. MaLT8 items serve a linking purpose and will therefore not be discussed in this paper.

METHODOLOGY-CONSTRUCTING THE HIERARCHY

For the purposes of this study 15 measurement items were used (that were included in the 5-7 year old MaLT tests) which assessed aspects of length, mass and capacity and they were taken by 5120 pupils. MaLT item responses were analysed using the Rasch model.

The Rasch model is a mathematical representation of what happens when a person responds to an item. The cornerstone of the Rasch model is the invariance principle: that person–item interaction can be modelled by independent parameters for items (one 'difficulty' parameter for each item) and persons (one 'ability' parameter for each person). These two parameters represent the positions of the items and the persons respectively on the same latent trait. The Rasch model uses these parameters to determine the probability of a certain person succeeding on a certain item (more able/developed persons have a greater likelihood of correctly answering all items and easier items are more likely to be answered/reached correctly by all persons) (Bond & Fox, 2001). Both person ability and item difficulty parameters have a common measurement unit, the logit scale (an interval scale). This enables us to plot the items on a scale according to their difficulty and to locate pupils according to their 'ability/development' on the same scale (usually called item-person map).

In this study we were interested in plotting the pupils' ability and the items difficulty on the same scale and also to plot the most significant common errors that were observed. In order to achieve that, we employed the methodology suggested by Doig et al. (2006). Instead of using a simple dichotomous Rasch analysis we used the Partial Credit model (Masters, 1982). While the dichotomous model allows us to have only correct or incorrect responses (all incorrect responses are treated as equal), the Partial Credit model allows us to have one or more intermediate levels of success between complete failure and complete success. This model however requires that part marks to be awarded in an ordered way i.e. each response to represent an increase in the underlying ability/development. In this study partial credit was given only in cases of items with significant common errors. Erroneous responses were ordered according to the mean ability of students making that error (obtained from a dichotomous Rasch analysis) i.e. errors made by students of higher ability were given a higher order. For more details about the specific methodology employed see Doig et al. (2006).

RESULTS AND DISCUSSION

The Rasch analysis of the data suggested that the data fits the model well (according to the fit statistics). This indicates that items and children behaved in a consistent way with Rasch measurement assumptions.

The hierarchy

Table 2 shows a scale of performance, common errors and levels of understanding of measurement. Items that were harder for the pupils are at the top of the map while easier items are at the bottom. Similarly more able pupils i.e. those that correctly answered more items are at the top of the map while less able pupils are at the bottom.

Logits	High Ability Pupils	Harder Items	Common Errors	Levels
5.0				**L4** Making estimations Read measuring instruments accurately
	XXXXXX	8q11.2-M		
4.0				
	XXXXXXXX	8q23.3-L		
3.0				**L3b** Suggesting suitable measurement equipment
	XXXXXXXXXX		8q11.1	
	XXX			
	XX	8q24-C		
		6q13.4-L,M,C	8q23.2	
2.0	XXXXXXXXXX	**6q12.3-L**		
	X		6q12.2	
	XXXXXX			
	X	7q16a-C	8q23.1	
	XXXXXXXXXXXXX			**L3a** Suggest suitable units for measurement
1.0	XX		6q13.3	
	XXXXXXXXXXXXXXXXXXXX	**6q11.2-L**		
		7q20a-L	6q13.2	
	XXXXXXXX			
0.0	XXXXXXX	7q16b-M	6q13.1	
			6q12.1	**L2** Measuring using non-standard and standard units
	X	**6q10-L**		
	XXX			
	XX			
-1.0	XXXXXX		6q11.1	
	XX	7q04-L		
	X	5q06-M		**L1** Ordering objects by direct comparison
-2.0				
	X			
		5q12-L 6q09-L		
	X			
-3.0				
	XXXXXXX			**L0 → L1** Simple ordering of two objects by direct comparison
-4.0				
-5.0				
		5q08-L		
-6.0				
	Low Ability Pupils	Easier Items		

National Curriculum Levels

L4: They choose and use appropriate units and instruments, interpreting, with appropriate accuracy, numbers on a range of measuring instruments.

L3: They use non-standard units, standard metric units of length, capacity and mass, and standard units of time, in a range of contexts.

L2: They begin to use everyday non-standard and standard units to measure length and mass.

L1: They measure and order objects using direct comparison, and order events.

Table 2: A developmental map for MaLT measurement items for ages 5-7

The items are numbered with their test number, question number and the aspect of measurement which they assess (L stands for length, M for Mass and C for capacity). For example 6q10-L indicates that this item was question number 10 in MaLT6 test and it assessed length. Some items have also an extra component i.e. the step level. These items are the items which had more than one step towards a correct answer (i.e. included steps for diagnostic errors). Therefore the step level component indicates the step of the correct answer e.g. 6q13.4 indicates that question 13 in MaLT6 had 4 steps and the fourth step was the correct answer. The erroneous responses of these items appear in the 'Common errors' column: thus the first, second and third step of question 6q13 appear in the 'errors' column. These erroneous responses listed in Table 2 are most likely to be made by children near the ability adjacent.

According to Table 2 the hierarchy is defined by five levels from Level 0 (L0) to Level 4 (L4). The cut-off points for the levels were assigned where groups of items and pupils seemed to cluster, therefore forming empirical gaps between the item difficulty allocations on the logit scale. These clusters of items and responses were then examined qualitatively in order to identify any commonality in relation to their content. It is important to note here that we acknowledge that the cut-off points between levels are somewhat arbitrary.

L0 and L1 items deal with the primary purpose of making a measurement i.e. to make comparisons between items according to the magnitude of some attribute. L0 items involve simple comparisons between two objects and the use of comparative language. L1 items involve comparisons of more than two objects. There is also one item on conservation of length (i.e. the length of an object is rigid when the object moves). Therefore L0 and L1 deal with two principles of measurement i.e. comparison and conservation. L2 items introduce the idea of the unit. At L2, the children are provided with the measuring unit (non-standard and standard) and they are asked simply to perform measurement. Three important principles underlie the understanding of iterated units of measurement according to the literature (Bladen, Wildish & Cox, 2000): ordering, transitivity and conservation. This suggests that pupils operating at L2 and above of our hierarchy should master these three important principles. Level 3 consists of two sub-levels: L3a and L3b. At L3a, pupils have to choose and suggest the appropriate unit to be used for a specific aspect of measurement, whereas at L3b pupils have to choose and suggest the appropriate measuring instrument in order to perform measurement of some aspect. These were mostly multiple choice items as pupils were usually provided with choices to choose from. L4 items involve estimation and reading of measuring instruments accurately. However this level of the hierarchy refers to items that were targeted for eight year-olds, so will therefore not be discussed further.

Generally, capacity items were found to be the hardest compared to mass and length items within each level whereas length items were found to be the easiest ones. This is in agreement with current literature. This is also due to the fact that children at early primary school years are firstly introduced to length and mass and then to capacity. Therefore they have more experiences at this stage with length and mass.

Comparison with the English National Curriculum hierarchy

Next to the empirical hierarchy we also present the National Curriculum level descriptors for the area of measurement. The curriculum level descriptions set out the standards that pupils are expected to reach by the end of compulsory education. The National Curriculum provides a description of the types and range of performance that pupils should characteristically demonstrate for each level they are working at. In total, there are eight levels of increasing difficulty (plus a description for exceptional performance above level 8). In Table 2 only the first four National Curriculum level descriptors are presented as the majority of pupils between ages 5-7 are expected to work between levels 1 and 3. The teaching progression and assessment of measurement in England is based on the National Curriculum hierarchy. Therefore, we were interested to compare the two hierarchies in order to identify any discrepancies.

In general our empirical developmental sequence matched the one described in the National Curriculum (NC). MaLT items were written to assess the National Curriculum as it is taught and assessed in England and as expected our empirical hierarchy confirmed the Curriculum hierarchy. This finding is consistent with the criticism on curriculum and assessment circularity noted by Noss et al. (1989) who commented that "observed hierarchies reflect not the differences but an underlying homogeneity in the mathematics curricula-that is similarity in topic coverage between schools. In this scenario, the hierarchies would reflect what actually happens in schools, rather than any universal levels of understanding. They say nothing therefore as to what might be achieved given a different set of experiences" (p.112).

The progression in measures in both hierarchies starts with direct comparison and then moves to the use of a physical unit and then on to the use of standard units. However there is no direct correspondence between levels. Specifically our L0 and L1 correspond to NC L1; our L2 incorporates NC L2 and NC L3 while our L3 corresponds to NC L4. One could argue that our tests were lacking of items at L2 and L3 of the National Curriculum as the majority of pupils in these early years are expected to work between these two levels.

Interesting Items

Although our empirical hierarchy confirmed the curriculum hierarchy for England there was a chain of three items (of increasing difficulty) that did not 'fit'. These items contradicted the curriculum hierarchy in ways which we find interesting and therefore discuss further in this section.

These items are marked in bold in Table 2 while Figure 1 presents their content. These three items had a common objective i.e. measuring using uniform non standard units. According to our hierarchy these items were expected to fall within L2. The empirical evidence though contradicted this expectation as only one of the three items fell within L2 i.e. item 6q10.

Item 6q10 was the easiest and most straightforward of the three as the paperclips were aligned to the ends of the toothbrush being measured. Seventy-two percent of pupils in the sample correctly measured the toothbrush to be 7 paperclips long. In item 6q11 the paperclips were not exactly aligned with the beginning and end of the comb. This resulted to an increase in difficulty. Only 46% of the pupils in our sample were able to answer correctly this item. Thirty-seven percent of pupils in the sample counted all the displayed paperclips in order to measure the length of the comb. This significant error is noted on our hierarchy in Table 2 (i.e. 6q11.1). This error (or the fist step of this item) falls within L2 of our hierarchy which suggests that pupils at this level lack of the concept of 'ends' in measurement of length.

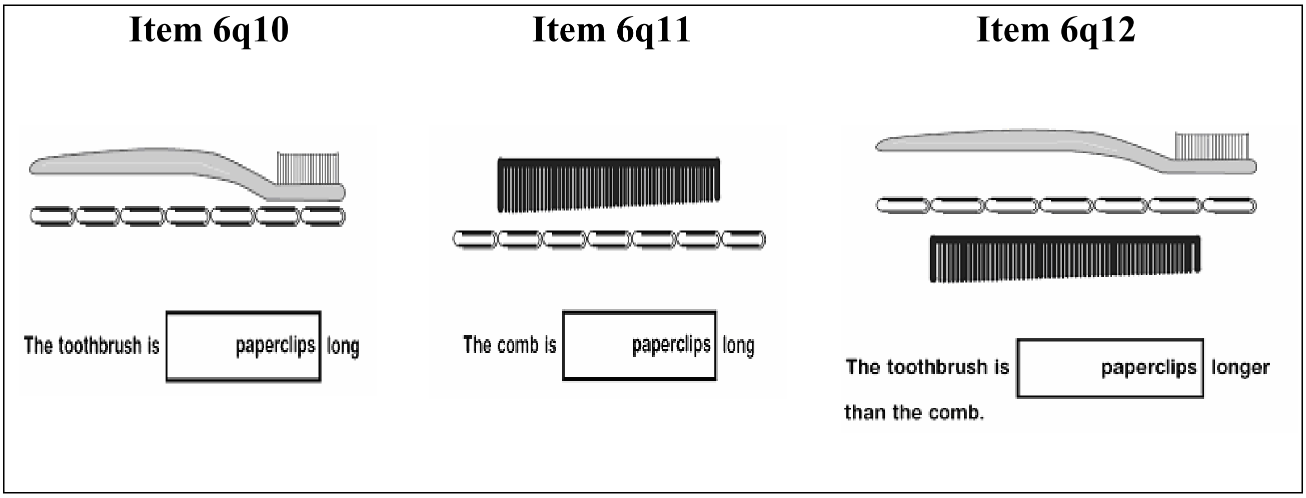

Figure 1: Three interesting items

Item 6q12 was even more difficult as only 20% of the pupils were able to provide a correct answer. Two significant errors-steps were identified in relation to this item (i.e. 6q12.1 and 6q12.2 in 'common error' column in Table 2). The first step (6q12.1) and less sophisticated error involved pupils counting only the length of the toothbrush i.e. the longer of the two items provided. Most of the pupils fell in this category. The second step (6q12.2) involved pupils attempting to count the difference in lengths, but they noticed only the difference in one of the 'ends' of the objects. This was a more sophisticated error and this is reflected in its position in the hierarchy. It is obvious that item 6q12 is cognitively more demanding as it involves a two-step process and as a result children in the early years of primary education found this two-step task overwhelming.

These findings suggest that children at early primary school years have not mastered the **concept of measurement**. Measuring using non-standard and standard units is considered among key objectives in Years 1 and 2 (see Table 1). Our findings contradict educational expectations. These findings are also in agreement with recent literature. Bladen et al. (2000), in particular, also found that there was a discrepancy between educational expectations in the early primary grades and the development of the concept of measurement.

CONCLUSIONS

In conclusion, our national large sample study has confirmed Bladen's (2000) view that "organizations that develop content standards need to reconsider expectations of students in the lower grades as they relate to measurement". Children do not master the concept of measurement in the early years, despite showing they can perform reasonably successfully on many of the standard tasks that are set in tests and offered in the curriculum related to measurement curriculum objectives. This result reveals the limitations of the curriculum and assessment regime and indicates the importance of tasks like the anomalous one we described.

Furthermore, we draw two further conclusions related to methodology. First, this analysis confirmed (cf Doig et al. (2006), and a few other studies) the utility of the method of awarding partial credit to certain errors such as those in question 6q12, which thereby obviates the need for asking questions such as question 6q10 (since the errors in question 6q12 are indicative of the same level of thinking as correct answers to question 6q10 and 6q11 afford).

Finally the study again raises the alarm sounded by Noss et al. (1989) about the potential for self-perpetuating hierarchies in assessment, since most of the items in this scale simply confirmed the hierarchy in the curriculum, while just three items and their errors revealed the weakness in such an assessment. This therefore lends weight to the Noss et al. (1989) argument but also points to the way forward, which is the development of better diagnostic assessment tasks.

Future work involves administering the same measurement items to children that are exposed to a different curriculum (i.e. in a different country) and then construct again a similar empirical hierarchy. In this way we will be able to compare the two hierarchies and study their similarities and differences in the development of measurement concepts.

NOTES

1. Key objectives are considered more critical than others if children are to become numerate according to the English National Curriculum and teachers are instructed to give priority to these objectives when they plan and assess work.

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HOW DO STUDENTS FROM PRIMARY SCHOOL DISCOVERY THE REGULARITY

Marta Pytlak

University of Rzeszow, Poland, mpytlak@univ.rzeszow.pl

INTRODUCTION

Regularity is one of basic idea of mathematics. Rhythms and regularities can be practically found in every domain of mathematics: analysis, arithmetic, algebra, geometry and statistics.

Teaching how to discover and use regularity means developing in the students an active approach towards mathematics. The ability to discover regularity is a starting point for a child to understand mathematics. Regularities stimulate the way of thinking that goes beyond particular cases (thinking about general regularities). One of theories which says about development of mathematical knowledge based on discovering the regularity is TGM theory described by M. Hejný (Hejný, M., 2004; Hejný, M., Kratochvílová, J., 2005). According to this theory cognitive process is decomposed into two levels: generalization and abstraction. Connecting point of both levels is generic model – the pivot term of TGM theory. For Generalization level the generic model is the final stage and for Abstraction level – it is a starting point.

Generalization level – a student gets experience which is stored in memory as an isolated model of the future knowledge. When these models start to refer to each other then the set of models is arranged into a group and it is changed into a new mental object, which plays a role as a representative of the whole group – generic model.
Abstraction level – the generic model covers a wide area of object experiences but still it is connected with specific actions. Therefore the next step of knowledge development is abstraction – disconnection from an characteristic object of generic model. This shift is accompanied by a change of language and the symbolic representation appeares in the place of the previous object representation. Transfer from generic model through abstraction to abstraction knowledge is difficult for student – it demends a lot of time and effort. Discovering, perceiving the regularity by students is a very important problem and it is present in the world trends of mathematics teaching.

In many countries, teaching mathematics closely connected with to the rhythm and regularity. We can find references to the description of researche concerning discovering and generalization the noticing rules. (Littler, G. H., Benson, D.A. 2005; Zazkis, R., Liljedahl, P. 2002). The regularity problem was considerated in PISA (Białecki I., Haman J., 2003). Mathematical contents appearing in tasks were divided into four areas, where the group named as "change and connection" was distinguished (this issue made up about 26% of whole). Solving these tasks was based on i.e. perceiving and using certain recurention regularity.

In Polish teaching mathematics practice children meet with regularities and rhythm in their early stages of education that is pre-school and primary school. These are mainly geometrical regularities connected with drawing patterns. A child is supposed to finish a given pattern. It is not expected that s/he discovers any mathematical rule behind it, s/he just completes the task neatly. The primary school students sometimes encounter arithmetical regularities (e.g. magic squares, triangular numbers) or geometrical (mosaics). However, most teachers treat these tasks marginally and they underestimate their importance and usefulness.

I became interested in regularity issue after looking at world's trends of teaching mathematics and Polish practice of teaching mathematics. It inspired me to carried out my own research concerning discovering the regularity by students on different levels of education. I forlmutated my research problem as: will students from the fifth class of primary school be able to perceive mathematical regularity and if yes – what is their thinking process about solving the task in which they have to discover and use the mathematical regularity.

METHODOLOGY

The research I'm going to present was carried out in November 2005. 38 pupils (11-years old) from two fifth classes took part in this research. There were 20 girls and 18 boys. The research was carried out by me during the maths lesson (teacher who teaches mathematics in this classes was present and only helped with videotaping pupils' work). This lesson took place directly before discussion about "algebraical expression"; it was a single subject from outside curriculum.

Pupils worked in pairs. Each group received one sheet with the task and additional sheet of paper for solutions. Pupils worked for the whole lesson (45 minutes). Pupils' work was being videotaped. Before pupils started work they were informed that:

- they would work in pairs
- their work wouldn't be marked
- they can solve this task in any way they would recognize as suitable
- their work would last one lesson hour
- teacher would be videotaping their work
- teacher would talk with every pair of pupils about their work and this conversation would be recorded

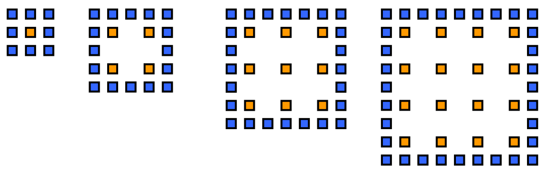
The work in pairs was intented as solving the task at the same time gave students the opportunity to share ideas between themselves. I also wanted to get students' conversations in order to follow their way of thinking.

At least one conversation was made with every pair of pupils about their work. In the next analysis I used both pupils' written work and film showing their work.

The following task was the research tool.

Bolek and Lolek thought up a new game: making figures from colour blocks. Bolek arranged yellow blocks and Lolek arranged blue ones. Their work looked like this:

Figure 1 figure 2 figure 3 figure 4



1. Complete the table:

Number of figure	Number of yellow blocks	Number of blue blocks
1	1	8
2	4	
3		
4		
5		

2. If boys wanted to arrange the seventh figure which blocks of both colors they would use?

3.Is the figure with identical number of yellow and blue blocks possible? Why or why not?

4.Bolek and Lolek decided to arrange very big figure. Which blocks would they need more: blue or yellow?

Pictures of figures look as following: the yellow blocks are inside the figure and the blue once are around the yellows. This task is geometrical-arithmetical and it allows variety of interpretation. It wasn't mentioned that pictures illustrate a pattern which should be continued. The aim of such a designed task was to check: if pupils would be able to discover appearing regularity and if they would use it in the further work or if they rather would behave in a completely different way.

Pupils willingly started to solve the exercises. Way of solving this exercises was in both classes similar: pupils began with calculation of elements placed on the small pictures of figures and values which they got were put into the table. Filling first four lines of the table took pupils relatively little time (about 5-10 minutes).

On this stage of solving the task pupils did not show any attempt at discovering any dependence presented in this task. By filling the fifth line of table (there was no right "illustration") pupils started to think and look for the right solution.

18 on to 19 pairs of pupils' which took part in this research was successful in perceiving the relationship occuring in the task and they were able to use it in the futher work. Only one pair of pupils did not make an attempt to discover the regularity occuring in the task.

Way of searching was different. The initial analysis of research material (relate mainly work above question 1) let me distinguish the following ways of pupils' work:

I. Analysis of numerical columns of the table filled to the fourth level;
   - discovery of the rule for blue blocks and transfer of the rule concerning blue blocks to the yellow ones,
   - discovery of two different rules: one for yellow blocks and another for blue blocks,

II. Analysis of picture of the figure and preparing the figure's picture No. 5 and continuing the task with drawing next figures;

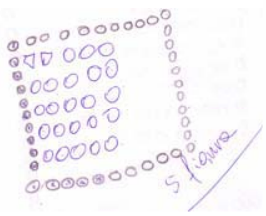
III. Analysis of existing figure's pictures and filling the table on this basis.

I moved atomic analysis (Hejný, M., 2004) of work of chosen pairs of pupils (running across all stages) which I treated as representatives of earlier settled strategy (category). Detailed analysis showed, that preliminary classification does not give back in full processes of pupils' thinking.

OLA AND KAROLINA'S WORK AS AN EXAMPLE OF USING THE THIRD METHOD

The girls started their work from counting elements and putting down the results in the table. In order to complete the fifth line of the table they had to refer to the figures drawings. Analyzing them they tried to find out in what way each of them comes into being. They made a picture nr 5 on the additional sheet of paper.

The girls started with drawing a frame consisting of circles. Next in the middle they drew 5 rows consisting of five elements each. Only after drawing illustration they counted the elements, firstly blue blocks -they put down natural numbers in the circles representing them. The results obtained in this way they put down into the table.



Numer figury	Liczba klocków żółtych	Liczba klocków niebieskich
1	1	8
2	4	16
3	9	24
4	16	32
5	25	40

When the girls started work in order to answer the second question, teacher came up to them and started a conversation.

1. Ola: How many is together? Count. [She turns to the friend to coun how many

elements she has already drawn]

2 Karolina: [She is counting elements of figure no. 7]

3 Teacher: What are you doing now?

4 Ola: To this second question.

5 Teacher: And how do you know how picture has to look like?

6 Ola: Just because as here it was 11 in the fifth figure in one row [she showing at first on fifth line of table and then on column of blue blocks in figure No. 4 and draws up 2 blocks], then for sixth one would be 12, and for seventh one will be 13.

7 Teacher: How do you know it will look like this?

8 Ola: Because we noticed that [points all the figures, thinks] here it changes, about 2 blocks, so it will be 11,13,15 [turns to Karolina] so draw 15 blocks

9 Karolina: [continues drawing the frame for the figure no 7 consisting of blue blocks]

During the conversation the teacher returns to the first question and asks about the way of completing the table. There are no additional pictures made by the girls on the sheet of task paper. There is only an extra paper with the fragment of the figure no 7.

10 Teacher: How did you complete the table?

11 Ola*:* Here? [points the last line in the table] we counted the squares [shows the fig. 1-4]

12 Teacher: You could count only these four figures, what about the fifth one?

13 Karolina: The fifth one is ...

14 Ola: We added 2, because we noticed that they increase by every 2. We added this to the blocks [points to the perimeter of the 4th figure], here 2 [adds 2 squares to the left column of blue blocks] and here...[tries to draw squares on the left side of the top row, after a moment of hesitation] no, it can't be here

15 Karolina: Ola, it was sufficient to add eights here [points to the third column of the table], then it would be together 40.

16 Ola: [looking at the numbers from the columns 'the number of blue blocks'] oh, that's true.

17 Teacher: So was it sufficient to add 8 in blue blocks?

18 Girls: Yes,it was.

<u>Comments:</u> Filling the fifth four line of table by the girls was a collecting experiences about consecutive figures and how the next one will be made. Collecting experiences

allowed girls to perceive occuring recurention in sequence of figures: all the next figure have about 2 blue blocks more on one side than the previous one.
It did not make any difference how blue and yellow blocks are arranged one to another. Drawing the picture they did not keep any proportion between the elements. They drew circles instead of squares and they quickly realized that the shape of the figures was not important. Therefore they drew circles since it was quicker. They also diversified colours, but they did not copy the colours from the task. The picture is not an illustration of a given figure, but it only shows a quantity expressed in numbers*,* which corresponds to this figure.

Here the girls made the first generalization, that discovered by them recurention works regardless of which kind of object they operationed (girls creating their first “generic model”).

At firs Ola while responding to the teacher (6) made a mistake. Maybe she did it unconsciously, and she started exchanging natural numbers successively: 11,12,13.... However, she discovered the rule behind it and put it into practice. Discovering the rule thanks to which the following figures could be made, let the girls draw the figure without the need to refer back. Ola's last remark proves that she can notice some regularity: here two blocks more -this is a certainty, which they can make use of, there will be: 11,13,15 -these are the results of the next addition of 2.

The picture of the figure no.7 is made in a characteristic way. Firstly, the column of blue blocks is drawn and next the girls count if the number of elements is right. Then they draw a horizontal row in which the 15th element from the column is treated as the first in a new row. What can be noticed is that the strategy recognized is applied to one side. Therefore four elements are taken into account twice, which does not disturb the girls in their task. On the other hand the element that is a resultant of the "two more in one side"

While explaining to the teacher how she completed the 5th line of the table, Ola presents the discovered rule and then demonstrates its application on the example of the figure no. 5. This rule is the enlargement of each side of the square by 2 elements. In order to do that she refers to the figure no. 4 and by means of it she demonstrates what should be done to obtain figure no 5. The elements drawn by her stick out of the frame, so the enlarged figure does not keep the shape. It does not show the picture of the figure no 5 drawn by Karolina. Therefore the picture itself is not as important as the way it was made. The girl rather wants to present the function of newly discovered rule for blue blocks. All the time she refers to the rule "2 more in each four sides".

Analysing pictures 1-4 girls did not take it as a whole but they considered it side by side, taking into consideration extreme rows and columns each of figures. The girls discovered local relationship applying to one side of the figure and they observed that this rule was suitable for another sides of figure. On the basis of the pictures 1-4 Ola can imagine the further sequence of sides for the figures constructed hypothetically. In order to move on to the figure no 7 she starts with the figure no4, but she does not express it as the whole, just looks at one side (which has 9 blocks). In this way she can generate (count) the number of elements in each side of the figure controlling the same the number of steps: from figure no4 to the figure no 7 we must add 2+2+2 (11=9+2, 13=11+2, 15=13+2). Another research is made during the talk with the teacher. The girls realized that instead of drawing a figure and counting its blocks it is enough to add to the previous result 8. Karolina comes across these discoveries while looking at the data from the table. She presents her point of view to her friend, and Ola agrees with this totally. Here comes the replacement of functional geometrical rule concerning blue blocks with arithmetical one. At the same time both rules do not exclude one another, but act irrespectively.

19	Teacher:	And the yellow ones? How did you find out how many of them there are?
20	Ola:	Because we noticed, that if there is one in the first figure [points yellow blocks in the figure] here in the figure no 2 there are 2 [points to the first column of yellow blocks in the figure no 2], in the figure no 3 there are 3,and here 4 [saying this points to the first columns in each of the figures]
21	Karolina:	[points to the second column] here increased by 3, and here by 5…
22	Teacher:	Did you draw the figure and then count the blocks? It is a very good idea.
23	Teacher:	[turns to Karolina] Did you have any idea?
24	Karolina:	That's right, here it increases by 8 [points to the third column], and here by 3 as we counted[points to the second column (1 and 3)], and here by 5...[stops for a while, a moment of hesitation]. Yes since there was 1 here [points yellow blocks in the figure no1], there were 4[points with circular movement to yellow blocks in the figure no2] and here 2 [points to the first column second line], here 9, and there 3 [points 9 and 3].
25	Ola:	Or maybe not...
26	Karolina:	We counted in this way: here 1, here 2, here 3, here 4 [points to the following figures in one column], in the next one should be 5
27	Ola:	And in the sixth one 6 and in the seventh one 7.
28	Teacher:	So there would be 5 yellow blocks in one row in the figure no. 5 . And how many altogether?
29	Karolina:	We can easily count this [points to the picture of the figure no 5 made by her on the separate sheet of paper]

<u>Comment:</u> Initially the only rule girls discovered and applied was matching the number of yellow blocks in one row with the number of the figure, which proves Ola's words

(20). The girls made an illustration in this way: for the figure no.5 they drew 5 rows five elements each, for the figure no.7 seven rows 7 elements each, analogically. To support her point of view she presents to the teacher the picture of figure no 5.
Ola while talking with teacher seems to refer the actual work only, whereas Karolina represents more reflexive approach. The reason may be that she was supposed to draw the figures before but she did not have a chance to analyse the data from the table. While Ola is talking to the teacher, Karolina tries to analyse the columns from the table and find some relationship between following numbers. The discovery concerning blue blocks that was positively accepted by friend is an additional incentive for her to search for existing regularities.

Karolina tries to find some relationship between the number of yellow blocks in the following figures. She starts analyzing all the data from the table in the column 'the number of yellow blocks'. She examines the differences between the following numbers. In this way she wants to find a similar rule to that in the case of blue blocks: we add something, too. She notices that numbers increase in 3, 5, 7, respectively. Karolina is not sure if it is the right reasoning. Therefore she tries to find some connection between the number of yellow blocks from one figure with its number. If it was not because of her friend's interruption she might have not failed to formulate an appropriate conclusion. Unfortunately, after Ola's interruption Karolina quits investigating and reports only the actual course of their work. She shows what they have noticed: The number of yellow blocks in one row equals the number of the figure.

The further part of their conversation concerned task no.4.

30 Ola: There always will be more blue blocks.

31 Teacher: Why do you think so?

32 Ola: Because yellow as if on this basis, in the fifth there would be 5 each, in the sixth one 6, so in the fifth figure they double.

33 Karolina: It can't be more here, there can't be equal number of the figures [points to the blue and yellow row; points to the blue blocks in the corner]

34 Ola: There can't be more yellow blocks than blue ones.

35 Karolina: If we counted the figures, these small squares, would be the same here. [points to the row of blue blocks and adjacent to it the row of yellow blocks as well as the two blue blocks being in the same row as yellow ones.]

36 Ola: Nothing can be done, nothing, then it would be...

37 Girls: No. Nothing can be done.

Comment: Both girls agree that blue blocks will always be in the majority. They take into consideration only the existing pictures of the figures, as well as those drawn and made by themselves. Studying the figures they spot that blue blocks are arranged next to each other, contrary to yellow ones. Moreover, yellow blocks are surrounded by blue ones, for that reason there will never be more yellow ones than blue ones (because there are always 2 blue blocks in the yellows' row).

The last talk as well as chosen strategy can testify that they did not analyze the figures as a whole, but only paid attention to extreme verses and columns. The strategy they applied is called "local visualization".

Despite Ola and Karolina noticed some regularities, they could not specify and define them fully. They only worked within real objects (that is the pictures of figures). They failed to go beyond the data from their task. They are not prepared to work in the level of abstraction. It is likely that with some help from the teacher they would manage to succeed and define some regularities.

SUMMARY

The following conclusions can be drawn from the research.

Using the third method could show about mathematical maturity of student. However,a pupil without suitable knowledge and ability could experience a failure during solving the task. It shows the example of Ola and Karolina’s work.

At the beginning girls focused their attention on analysis of existing pictures of figures and discovering geometrical regularity. After discovering by Karolina arithmetical correlation they tried using it in the further work. Unfortunally they kept to the geometrical representation of particular figures too tightly. When they tried to answer next questions mainly basing on an analysis of neighbouring rows consisting of yellow and blue blocks. Certain duality appears between the girls: on the one hand arithmetical correlations are discovered; on the other hand - geometrical correlations between neightbouring rows of blue and yellow blocks.

Students' capabilities need to be assessed very carefully. Sometimes teacher can jump to wrong conclusion while observing students' work. For instance, on the basis of the talk concerning question number 2 the teacher could draw a conclusion that girls can apply geometrical and arithmetical representation and are able to notice and verbalize some relationships. However,their worked on the last question showed that their argumentation concerns only specific cases, i.e. these ones which were analised by them. They are not ready yet to make some generalizations and formulation of self-contained whole.

The girls in my study are not able to make generalization on the abstraction level. They created their own generic model connected with picture of figures and they used it very well. But they were not able to break away from real representation and go to symbolic representation, to abstract level.

Finding the solution of the task does not lead to discovery of something new or to creating any new element of mathematical knowledge. Necessity of verbalization of executing activities and explanation of using procedures show that pupils are able to notice new things. The verbalization forces to look at the own work from a different perspective.

In teaching mathematics, interactions between the teacher and the student and among students play a vital role. Through making students formulate and defend their points of view we develop in them their self-assement.Thanks to it during solving problem a child is more responsible and conscious of what s/he does. The verbalization and explanation of their own thoughts develop the ability of searching for regularity and discovering new mathematical knowledge.

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REFLECTION ON ACTIVITY-EFFECT RELATIONSHIPS IN SOLVING WORD PROBLEMS

Ana Isabel Roig & Salvador Llinares

Department of Innovation and Teacher Training. University of Alicante, Spain

Abstract. *This study focus on how the mechanism "reflection on activity-effect relationships" (Simon et al., 2004) is involved in the students' problem-solving processes. The participants were 71 secondary-school students who were in the final year of compulsory education (15-16 years old). Four levels of development of constructing and using mathematical concepts as tools for problem-solving were identified using a constant-comparative methodology to analyse the reflection on activity-effect relationships in problem-solving processes. Different components of the mechanism were identified revealing that "reflection on activity-effect relationships" is a key component when a mathematical concept (used as a tool) for solving the problem is not identified.*

RESEARCH PROBLEM

The problem-solving process involves the use of mathematical concepts in order to think about the situation, explain it or formulate predictions based on its structural characteristics (elements, relationships, patterns or operations etc) (Lange, 1996). In this sense, problem-solving process is guided by a goal and the mathematical knowledge (concepts, structures…) that can be used to organise a situation can be considered as tools for thinking and acting. From this, mathematical knowledge can be seen as a set of tools for problem-solving.

Research indicates that students use informal strategies for solving problems in their attempts to give meaning to stated situations. The strategies they use reveal their understanding of the relationships expressed in the situations and how they use this information to solve problems (Johanning, 2004; Nesher et al., 2003; Tzur, 2000). In this sense, the student identifies the quantities and variables in the situation as well as the relationships between them, in order to come to a decision regarding the situation and to communicate that decision.

This kind of problem-solving processes has as a result a structural organization of the different elements implied that can be of greater or lesser sophistication. The idea of a "degree of development" of the problem-solving process is therefore meaningful when the student is constructing and using mathematical tools in solving problem situations (Lesh & Harel, 2003).

The research reported here was carried out to contribute to two lines of research. First, it helps to explain the problem-solving process as evidence of the students' mathematical competence. Second, it obtains information that helps us to understand better the construction and use of conceptual entities. In this paper, the specific goal is to show how the mechanism *"reflection on activity-effect relationships"* (Simon et al., 2004) is involved in the problem-solving processes and how its different components can be identified in the secondary school students' resolutions.

THEORETICAL FRAMEWORK

Simon et al. (2004) elaborate a mechanism for mathematics conceptual learning called *"reflection on activity-effect relationships"* that is characterized from Piaget's (2001/1997) reflective abstraction. They argue that a learner's goal-directed activity and its effects (as noticed by the learners) serve as the basis for the formation of a new conception. This new conception can be a tool constructed and used for solving a problem situation. They identify the following components of the mechanism:

> *"the learners' goal, the activity sequence they employ to try to attain their goal, the results of each attempt (positive or negative), and the effect of each attempt (a conception-based adjustment). Each attempt to reach their goal is preserved as a mental record of experience"* (Simon et al., 2004, p. 319)

This process is supported by the capacity of the learner to compare the effects of his/her activity sequence (identifying invariant relationships in the situation) with his/her goal. So, the identification of invariants in each comparison is the product of an abstraction of the relationship between the activity and the effect. In this sense, *"an abstracted activity-effect relationship is the first stage in the development of a new conception"*. In this process Simon et al. (2004) elaborate on the two phases of Piaget's (2001/1977) reflective abstraction, the projection phase and the reflection phase. Students in the projection phase sort (not necessarily consciously) records of experience based on the results (positive or negative). In the reflection phase, the mental comparison of the records leads to the recognition of patterns and then, regularities are abstracted. These regularities are the activity-effect relationships. Simon et al. (2004) note that:

> *"the regularities abstracted by the learners are not inherent in the situation but rather a result of the learners' structuring of their anticipation-based observations in relation to their goals and related (existing) assimilatory structures"* (Simon et al., 2004, p. 319).

The students' records of experience can be traced by their talk or their written answer on the paper. The mental comparison of these records is a mental activity whose results can also have a written record or a justification during an interview on the activity done.

METHODOLOGY

Participants and tasks

71 students participated in the study. Their ages were ranged between 15 and 16 years. A five-question test paper was prepared. Three of the questions were word

problems in which the students were asked to make a decision regarding the situation and to express the reasons why they thought their decision was the right one. The idea behind using this type of problem is to avoid encouraging students to simply look for an operation they think suitable for solving the problem.

One of these problems is presented here:

A floor-tile manufacturer has donated a quantity of floor tiles to the festival committee. Each tile is 33 centimetres long and 30 centimetres wide. The committee has decided to lay a square dance floor within the festival enclosure, but you have to tell them:

a. *the length of each side of the smallest square that can be made with this size of tile without cutting any of them*
b. *what other sizes of square dance-floors could be laid using only uncut tiles of this size, and why?*

In your reply to the committee, explain what you have done.

All five problems could be interpreted and manipulated using the mathematical knowledge that final-year secondary-school students are assumed to possess. In this task of problem solving we assume that the students' goal is to provide an answer and justify it. Given the task, the student tries to reach a solution, that is the learner's goal. For achieving the goal the student performs an activity sequence that can be considered as a goal-directed activity. In this process, the students' mathematical knowledge can be used as tools allowing solve the task in some performance level.

One mathematical concept that the students can use to solve the problem in figure 1 is the idea of the lowest common multiple. The meaning of the common multiple comes into play due to the need to establish the relationship between the length of each side of the tiles and the length of each side of the finished dance-floor. In this situation, the student should realise that if rectangular tiles whose area is a×b are used, a square floor can be laid each side of which will be a common multiple of a and b. Another multiplicative relation that applies to this situation is that starting from a square of side c other larger squares can be formed whose sides will be multiples of c (2c, 3c, 4c …) and, therefore, common multiples of "a" and "b". Although the lowest common multiple can be considered a mathematical tool in this situation, students can use other tools.

Finally, and in order to obtain further information on the ways in which students solved the problems, the students were interviewed. The aim was to get the students to verbalise the thought-processes they had used in solving the problems (Goldin, 2000) in order to look for evidences of how the pairs activity-effect are mentally preserved and then compared looking for regularities.

Data coding. Procedure and analysis.

In a previous phase of that research (Roig, 2004) carried out with a bigger sample, the students' answers were analysed from a descriptive point of view taking into account the way in which each student set up and used mathematics knowledge as tools in order to interpret the situation and then made a decision. Using a constant-comparative methodology (Strauss & Corbin, 1994), two researchers characterised from a small sample of the problems the way in which the students' answers indicated the construction and use of mathematical knowledge as tools. We identified characteristics in the students' approach to the solving of the problem and these characteristics led to definition of levels of development in the process of constructing and using mathematical tools in word problem solving. We continued to review answers, observing how each one aligned or did not align with the initial descriptions. Thus, when we noticed an anomaly in the description of a characterisation, we modified this description or developed a new characteristic and re-examined several earlier answers for similarities and discrepancies. Following this process we characterized four levels of development in the solving processes carried out by the students (e.g. in this problem the 71 students were categorized as follows: L0=71.8%, L1=0%, L2=15.5%, L3=12.7%)

Level 0	Although the student sometimes appears to identify some of the variables which may be relevant in the situation, (s)he is unable to establish any meaningful relation between them.
Level 1	The quantities and relationships involved in the situation are identified, but "global" comprehension of the situation is incomplete, which prevents the student from developing effective tools with which to interpret the situation and justify the decision taken.
Level 2	Some relevant aspects of the situation are identified and the relationships between them are established, thus revealing a structural understanding of the problem. Effective tools are constructed in order to facilitate the search for an answer, but these tools are not used appropriately for decisions to be made.
Level 3	Tools for solving the problem situation are constructed or identified and is used in an appropriate manner so that decisions can be made and justified.

Table 1: Levels of development in the process of constructing and using mathematical tools

Once we had characterized the levels of development from a descriptive point of view, we focus our interest in the mechanisms that can explain the pass from one level to another. For that purpose we analyse the student's answers from the mechanism of *"reflection on activity-effect relationships"* (Simon et al., 2004). The student's *"reflection on activity-effect relationships"* in solving the word problems was inferred from the written text supplied by the student, and by the way in which he or she justified his/her final decision, in an attempt to theoretically explain the different levels of performance.

SOME RESULTS RELATED TO "THE DANCE FLOOR" PROBLEM

Using of mathematical knowledge as tools was revealed in the way in which students were able to think beyond the particular cases in order to incorporate the influence of the variability of the quantities (Llinares & Roig, in press). The way in which students used the information provided by the particular cases was different in each situation. In "the dance floor" problem students relied on the general structure of the situation to generate particular values when they could not remember a suitable algorithm, from the use of these particular cases a *"reflection on activity-effect relationships"* can be carried out which allows students to take a decision and justify it.

The idea of the common multiple is the key to the resolution of this task, linking it to the problem's two conditions of "square floor" and "without cutting any of the tiles". The use of the idea of a common multiple as a general tool in this situation made it possible to offer a correct answer, though the procedures followed by the students were different depending on whether they remembered an algorithm for the calculation of the lowest common multiple.

MH/35 (see Annex, Figure 1), guided by his goal of finding the dimensions of the smallest square that can be laid (the student's goal), recognised the role played by the idea of the lowest common multiple, but could not remember the algorithm required to calculate it. The first step for solving the problem consists in finding any common multiple of 30 and 33. These students then resorted to trial-and-error procedures. MH/35 looked for multiples of 33 (33×2 = 66, 33×15 = 495 and 33×40 = 1320) in order to see if they were also multiples of 30 (66 is out of the question, it is obvious that 495 is not a multiple of 30 because the division does not give a whole number, while 1320 is a multiple).

Researcher: What were you working out when you wrote 33 per 15?

Student: Well… the side of the tail is 33 per 15 tails, it's 495 and I divided it by 30 which is the other side of the tail and it was 16'5, and I said no, because this has to be cut.

Researcher: Well.

Student: Then I tried per 40 and it was 1320, I divided it again by 30 and it was 44. I said yes, it was it.

The student searches among multiples of 33 one which is also a multiple of 30. This manner of proceeding seems to show that MH/35 was seeking by trial and error a common multiple of 30 and 33, as can be seen in the written answer "the number 40 has come up purely by chance [by trial and error] and I have hit the nail on the head. It was a bit of a fluke [i.e. good luck]".

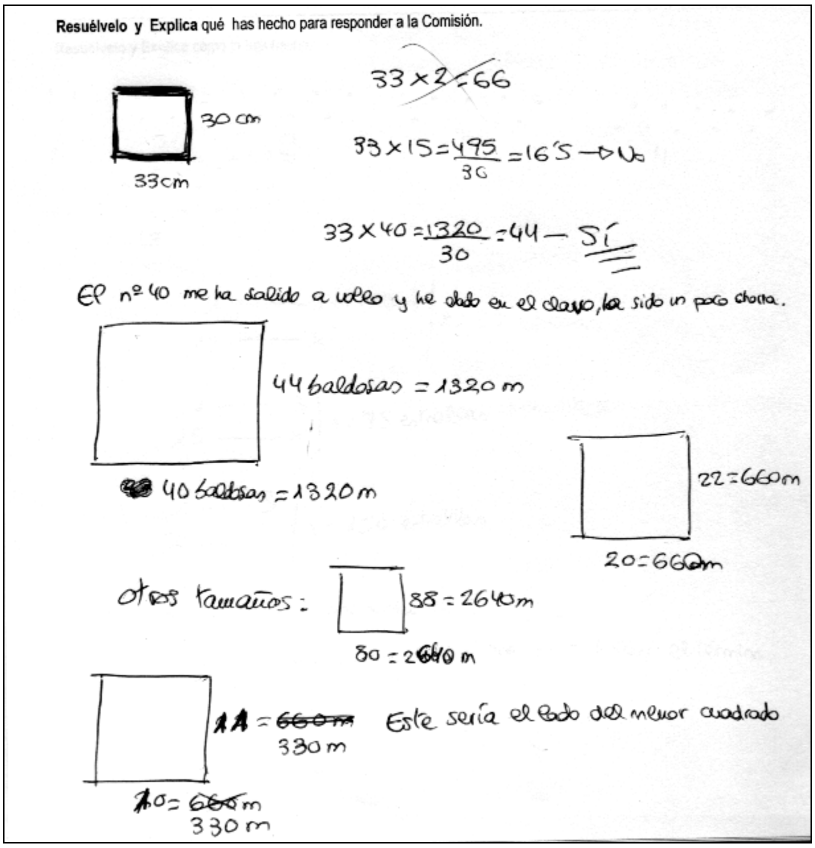

Figure 1: MH/35's answer for word problem 4 from a general model through particular cases to make sense of the situation

Although the way in which the student writes the operations are not arithmetically correct, they do reveal the mental process that he appeared to follow. MH/35 then suggests as a possible solution laying 40 tiles along one edge (the 33 cm ones) and 44 30 cm tiles along the other. From this starting-point MH/35 carried out the search of the lowest common multiple through other particular cases. For doing so, the initial number of tiles was doubled (80 by 88) or halved (20 by 22).

Student: Here, I multiplied per 2.

Researcher: You multiplied the one of 40 per 44 per 2.

Student: Yes.

Researcher: Then, what did you do to get the 10 per 11 one?

Student: From 44 and 40 I thought about a divisor and 20… 22 and 20. And from 22 and 20 y thought about other divisor and it was 11 and 10 and after that trying, although it isn't here, I tried from 11 and 10 and the result wasn't less.

The search of each of these particular cases can be seen as an <u>activity</u> which result was positive or negative depending on the dimensions of the resulting square. If the length of the side is bigger than the initial one the result is negative, in other case the result is positive because it's closer to the solution. In this sense that process can be considered as an activity sequence were the comparison of the pairs activity-result (records of experience) seems to make MH/35 aware that multiplying leads to negative results and dividing leads to positive results. This relation showed the student the path to follow in order to discover the dimensions of the smallest possible square floor. From here, he was able to anticipate the actions that could be done for achieving the goal. This number was obtained by dividing by two the number of tiles laid in the case of a rectangle measuring 20 tiles by 22.

On the other hand, students who remembered a method for calculating a lowest common multiple came to a much more direct conclusion. What the answers seem to have in common is the fact that the students identified the idea of the lowest common multiple as a tool for solving the problem situation, the difference being that some knew, and others didn't, a procedure for calculating it. Students who could not remember the procedure resorted to a trial-and-error method using particular cases and obtaining the answer by reflecting on effect-activity relationships. In this situation, students who were aware of the idea of the lowest common multiple as the general idea that could organise the situation, resorted to particular cases in order to organise their search when they could not remember an algorithm. From this point of view, we may conclude that in this situation it was the general structure which governed the use of particular cases, giving rise to *a "reflection on activity-effect relationships"*

DISCUSSION

The aim of this study was to find out how secondary school students constructed and used mathematical tools in order to make decisions in solving word problems. Firstly, we sort the students' answers from a descriptive point of view and identified four performance levels; then we focus our attention on the role played by the mechanism *"reflection on activity-effect relationships"* (Simon et al., 2004) and how its different components can help to explain the different levels of performance in the secondary school students' resolutions. Two ideas were shown to be important in this process: firstly, the relationship between the general and the particular revealed in the different ways in which students used particular cases; and secondly the difficulty students encountered in using mathematical knowledge as a tool to solve the problems (Llinares & Roig, in press). These two characteristics can be explained from the process of *"reflection on activity-effect relationships"* (Simon et al. 2004) particularly in the two phases of the reflective abstraction, the projection phase and the reflection phase (Piaget, 2001/1977; Simon et al., 2004)

In the process of recognising the underlying structure of the problematic situations in order to achieve the student's goal, the students used particular cases for different purposes. In some situations, when the student doesn't recognise a mathematical tool that allows him(her) to handle the situation, the use of particular cases provided information which facilitated the construction of appropriate tools through a *"reflection on activity effect relationships"*. In "the dance-floor" problem students made use of particular cases for a different purpose. In this situation, once the idea of the lowest common multiple had been identified as a useful general tool in order to interpret the situation, the use of particular cases helped in particularising that structure when no appropriate algorithm could be remembered.

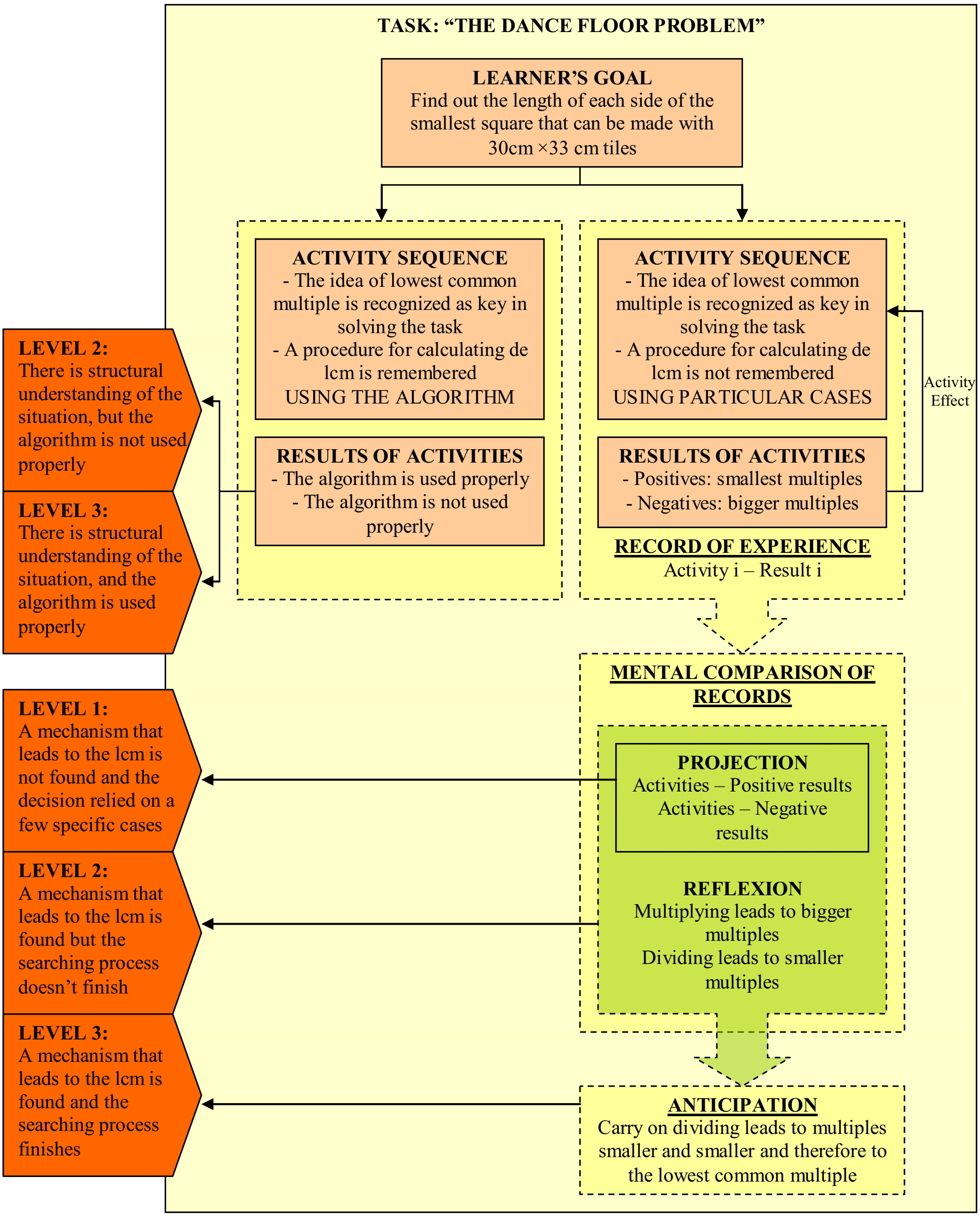

Figure 2. "Reflection on activity-effect relationships" in "the dance floor" problem

This difference in students' behaviour may be caused by differences in the levels of conceptual development of the mathematical concepts involved. From this standpoint, the activity sequence (based on the use of particular cases) carried out by the students, led them to the comparison of records of experience. The information obtained from such comparison on particular cases seems to be a determining factor in the characteristics of the solving process. This information enabled the student to go beyond the particular when the abstraction of the relationship between activity and effect takes place. These students are at the reflection phase and, consequently, can anticipate the next stages in their solving process.

If the comparison remains at the projection phase, the underlying structure of the situation is not (or is partially) recognized. Students fail to perceive this influence and a particular-cases-based Level 1 solving process for the situation is developed, leading to decisions that relied exclusively on a few specific cases. We interpreted this fact as that student has not got *an abstracted activity-effect relationship*. In these sense, these students are at the participatory stage but not yet at the anticipatory stage. A solving process of this kind is devoid of a complete structural understanding of the situation.

Students who abstract the activity-effect relationship (Level 2), had achieved this complete structural understanding of the situation. In these cases, students anticipate what he/she needs to accomplish the goal making goal-directed adjustments. At the same time, appropriate search of the result was determined by the correct handling of their anticipation-based observations in relation to their goals and related (existing) structures (Level 3).

In "the dance floor" problem students relied on a general realisation (they identified the idea of the lowest common multiple as an appropriate tool) and resorted to particular cases as a process by which to calculate the lowest common multiple when they could not remember an algorithm. The knowledge of the structure of the situation enabled the student to carry out a meaningful search among the particular cases. This search for the lowest common multiple was sometimes carried out by a process of trial and error and on other occasions by a more systematic method.

Our results point out that it is possible to understand different performance levels in solving word problems from the *"reflection on activity-effect relationships"* mechanism and provide us with information about as the students uses mathematical knowledge as tools in word problem solving process. However this approach might be limited if the research focus had been the heuristic strategies followed in problem solving.

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CHILDREN'S PERCEPTIONS ON INFINITY: COULD THEY BE STRUCTURED?

Mihaela Singer*, **<u>Cristian Voica</u>****

*Institute for Educational Sciences, Romania, m.singer@ise.ro

**Dept. of Mathematics, Univ. of Bucharest, Romania, voica@gta.math.unibuc.ro

Abstract. *The paper is focused on explaining what kinds of structures are activated when dealing with the intuition of infinity in school context. The research revealed that some children are able to develop a structured representation about the infinite sets even at the age of 10-11 years old, or earlier, based on their processional perception. Moreover, when students' arguments are consistent, they seem to be based on connections between algebraic and geometrical thinking, facilitated by their topological perception.*

INTRODUCTION

According to Fischbein (1987), intuitive knowledge is a self-explanatory cognition that we accept with certainty as being true; it is a type of immediate, coercive, self-evident cognition, which leads to generalizations going beyond the known data. Fischbein differentiated between primary intuitions and secondary intuitions. Primary intuitions were defined as intuitions that "develop in individuals independently of any systematic instruction as an effect of their personal experience" (Fischbein, 1987). Secondary intuitions were defined as "those that are acquired, not through natural experience, but through some educational intervention", when formal knowledge becomes immediate, obvious, and accompanied by confidence (Fischbein, 1987). The research findings in the literature indicate that the methods students apply for the comparison of infinite sets were largely influenced by methods they had used when comparing finite sets (e.g. a set has more elements than its proper subsets). Students usually did not use 1:1 correspondence, the criterion that should be used to determine the equivalence of two infinite sets within Cantorian set theory (Tall, 1990; Tsamir, 1999).

METHODOLOGY

To identify children's primary and secondary intuitions about infinity, we covered a broad range of ages. The participants in our study were students from grades 3 to 12 (9-10 to 18-19 years old) and undergraduate students – prospective mathematics teachers; 262 students (143 girls and 119 boys) answered to questionnaires, and 31 students from the sample were interviewed. The 31 interviewed students where selected based on the way they formulated, explained, or illustrated their answers in the questionnaires. During the discussions, the students looked at their solutions on the questionnaires, and explained what they have been thinking when they were designing their solutions. At the same time, interesting remarks have been generated by a two hours interactive discussion with 30 students in 5th grade (11-12 year olds). This discussion took place in the classroom, in the absence of the current teacher; thus, the authors interviewed the children in their natural school environment. We consider this discussion as focus-group contribution to the data collection. We were interested to stimulate divergent opinions in students through open questions; consequently, the range of answers was very large in all the three categories used for collecting data. In the present paper, we comment on the reactions of 7 to 15 year olds.

To get a better view of the students' insights about infinity we used a variety of questions. Generally, we redo the same type of question in different contexts: algebraic versus geometric, discrete versus continuous, static versus dynamic, etc., in order to identify to what extent the students construe coherent arguments and continue to use them consistently. We adapted the questions as we progressed in exploring students' ideas. Generally, the questions covered the following categories:

Vocabulary, intuitive representations. Within these questions, we wanted to spot on students' own ideas about the words 'infinite' and 'infinity'. Sample question: "*Use words, expressions, comparisons, metaphors to describe what you understand by the word infinity"*.

How does intuition work? Within this category of questions, we have tried to grasp students' primary algebraic and geometric intuitions, trying to avoid as much as possible a formal approach. Addressing these questions to various ages, we could compare to what extent the primary intuition is or is not affected by formal knowledge. Example: *From a ray, someone cuts 1 km starting from the origin; will the new ray be longer (or shorter) compared to the initial one?*

How does one prove the infinity of a given set? This set of questions is meant to identify the way in which children of various ages build arguments to support their intuition. We were interested to see if the students understand the difference between finite and infinite sets, and if so, in what way they do understand this difference. Example: *Which of the following sets are infinite: the set of divisors of 34456348287; the set of even numbers {0; 2; 4; 6;...}; the set of rational numbers between 1 and 2?*

How does one compare infinite sets? Through this set of questions, we have tried to see if there is a primary intuition for comparing the cardinals of infinite sets. In phrasing the questions, we have tried to use informal language, avoiding speaking explicitly about functions or cardinal equivalence. Example: *Which set of the following pairs of sets has more elements: {0; 2; 4; 6; 8;...} and {0; 1; 2; 3; 4;...}; the rational numbers between 1 and 2, and the rational numbers between 2 and 3?*

CHILDREN'S METAPHORS: HOW TO DESCRIBE INFINITY?

Because children lack technical means to operate with the idea of infinity, they construe ad-hoc metaphors through which they describe infiniteness. The study of these metaphors might be done from various perspectives. Thus, Boero et al. (2003) addressed fifth graders the question: *How many numbers are there between 1 and 2?* They classified students' answers in three categories: metaphors where the source domain was mathematical, metaphors related to ordinary life experience, and metaphors related to religious ideas. From another perspective, Jirotková and Littler (2003) have classified the perceptions about infinity revealed by the definitions given by primary school student teachers on grammar criteria.

For the first stage of our study, we wanted to keep a range as large as possible of children's spontaneity and, therefore we avoided to narrow the gamut of perceptions through structured interviews. The open-ended question: *Describe the idea of infinity in your own words*, received various answers which we have later classified as follows (see also Singer & Voica (2003) for a more detailed discussion).

a) Metaphors emphasizing **a processional** dimension (e.g. Xena (grade 8): *Infinity is the term we use for something that does not stop; it continues to rise.*). These processes:

- are seen in terms of **change** (e.g. Bissan (grade 8): *The number of desks in a classroom is considered finite because they have an amount that can't change in the same sense as an infinite number*.)

- are seen in terms of **counting** (e.g. Xim (grade 8): *Finite is like the number of pencils in a room, but infinite is like numbering all numbers in the world*.)

- are emphasizing **a temporal** dimension (e.g. Bissan (grade 8): *Infinity is something that never stops. It will go on and on forever.*)

- are emphasizing a **spatial rhythmic** dimension (e.g. Rebecca (grade 8): *Infinity is when something doesn't finish and it keeps on going and going and never ends*.)

How do children represent continuous processes? Lakoff and Núñez argue that human beings conceptualize indefinitely continuous motion as repeated motion: "continuous walking requires repeatedly taking steps; continuous swimming requires repeatedly moving the arms and legs; continuous flying by a bird requires repeatedly flapping the wings. This conflation of continuous action and repeated actions gives rise to the metaphor by which continuous actions are conceptualized in terms of repeated actions." (Lakoff and Núñez, 2000, p. 157). They concluded that infinite continuous processes are conceptualized via this metaphor as if they were infinite iterative processes.

b) Metaphors emphasizing **a topological** dimension. We consider that the **topological perception** manifests when the child evokes, in his/her description, properties and transformations that are invariant to the change of shape. The topological perception is of a continuous nature; it supposes evoking density/ jam/ accumulation of the elements of a set to describe the perceived reality. Instead of *recursive* as a main attribute, the topological perception is characterized by *diffused*: the order relation is not evocated for arguing the infinity of a given set; instead, it is distance that appears, as function with values in **R**.

Students use spontaneously intuitive descriptions of open sets, bounded sets, or of the frontier of a set (e.g. Cristina Maria (grade 8): *Infinity is something enormous… Big... large... err... very big*; Loredana (grade 8): *I mean something unfinished… without end…*; Mihai (grade 9): *Infinity is ineffable. Endlessly ... Huge…*)

c) Metaphors emphasizing **emotions and spirituality** (e.g. Octavian (grade 4): *Infinity is something the secret of which we cannot grasp. Our mind is bound and we can't say many things. It is not able to understand everything about infinity. This is a word that is endless in numbers, and love, etc. But not everything is endless. We can't get this secret but with the help of God. He can help us find the key to understand infinity. With the help of people we can't get this marvelous mystery of understanding with the help of people. Not even the greatest scientists can understand this mystery. It is only God Who can uncover this. It is only when we get in heavens that we can fully understand infinity*.)

The spiritual dimension is spontaneously expressed by children within discussions about infinity. This might be seen as being a prior component of a primary perception of infinity. As Lakoff and Núñez (2000) emphasized, the metaphorical concept of infinity as a unique entity – the highest entity that encompasses all other categories – was naturally extended to religion.

The demarcation among these categories was developed and confirmed along the study: we realized that students' metaphors bear significant information to a larger extent than we supposed at the beginning. Thus, the processional dimension appears frequently when students bring arguments for the infinity of sets, while the topological dimension is spontaneously activated when the students compare the cardinals of sets. We found that these categories are not disjoint in child's mind: on the one hand, the same child usually uses (simultaneously and/ or successively) more than one type of representations; on the other hand, a metaphor is independently interpreted from one or the other of the perspectives.

CHILDREN'S ARGUMENTS: WHY SOME SETS ARE INFINITE?

Counting supposes to construe the order and the recurrence; this is why the processional perception of infinity appears to be natural. The primary intuition of infinity is strong enough at the age of 6 – 8 years old, so that students might be able to construct arguments for the infinity of the set of natural numbers based on this intuition. The students justify the infinity of ***N*** usually using recursion. The correct reasoning is made mostly by using sequences – this shows that the Peano's axioms are fully internalized at this age and students are able to extend their knowledge to build arguments in situations that are not familiar to their knowledge level. A functional dynamic thinking (Schwank, 1999) in the early ages seems to be a pre-condition for a later good understanding of the way infinity works. Going further, the concept of sequence seems to precede the idea of set and the concept of recursion is a primordial one. In contrast with sequences, which have a rhythmic structure, sets have an amorphous structure (given essentially by the possibility of randomly listing the elements). The preference of students for a processional description of infinity could be interpreted as an evidence for the fact that the "sets as containers" metaphor is less embodied than the "sets as graphs" metaphor (e.g. Lakoff and Nunez, 2000). Children's arguments to show that a set is infinite, confirm the categories of perceptions we identified in the previous section. As we have seen, primary graders show a processional perception. Children in higher classes construct arguments referring to the infinity of ***N***, generally, based also on a processional perception. However, arguments that belong more to a topological perception are also present. Some answers to the question "*How do you show that **N** is infinite?*" are given below.

Alice (grade 6): ***N** is infinite because we can count 1, 2, 3, and anywhere we arrive, we know that it goes on.* (Obviously, this vision is of a processional type, as it frequently appears in children grades 1 and 2.)

Andreea (grade 5): *There are infinitely many natural numbers because if I pretend that I found the biggest, I can add 1 and I get a bigger one*. (The reference to the fact that the set could be bounded is of a topological type, and the way to get the contradiction is of a processional nature.)

Tiberiu (grade 9): *The set of natural numbers is infinite because it is constructed following a rule: start from 0, add a number…add 1 each time.* (The process is described through the mathematical induction procedure.)

We found that, for many children in the lower secondary school, the concept of infinity of ***N*** is strong enough to allow reasoning. For example, some students found necessary to apply the negation to argue that a given set is finite. Some quotations:

Rebecca (grade 8): *The divisors of 24 are not an infinite set, nor the divisors of 32561784937289463785, because the last divisor is the number itself and there are bigger numbers than them.*

Vrit (grade 8): *The biggest fractional number in the interval (2; 5) is 4.9…., no, it doesn't exists because it is an infinite set.*

Moreover, starting from ***N***, students manage to make transfers of reasoning to justify that various sets of rational numbers are infinite. We found that at the age of 10-11, when students start to learn decimal numbers, some children are able to build analogies for ***Q*** with the way infinity works on ***N,*** without any formal training on related concepts.

CHILDREN'S ARGUMENTS: HOW TO COMPARE INFINITE SETS?

In our questionnaires and interviews, we asked students to compare the cardinals of some infinite sets. It was found that when undergraduate students used more than one method for comparing infinite sets, they reached contradictory conclusions, of which they were usually unaware (e.g., Tirosh & Tsamir, 1996; Tsamir, 1999). However, some of the students we questioned were able to formulate pertinent arguments.

Particularly, we have been surprised by the accuracy in reasoning of some students in grade 5. Without any formal training on infinity or functions, they effectively found rules to associate one-to-one the elements of the following pairs of sets: {1; 2; 3; … } and { – 1; – 2; – 3;…}; {0; 1; 2; 3; 4;…} and {0; 2; 4; 6; 8;…}; {0; 2; 4; 6;…} and {1; 3; 5; 7;…}. In order to prove that the given pairs of sets have the same cardinal, the fifth graders easily and readily identified the generating pattern of each numerical sequence. They had some difficulties in finding the connections between the elements in each pair, but finally they discovered by their own wits the correspondences. Next, we addressed the question: *Which of the sets*: {0; 3; 6; 9; 12; 15;...} *or* {1; 2; 4; 5; 7; 8;..} *has more elements?* This has generated strong controversial discussions. The first reaction in the class was to reason on the finite case: "from 0 to 15, the first set has 6 elements and the second has 10 elements, because, successively distributing the elements from ***N***, one element enters the first set and two elements enter the second". Within this type of argument, the majority of children agreed that, taking away 0, the second set has two times more elements than the first. This confirms the results of other studies that recorded the finite reasoning on the infinite sets (e.g. Tsamir, 1999). Yet, some students disagreed with this conclusion and started to ask themselves how to proof/ demonstrate one or the other of the assumptions. In what was followed, three of the students' interventions were fundamental.

First, Roxana concluded:

Roxana: *There are no rules, so these sets cannot be compared.*

Then, Anca remarked:

Anca: *At the beginning, the numbers in the initial sets looked to be at random, then we found a rule, so, if we do not have the rule yet, this does not mean it doesn't exist.*

Their reactions have been completely spontaneous and unexpected within the discussion. The rules at which Roxana refers to are the ones by which they described the sets in the previous examples as ordered sequences (for instance, the elements of the set {0; 2; 4; 6; 8; …} „go by twos"). By this comment, Anca has actually expressed the conviction that the sets of numbers are *structured* (in Roxana and Anca's terms that mean that the elements of the sets can be built sequentially, that they "have rules"). The reactions of the two girls led us to the following hypothesis: in arguing about the infinity of a set and in comparing cardinals of infinite sets, students try to identify *structure*(*s*) of that/ those sets.

The third spontaneous reaction was the one of a girl who discovered an associating rule one-to-one between the elements of the given sets:

Mădălina: *I noticed that* $2 \times 2 - 1 = 3$; $2 \times 4 - 2 = 6$; $2 \times 5 - 1 = 9$; $2 \times 7 - 2 = 12$.

It is to observe that in the previous three examples of the pairs of sets to be compared, the associative rules started from the first set, but here the starting point is the second set. Mădălina made an implicit transfer; the one-to-one correspondence between the two sets in each pair seems to be obvious to her. The discussion in the class continued, without intervention from the interviewers. Doubting about the rule, one boy argued that it is not good for 0. Another boy remarked the formula: $2 \times 1 - 2 = 0$. In this way, the fifth graders actually constructed, between the two sets:
A ={0; 3; 6; 9; 12; 15;…} and B = {1; 2; 4; 5; 7; 8;…}, a bijection from B to A of the type: $x \rightarrow (2 \times x - 2) \vee (2 \times x - 1)$.

ARGUMENTS FOR INFINITY: IDENTIFYING STRUCTURES

The fact that students try to determine the structure of certain sets in order to compare their cardinals, led us to resume their arguments from another perspective. We noticed that, when young students can use geometrical arguments, they become very confident in their statements and they seem to have no doubts about the equivalence of the cardinals of some sets. An example of this type follows below.

Interviewer: Which set do you think has more elements: the set of even numbers or the set of odd numbers?

Simona (Grade 6): *I think they are equal…*

Interviewer: Why?

Simona: *Because they are by twos…*

Interviewer: Well, the evens… but the odds?

Simona: *By twos, too…*

Interviewer: So, the sets whose elements increase by twos…

Simona: *Not necessarily by twos! They are* ***congruent*** *... so they have the same number of elements, they are equal…*

How did Simona move from algebra to geometry? In the same interview, she characterized infinity in a topological way (*Infinity that is… it does not have limits… something without end…*) and referred to density as a way to compare the cardinals of the two sets. We assume that the topological perception is the one that facilitated this passage. We also noticed that Simona based her argument on finding a common structure for the two sets.

Mathematical concepts are included in hierarchically organized systems, with multiple relations of subordination, coordination and super-ordination. Internal structures seem to be natural ways to connect concepts (Singer, 2001). Students refer to various structures to argue the infinity or the cardinal equivalence of some sets: it also seems that, by identifying a structure, children are able to cross in – between processional and topological perception of infinity. We therefore advanced in describing some structures emphasized by students.

An interesting way of arguing the infinity of a given set was to highlight a tree structure. We identified algebraic tree structures and geometrical tree structures.

An algebraic three structure is emphasized in the following excerpt:

Interviewer: Is the set of rational numbers between 2 and 5 infinite?

Alice (grade 6): *Yes!*

Interviewer: Why? Look, I have the smallest number and the biggest… why should be this an infinite set?

Alice: *Well, yes, could be 2.1; 2.11 ... I mean 2 point ... 111 and so on, I mean ...*

Interviewer: And you say they are infinitely many…

Alice: *Perhaps they are not quite infinitely many, because finally we still get to number 3, but they can be said as a sequence ... it might be ... number 2.1., it might be 2.11 to 2.19, and so on ... number 2.11 might be 2.111 and 2.119 ... and so on…*

The next excerpt reveals a geometrical tree structure:

Interviewer: What about if I consider the set of points from a segment of 2 cm … and I consider the set of points of this segment … what do you think, is a finite or infinite?

Simona (Grade 6): *It is in… infinite, because there could be many points and ... one point could contain many smaller points…so, in a way it is finite, but it contains very many ...*

Interviewer: What does that mean, a point contains very many smaller points?

Simona: *I mean…*

Interviewer: … I did not understand you…

Simona: *…a smaller piece [of a segment] contains more points…*

Interviewer: Well, then I take a piece as small as a point …

Simona: *Yes… that one, too, contains many smaller points ...*

These two examples show that the two girls have a definite structural representation for the entities under discussion. The structures they highlighted are of a *fractal type*. (A definition of fractals can be found in Mandelbrot, 1975.) Thus, children suggest sequential steps where each step is seen through a lens that repeats the model of the previous step at a different scale. We notice that in Alice's case, the primary perception "margins equal finite" („*perhaps they are not quite infinitely many, because finally we do get to the number 3…"),* interfere with another primary perception, the spatial-rhythmic one ("*and so on ... and so on…").* Alice's argument is not contradictory because the fractal structures are of spatial-rhythmic type.

SOME CONCLUDING REMARKS: STRUCTURES RESPONSIBLE FOR PROCESSING INFINITY?

Two findings could be synthesized as results of this study. The first is telling us that some young children have a structured representation about the infinite sets. This is happening as soon as they learn about the set of natural numbers, in primary grades. At the age of 10-11, when students learn about the decimal numbers, they are able to identify structures that are helpful in arguing about the infinity of some sets or in suggesting hints for the cardinal equivalency. The second finding refers to the fact that, when the students' arguments are consistent, they seem to be based on connections between algebraic and geometrical thinking. It is necessary to make a distinction. In early grades, the interference between algebra and geometry is based on measuring, while in later grades the correct reasoning is based on geometrical transformations. As the traditional teaching separately deals with algebra and geometry, this natural tendency of the students could be interpreted as an argument for the presence of a mental structure able to process the concept of infinity.

A conclusion that could be drawn from these findings is that, infinity being an important intrinsic concept connected with the number formation and with the sets of numbers structure, the aspects connected with the idea of infinity should be part of the training very early in concepts' learning. We did not say as explicit part of the curriculum because here any attempt to formalize is dangerous; on the contrary, the approach has to be made using various examples from different contexts and emphasizing various points of view, outside mathematics (Singer, 2003). Here, lots of precautions are necessary, because, on the one hand, only through numbers infinity could be defined and explained, and, on the other hand, the trap of paradoxes is always near by, when we are dealing with infinite sets. If we take into consideration recent research in mind and brain, there is a close interrelationship between some natural predispositions-intuitions and the learning process, which rebuild connections and structures. This is why, bringing intuition into the math class and starting building new knowledge from here might be the way to diminish misunderstandings and misconceptions in fundamental areas of mathematics learning.

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THE ROLE OF SPATIAL CONFIGURATIONS IN EARLY NUMERACY PROBLEMS[1]

Fenna van Nes, Prof. Jan de Lange

Freudenthal Institute for Science and Mathematics Education, the Netherlands

Little is known about how early spatial thinking and emerging number sense may be related in the development of mathematical thinking. In this explorative study we examine how a child perceives a structure in a spatial configuration and applies it to determine an amount, and how this ability may be related to a child's number sense and mathematical performance. Fifteen 4-year olds, fifteen 5-year olds and fifteen 6-year olds were interviewed as they performed a series of number sense and spatial tasks. An association was found between a child's ability to apply spatial structures, and their level of mathematical performance. This triggers questions for further research about how spatial thinking may influence the development of number sense.

INTRODUCTION

Many studies have focused on how remarkably young children augment their knowledge as they explore and discover the world around them. Gopnik related children's behaviour to that of scientists in her Theory Theory (e.g. 2004). She suggests that children are born with certain theories about the world, which are continuously tested and amended as children gain new insights from daily experiences. It is these apparently natural abilities that allow for children to learn and develop so easily.

One of the areas in which young children develop faster than is generally assumed, is in mathematical ability. Recent research on children's numerical abilities has provided evidence that infants as young as six months can differentiate between amounts of objects that differ by a 2.0 ratio (i.e. 8 versus 16 objects, Lipton and Spelke, 2003). This ability improves within months as 9-month old infants can differentiate sets that differ in number at a 1.5 ratio (i.e. 9 vs 6 objects). In another study, Berger, Tzur and Posner (2006) found that 6-month old infants can recognize simple addition errors and that the corresponding brain activity can be compared to that of adults detecting an arithmetic error.

As for children's spatial abilities, studies have shown that 16-24 month old infants can use the concept of distance to localize objects in a sandbox (Huttenlocher et al., 1994). This indicates an early ability to judge distances that is manifested regardless of the presence of any references in the direct surroundings of the child. Other studies involving four- and five-year olds provided evidence that at this age children can

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compare proportions and figures (Sophian, 2000). The children in this study were able to match the correctly shrunken picture to the original picture without being distracted by pictures that not only were smaller, but also disproportional to the original picture.

These studies are representative for the research on the development of numerical and spatial abilities in young children. What lacks in much of this type of research, however, is any speculation about an association between the development of numerical and spatial abilities as children gain mathematical understanding. We propose that these two domains are related and that this may greatly influence the development of young children's mathematical thinking. This paper discusses an explorative study in which we aimed to better understand the nature and characteristics of such a relationship.

Number sense and spatial thinking

Before describing association between number sense and spatial thinking, we first define the two concepts. Number sense implies the ease and flexibility with which children operate with numbers (e.g. Gersten and Chard, 1999). It involves an awareness of amounts, giving meaning to numbers and being able to relate the different meanings of numbers to each other (Van den Heuvel-Panhuizen, 2001). As children progress in their ability to count, they discover easier ways of handling numbers, and they come to understand that numbers can have different representations and can act as different points of reference (Van den Heuvel-Panhuizen, 2001).

Two key concepts in number sense are ordinality and cardinality. Ordinality concerns the order of numbers, and cardinality has to do with understanding the meaning of a number which is usually developed as children reach the age of five. This is when children learn to count resultatively; they count a set of objects and understand that the last named number stands for the total amount of objects in the set (Gelman and Gallistel, 1978).

Most early mathematics curricula are mainly concerned with teaching young children how to count, but children tend to acquire many spatial skills both inside and outside the classroom. Spatial thinking involves grasping the external world (Freudenthal, in National Council of Teachers of Mathematics [NCTM], 1989). The children's experiences in discovering their environments help them to gain insight into relative positioning and sizes of shapes and figures. Children learn to orientate themselves, to describe routes, and to understand shapes, figures, proportions and relationships between objects, as well as to order, describe and compare physical sizes of objects (e.g. Van den Heuvel-Panhuizen and Buys, 2005). These abilities are typically manifested even before these children begin formal schooling.

The origins of human mathematical thinking

Much neuropsychological research has delved into a debate about the origins of human mathematical thinking. Dehaene and colleagues (1999), for example, describe two systems in the brain that represent amounts. One system is based on a circuit in the brain that is associated with language and in this sense helps store exact numerical information. The other system is based on a circuit that is activated by general visual-spatial functions and helps store and manipulate approximate calculations. The neurological foundations of this theory are described in the Triple Code Model (Dehaene et al., 2003) which unites various areas of the brain that are typically associated with spatial functions.

As such, Dehaene and colleagues suggest that the human brain can specifically represent and process numbers. Simon (1999), however, proposes that the origins of human mathematical thinking lie more in the general nature of the human perception and attention systems; numerical processes are not specifically represented in the brain but, instead, depend on areas that are specialised in visual-spatial processing. He bases his conclusions on research about how when children count using their fingers (an important step in the construction of a mental number line), the same areas in the brain are activated as associated with hand figures and finger movements. Likewise, Freudenthal expressed no doubt that the ability to judge similarity between objects precedes number in cognitive development (Freudenthal, 1991).

These and many other researchers propagate the discussion about whether the brain is specialized in working with numbers, or whether the brain is 'non-numerical (Simon, 1999) in that the processing of numbers depends on spatial brain functions.

The present study: using spatial configurations to help count

Compared to neuropsychological research, only a number of early studies in mathematics education have focused on a possible relationship between spatial skills and mathematics. These studies mainly found supporting evidence for a positive relationship between spatial ability and mathematics achievement (e.g. Clements and Battista, 1992; Guay and McDaniel, 1976; Tartre, 1990), but they do not consider how the relationship may be characterized in the earliest mathematical development.

As such, the present study is an explorative and descriptive investigation into the relationship between spatial and numerical thinking in the development of mathematical abilities. In particular, we seek to better understand how early spatial thinking may influence the attainment of number sense, by examining how the ability to recognize and apply spatial configurations to numerical problems can help children simplify and shorten the counting procedure. Ultimately, we hope to collaborate with neuroscientific research for a more vivid and in-depth understanding of how mathematical thinking develops.

The focus of the study complements research such as that on visualisation in mathematics. As Bishop (1988) notes, 'mathematics is a subject which is concerned with objectivising and representing abstractions from reality, and many of those representations appear to be visual' (p. 170). Arcavi (2003) goes on to state that such visual imagery can be 'at the service of problem solving' since visualisation 'may play a central role to inspire a whole solution' (p. 244). A problem where students had to determine how many matches are needed to build a particular n x n square, for example, lead to various kinds of decomposition that simplified the counting process (Arcavi, 2003). Indeed, the change of gestalt sometimes took the form of imposing an 'auxiliary construction', providing visual 'crutches' which supported and facilitated the visualization of a pattern that suggested a counting strategy (Arcavi, 2003, p. 229). This clearly illustrates the heuristic potential of distilling a structure out of a spatial configuration in order to determine an amount.

In the present study, we focus on 4- to 6-year old children in the first two grades of a Dutch elementary school. These children are of particular interest because this is the age at which children are first confronted with formal mathematics education as opposed to their informal and more intuitive ways of learning. Hence, the question is how and to what extent their developing number sense is influenced and supported by early spatial thinking skills. We investigate their ability to recognize and apply typical spatial structures (dot structures on dice, and finger images for counting fingers) to solve number sense problems in order to examine the effect on mathematical performance. This exploration should shed more light on the role that spatial thinking may play in the development of number sense and mathematical performance in school.

A second point of interest in this study is whether and how children may be differentiated in terms of the extent to which they recognize and apply structures to mathematical problems. We speculate that children who do not adequately apply a structure from a spatial configuration to determine an amount, may come to lag behind and increasingly experience difficulty in attaining number skills. In this study, we explore whether and how an association between recognizing and applying spatial structures in spatial configurations exists. We hope to gain a better understanding about what potential spatial skills may have to stimulate the mathematical performance of children with relatively weak mathematics skills.

In summary, this study addressed two main research questions:

1. How do children aged 4 to 6 years develop insight into spatial structures and use structures to solve (numerical) mathematical problems?

2. How is the extent to which a 4- to 6-year old child recognizes and applies structures related to the child's level of mathematical ability?

METHODS

Participants

Fifteen four-year olds, fifteen five-year olds and fifteen six-year olds with middle class social backgrounds from the first two grades of a local elementary school participated in this study. These children were identified by the head-teacher of the school's elementary department with the aim of forming an as representative a group as possible with respect to mathematical performance in school, age and gender. They participated with the consent of their parents and at the end of the study the whole class was rewarded for their participation.

Materials and procedure

As the aim of the study was to explore how children may make use of spatial structures in mathematical problem solving, the first part of the study involved the development of a set of tasks that would make the strategies that the children used evident to the researchers. The tasks were to cover general spatial abilities and numerical understanding, but they had to be complex enough to challenge the 6-year olds and still be accessible to the 4-year olds. Furthermore, the tasks had to be intriguing enough to capture the attention of the children and to trigger their problem solving interests. As such, paper-and-pencil tasks were avoided and, instead, context-rich, interactive activities were developed that would appeal to the child.

The tasks were originally inspired by literature (e.g. Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996) and meetings with experts, after which they were frequently pilot tested and improved. The final list of tasks included five number sense activities covering counting of structured and unstructured small amounts, applying own structures, comparing two groups, equalizing two groups, dividing a set into two groups, manipulating amounts and determining positions in a row. Figure 1 is an example of a task about counting a structured amount.

Figure 1: 'How many ducks are swimming in the pond?'

The five spatial thinking activities cover patterns, constructing, counting parts of a construction, taking on perspectives, and recognizing shapes and figures. Figure 2 shows a task about constructing a house of blocks by following an example.

Figure 2: 'Can you build a house just like the example with these blocks?'

Parallel to developing a list of appropriate tasks, based on literature and expert meetings, we created a list of strategies that children may use for each task. The strategies that were observed during the pilot tests contributed to making the list as elaborate as possible and to organizing the strategies from relatively basic (e.g. pointing to each object and counting out loud) to more complex (e.g. perceptive counting). The more complex strategies are usually the strategies that require recognizing and applying structure to complete a task. This list of strategies was a key instrument in the study for exploring whether and how children make use of spatial abilities to solve the different types of tasks. As such, the approaches that children used were evaluated both according to their relative complexity (i.e. use of structure) and how they coincided with accuracy in solving the tasks.

Next to the tasks, a set of sixteen flashcards was created in order to determine the degree to which children recognize structure. There were four dice cards, each with three to six structured dot configurations, four dot cards, each with three to six unstructured (defined to be taken out of context and unrelated to learned configurations) dot configurations, and eight cards with three to ten raised fingers each (see Figure 3). The cards were presented one-by-one for no longer than three seconds. This was enough time to determine whether the child recalled the structure or had to count each of the dots or fingers to determine the presented amount.

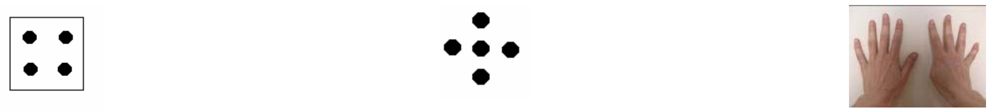

Figure 3: Examples of a die flashcard, a dot flashcard and a finger counting card

The last element of the study was the Utrecht Numeracy Test (UNT, van Luit et al., 1994). This test was included so that the results of the tasks and the flashcards could be interpreted against the backdrop of a normative numeracy test. The UNT covers comparing, classifying, correspondence, seriation, using counting words, synchronized and shortened counting, resultative counting, and applying number knowledge. The test is normed for children aged 4.5 to 7 years, and scores range from a maximum of 'A' to a minimum of 'E'.

The study was performed in three sessions per child. Each session took half an hour and occurred on three subsequent days. The first session covered the UNT followed by the flashcards, the second session covered the spatial tasks and the last session covered the number sense tasks. Importantly, the researcher prioritized the approach that children took to solve a problem in contrast to merely noting whether or not the given answer was correct. Therefore, the style of the study was interview-like, theory driven and interactive; the children were continuously stimulated to think aloud and to elaborate on the strategy that they used. In the meantime, the researcher noted as much of the child's verbal and nonverbal behaviour as possible and recorded the session with a voice-recorder for later reference.

The transcripts were analysed both qualitatively and quantitatively. As this study was designed to be descriptive and explorative, we prioritized the qualitative analyses in order to gain a vivid picture of the group tendencies without eliminating children whose results were outliers with respect to the general group. These analyses involved cycles of examining the interviews, categorizing the children's responses according to the list of strategies, discussing possible trends with experts, and comparing the strategies and accuracy results of children for each type of task with other tasks and other children. The quantitative analyses were performed to allow for large group comparisons with non-parametric correlations and tests for significance.

RESULTS AND ANALYSIS

From the initial analyses of the interviews, it became clear that the 5- and 6-year olds who recognized the structures also tended to apply them. Relatively few 4-year olds, however, either recognized or used the structures to solve the tasks. Those 4-year olds who did recognize structures were not always consistent in applying their knowledge to the tasks. Sometimes they counted the objects, other times they instantly recognized the amount. This coincides with Gopnik's Theory Theory (2004, as described above), as well as with what de Lange described as Conceptual Mathematization (1987).

Conceptual Mathematization is the process of developing mathematical concepts and ideas, which involves exploring situations, schematizing, visualizing, and developing a model that leads to a mathematical concept. By reflecting and generalizing, a child can spiral in to a more developed concept. Hence, the 4-year olds in this study may have recognized the structures and used them sporadically, but they are still to increase their experience and affiliation with the benefits of applying such structures. This first insight into how the use of spatial structures emerges with a child's development should be a topic for elaboration in further research.

What is especially intriguing, is that there were three 5-year olds and one 6-year old who, like many 4-year olds, recognized the structures (i.e. instantly read out the presented amount without having to count) but did not proceed to apply the structures to the tasks. In contrast to the four-year olds, however, these older children performed below average on the tasks. This is curious, since age was significantly related to performance on each of the types of tasks and older children tended to recognize and apply more of the recognized structures to the tasks than younger children did. Furthermore, performance on the tasks improved when the recognized structures were used. These four children, then, seemed to be delayed in learning to apply recognized structures, performing well under the average of their age group. Importantly, the results of this study do not imply a causal relationship, but they do accentuate the question about how the development of the ability to recognize and apply structures to mathematical problems may relate to the progression of mathematical performance.

The interviews and analyses indicated a positive relationship between the type of strategy usage of the 4- and 5-year olds and their level of performance. As a more complex level of strategy usage generally coincided with more use of structures, these results support our expectation that children with relatively strong spatial thinking skills would recognize more spatial configurations and would use them more regularly for facilitating the counting process. This allows for the differentiation between the mathematical performance of a child who recognizes and applies structures to mathematical problems and the mathematical performance of a child who has yet to develop such an insight. Hence, the ability to use recognized structures and the mathematical performance of young children seem to be related.

Compared to these results for the 4- and 5-year olds, the results for the 6-year olds seem less evident. The 6-year olds tended to apply more complex strategies to the tasks despite their below average general mathematical performance. It seems as though they experimented with various (perhaps newly learned) strategies, without yet having the expertise to always be able to apply the strategies correctly. This situation resembles the inconsistent behaviour of the 4-year olds mentioned earlier, once again illustrating the process of learning, experiencing and adjusting behaviour as described in research of, for example, Gopnik (2004) and de Lange (1987). Taken together, these results are highly notable as they, firstly, provide a more specific account of how spatial skills and mathematics performance may be related than earlier studies have, and, secondly, focus on development and the youngest elementary school children.

CONCLUSION

As an aside, we note that only careful conclusions can be drawn from this study. Many children responded differently to the various tasks and the number and representativeness of the subjects may have been limited. This can be traced back to the individual differences of children as well as to the difficulty of performing such research and generalizing the results. Yet, the design of the study was for it to be descriptive, and the methods of inferring the intentions and approaches of the children to the various tasks are still exploratory. Subject to these limitations, then, the results provide some insight into how the development of spatial thinking and number sense may be related in the emergence of mathematical thinking.

As this is a first explorative study into how early spatial thinking and emerging number sense may be related in the development of mathematical abilities of 4- to 6-

year old children, it has set the stage for future research to further examine the possible influential effects of the level of spatial thinking on mathematical performance. The result that not all 5- and 6-year olds in the study recognized and subsequently applied the structures is especially interesting in this context because it triggers the speculation about causality between the application of spatial skills and children's lagging mathematical performance.

For more practical purposes, this study has suggested that it could be useful for children to at least be familiar with basic spatial configurations so as to learn to perceive them in sets of objects that are to be counted. Our next intention is to design an intervention in which children learn to recognize and apply spatial structures to tasks that involve numerosity. This may help children with relatively poor mathematical skills to improve their counting skills and gain more insight into numerical relationships. The list of strategies from this study will help categorize, evaluate and monitor the approaches that children take to solving the tasks.

This study has taken an important initiative for mathematics education to contribute to the ongoing debate about the characteristics of mathematical thinking. The long-term plan for this research is to cooperate with neuropsychological research for a more all-round and in-depth perspective on possible influential effects of spatial thinking on mathematical abilities. This should contribute to a better understanding of the development of children's mathematical thinking.

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STUDENTS' ABILITY IN SOLVING LINE SYMMETRY TASKS

Xenia Xistouri

University Of Cyprus

The aim of this study is to propose and evaluate a model of 4th, 5th and 6th grade students' structure of knowledge in line symmetry. The model used is the Taxonomy of Structure of the Observed Learning Outcome (SOLO). The model describes the structure of students' aptitude to respond correctly to tasks of line symmetry, and thus it can be used by teachers to enhance students' learning.

INTRODUCTION

Symmetry is a fundamental part of geometry, nature, and shapes. It creates patterns that help us organize our world conceptually (Knuchel, 2004). It is important for students to grasp the concepts of symmetry while at the elementary level, by exposing them to things they see everyday that are not obviously related to mathematics but have a strong foundation in it (Knuchel, 2004). According to the NCTM (1991), grades 3-5 should be able to use symmetry to analyze mathematical situations. This includes predicting and describing the results of sliding, flipping, and turning two-dimensional shapes. They should also be able to identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs.

Many researchers (Knuchel, 2004; Mackrell, 2002; Hoyles & Healy, 1997) support that symmetry is all around us and even though it does not seem to be mathematical, its very roots are buried there. However, it appears that there is little research concerning the development of symmetry in the field of mathematics education, especially at the elementary level. Thus, the purpose of this study is to construct an understanding of the structure of students' knowledge regarding line symmetry, by developing an assessment model for which the students' knowledge and abilities for solving line symmetry tasks are taken into consideration. To this end, the Taxonomy of Structure of the Observed Learning Outcome (SOLO) was used as a means for exploiting students' development of knowledge in symmetry (Biggs & Collis, 1991).

THEORETICAL BACKGROUND

Developing the Concept of Symmetry

According to Leikin, Berman and Zaslavsky (2000), symmetry has a special role in problem solving. In their studies with secondary mathematics teachers, they underline that symmetry connects various branches of mathematics such as algebra, geometry, probability, and calculus, and present it as a useful problem-solving tool.

At the elementary level, research mainly focuses on teaching experiments, often involving technology (Edwards, 1992; Mackrell, 2002; Seidel, 1998; Hoyles & Healy, 1997). For example in Mackrell's (2002) class, while students were exploring the creation of polygons using geometry software, they created a lot of abstract patterns that were symmetrical and they were concerned with filling in the gaps of their patterns. She concluded that while she was not attempting to teach or emphasize any particular area of mathematics, "ideas regarding size, symmetry, tessellation and representation of 3D objects were arising spontaneously and, given more time, could have been further developed" (Mackrell, 2002). Another example is the one of Seidel's (1998) experiment, where he applied geometry and symmetry to everyday life for 2nd and 5th graders by means of technology. In this experiment, the 2nd graders were using a program to create symmetrical flowers to make a garden. This lesson gave students a chance to solve their own problems. The 5th graders were introduced to Geometer's Sketchpad and used it to create symmetrical snowflakes. It let them use the concepts of rotation and reflection, allowing them to tie in the idea of translations and how all three are related. Edwards (2003) also described a series of studies using a computer environment designed for exploring transformational geometry, which included some tasks of symmetry. In this series of studies, she studied misconceptions concerning transformation geometry at various ages and conducted teaching experiments with the help of microworlds. Her results suggest that misconceptions are not different among age groups.

Another experiment was conducted by Hoyles and Healy (1997), using a micro world tool called "Turtle Mirrors" to provide tools to help students focus simultaneously on actions, visual relationships and symbolic representations concerning line symmetry. According to these researchers, the rich set of meanings around symmetry which is developed outside school, shapes student responses in the mathematics classroom. In their study, they described some students' primitive and intuitional strategies for solving paper and pencil symmetry tasks, with special focus on 12-year old Emily. They described how Emily solved the tasks, by using a variety of strategies, which varied according to task features. They also described how Emily successfully reflected objects in horizontal or vertical mirror lines but when the mirror was slanted, she had difficulty completing the task and she used an approximation strategy which derived from paper folding (Hoyles & Healy, 1997).

The above information suggests that symmetry is given minor attention by teachers as well as by researchers. It is often de-mathematized and its teaching mainly focuses on what children already know intuitively and by experience, while in research it appears to be mostly a means for implementing technology. However, apart from Hoyles's and Healy's study, no other study seems to highlight the structure of students' knowledge of symmetry and its properties. Therefore, in order to support the mathematical development of symmetry rather than the intuitional, it might be useful to develop a structural model which will guide and support mathematics educators' instruction and assessment. For this purpose, this study will use the SOLO Taxonomy hierarchical model to describe the development of the concept of symmetry and to provide a structure for measuring students' learning outcomes in symmetry, with focus on line symmetry.

Solo Taxonomy

From the literature on SOLO taxonomy, it was suggested that SOLO is a hierarchical model that is suitable for measuring learning outcomes of different subjects, levels and for all lengths of assignments (Biggs & Collis, 1991). The SOLO taxonomy, initially proposed by Biggs and Collis (1991), evaluates and categorizes cognitive performance by considering the structure of students' answers. A response is prompted by a question, and is indicative of the difficulty of the question and the cognitive ability of the individual. A response varies between 5 levels of complexity, ranging from prestructural to extended abstract.

- Prestructural: The task itself is not attacked in an appropriate way. The student misses the point.
- Unistructural: One aspect of the task is picked up or understood serially, and there is no relationship of facts or ideas.
- Multistructural: Two or more aspects of a task are picked up or understood serially, but not interrelated.
- Relational: Several aspects are integrated so that the whole has a coherent structure and meaning.
- Extended abstract: That coherent whole is generalised to a higher level of abstraction.

In this study, the extended abstract level is not examined, assuming that fourth, fifth and sixth grade students cannot reach this level of performance.

METHODOLOGY

The purpose of this study is to develop a cognitive model for the assessment of 4^{th}, 5^{th} and 6^{th} grade students' structure of knowledge concerning symmetry. The subjects were 474 elementary students (150 fourth graders, 202 fifth graders and 122 sixth graders).

The students were given seven tasks to solve (see Appendix), in order to measure their ability in solving mathematical tasks of line symmetry. The tasks were presented in increasing order of difficulty, with task 1 corresponding to the described characteristics of the unistructural level and task 7 to the characteristics of the relational level. The first task was to identify shapes with lines of symmetry. It included 8 items, 5 of which had lines of symmetry. The second task was to draw all lines of symmetry for given shapes. Six items were given, of which: 1) one had a horizontal line of symmetry, 2) one had a slanted line of symmetry, 3) one had two slanted lines of symmetry, 4) one had five lines of symmetry (one vertical and four slanted), 5) one was consisted by two reflected congruent triangles attached by a point and 6) two reflected congruent shapes with a horizontal line inside the shapes and a vertical line of symmetry between the shapes. The third task was to identify which of the first eight letters of the Greek alphabet have exactly two lines of symmetry. The fourth task was to identify the symmetrical of a given shape, among three different alternatives with differences in shape, colour and size. This task included three items. The fifth task was to shade boxes to draw the symmetrical of a given shape, given the line of symmetry. The task included two items. The first had a vertical line of symmetry and the second had a horizontal line of symmetry. The sixth task was to draw the symmetrical of a given shape. This task included two items. The first had a horizontal line of symmetry, one centimetre away from the shape and the second had a slanted line of symmetry, attached to the shape by a point. The seventh task was the problem:

> John discovered the rule: "I can say whether a quadrilateral has lines of symmetry. If the triangles formed by the folding line are exactly the same, then the shape has a line of symmetry". Explain whether you agree or not.

The students were given 40 minutes to complete all seven tasks, during normal lesson time. Each correct response to an item in each of the seven tasks was assigned a positive point. The points were summed up for each task. The percentages of success in each task were calculated for each task. Success in a task was considered the achievement of all points. The percentages were used for structuring the model of students' knowledge in symmetry. The tasks were grouped according to the percentages of success and task characteristics. Percentages were divided into three equal ranges of about 15 percentiles each. The tasks with percentage of success in each range were considered to form a group, and then each group was matched to a level of the SOLO Taxonomy, according to the level of success. A deeper examination of the characteristics of the tasks followed to describe and determine which aspects of symmetry appear to be applied in each group of tasks and further on to confirm the matching between each group and the corresponding SOLO level.

RESULTS

The validation of the model is based on students' success in solving the symmetry tasks. Table 1 presents the percentages of success for each task and matches them to the SOLO Taxonomy levels, according to the levels of success. The analysis indicated that the items of task 6 were actually quite different; therefore they were considered as different tasks during the classification. Following are the descriptions of the aspects of symmetry appearing to be understood by the students in each level, to confirm the matching between each group and level.

A percentage of 16,3% of the students were matched to the prestructural level, since they did not manage to succeed in any of the tasks. This percentage is not included in the table, since it does not measure success in a task. Tasks 5 and 4, with percentage of success between 31% and 45% were grouped and matched to the unistructural level. Tasks 3, 1 and 6a, with percentage of success between 15% and 30% were grouped and matched to the multustructural level. Tasks 7, 6b and 2, with percentage of success between 1% and 14% were grouped and matched to the relational level.

Task	% of Success	Matching SOLO Level
5. Shade boxes to draw the symmetrical of a given shape	40,9	Unistructural
4. Identify the symmetrical of a given shape	36,1	
3. Identify which of the eight first letters of the Greek alphabet have exactly two lines of symmetry	30,4	Multistructural
1. Identify shapes with lines of symmetry	14,8	
6a. Draw the symmetrical of given shape (vertical line of symmetry)	14,8	
7. John says: "I can say whether a quadrilateral has lines of symmetry. If the triangles formed by the folding line are exactly the same, then the shape has a line of symmetry". Explain whether you agree or not	11,2	Relational
6b. Draw the symmetrical of given shape (slanted line of symmetry)	6,5	
2. Draw all lines of symmetry for the given shapes	1,5	

Table 1: Percentages of success for each task and matching to SOLO levels

Unistructural Level

The tasks grouped and matched to the unistructural level are 5) Shading boxes to draw the symmetrical of a given shape, and 4) Identifying the symmetrical of a given shape, among three choices.

Task 5 was included in this level of understanding, since it can easily be solved in an analytical way, by counting boxes from one direction to another. Apparently, it does not require the application of special mathematical abilities in symmetry. The main aspect taken into consideration for this task and consequently for students at this level, is the "same distance from the line".

Another main aspect of symmetry is the conservation of shape, size and colour of the reflected image (Hoyles & Healy, 1997). However, it should be noted that in task 4, each of these qualities was isolated among three separate items. It is assumed that combinations of the three properties could increase the level of difficulty and such task would probably be classified at a higher level of understanding.

Multistructural Level

Three tasks were grouped to match this level. These are 3) Identifying which of the eight first letters of the Greek alphabet have exactly two lines of symmetry, 1)

Identifying the symmetrical shapes, and 6a) Drawing the symmetrical of a given shape with vertical line of symmetry given.

It seems that identifying a shape with symmetry requires a combination of more skills and better understanding of symmetry. In task 3, students had some additional aspects to consider. They had to test and identify both a vertical line of symmetry and a horizontal line of symmetry, since the first eight letters of the Greek alphabet only have vertical and horizontal lines of symmetry. These had to be done while taking into consideration that the parts of the shape had the same shape and size, as well as the same distance from each line of symmetry. The same applies for the case of task 1, where children had to determine whether the given shapes were symmetrical. The additional aspect which made the percentage of success drop to 14,8% was the fact that some of the shapes had slanted lines of symmetry. These were also shapes that children are not as familiar with as the letters of the alphabet, which are often used by teachers in Cyprus as common examples for teaching symmetry.

The aforementioned properties of conservation of shape, size, and colour of the shape have to be also considered while drawing the symmetrical of a shape, such as in task 6a. Additionally, the ability to "flip" the image in order to obtain the reflection of the shape, while keeping track of the distance from the line of symmetry are also necessary.

Relational Level

For this level, three tasks were also grouped. These are 7) the problem [John says: "I can say whether a quadrilateral has lines of symmetry. If the triangles formed by the folding line are exactly the same, then the shape has a line of symmetry". Explain whether you agree or not], 6b) Draw the symmetrical of given shape with slanted line of symmetry, and 2) Draw all lines of symmetry in the given shapes.

In the case of task 7, all of the properties involved – conservation of shape and size, distance from slanted line – have to be considered and combined in order to determine whether the congruency of the triangles obtained by folding a quadrilateral is a valid indication of symmetry. Therefore it requires the integration of several aspects in order to extract meaning and apply it in a problem-solving situation. The integration of a number of aspects is a characteristic of the relational level of the SOLO Taxonomy.

Drawing the symmetrical of a shape with a slanted line of symmetry given is, as expected, more difficult than drawing with a vertical line given. Although it seems that the aspects of symmetry applied are the same as for the vertical line (as in task 6a), reflecting a shape on a slanted line of symmetry was more difficult to students. However, this might be due to lack of experience in such type of tasks, since common tasks of line symmetry often involve shapes with vertical or horizontal lines of symmetry, and much rarely with slanted lines of symmetry.

Finally, the smallest percentage of success was on drawing the lines of symmetry for given shapes. In this case, all properties must be taken into consideration and students' abilities and knowledge are required to be applied not only simultaneously, but also repeatedly for several times, in order to manage to find and draw correctly all the lines of symmetry – vertical, horizontal and slanted.

DISCUSSION

The aim of this study is to construct an understanding of the structure of students' knowledge regarding line symmetry, using the SOLO Taxonomy. As it appears from the percentages of success for the grouped tasks as well as the tasks' characteristics, the structure of students' knowledge regarding line symmetry can be described using the levels of the SOLO Taxonomy. The results suggest that more aspects of line symmetry are considered and applied for completing the tasks for each successive level, which is what makes the SOLO Taxonomy levels distinct.

Particularly, it seems that for the tasks of the first group, which was matched to the unistructural level of the SOLO Taxonomy, students apply only one property of symmetry; they only see one aspect at a time. However at the second group, students see more properties simultaneously to solve the tasks. This is in agreement with Biggs and Collis (1991) descriptions of the multistructural level; therefore the second group was matched to that level of the SOLO Taxonomy. As for the third group, all the properties had to be integrated in order to reach to a conclusion which was more complicated than in the other tasks. Thus, this group of tasks was matched to the relational level of the SOLO Taxonomy.

While examining the characteristics of each task and the aspects viewed at each level, it appears that while identifying a shape with symmetry was considered an easy task, the data suggest that it is grouped at the multistructural level, since it requires a combination of more skills and better understanding of symmetry than finding its symmetrical among others. In fact, it requires the ability to test and recognise all types of lines – vertical, horizontal and slanted – to determine whether a shape is symmetrical. Additionally, one has to keep in mind the conservation of the properties of shape, size and colour, all at the same time. Similarly, these properties also have to be taken into account in the case of drawing the symmetrical of a shape, such as in task 6a, while at the same time the ability to "flip" the image in order to obtain the reflection of the shape is also necessary. In addition, one has also to keep track of the distance from the line of symmetry, in order to get the correct solution.

As for the relational level, and particularly in task 7, one needs not only a combination of skills to determine whether the congruency of the triangles obtained by folding a quadrilateral is proof for symmetry. The strong characteristic of this task, which is crucial for the relational level, is that it requires for all the skills involved to be combined in order to reach to a meaningful decision, a general rule. However, the verbal form of the task as well as the requirement of types of quadrilaterals may have increased the level of difficulty for this task. One could also assume that finding the symmetrical of a shape for slanted line of symmetry, such as in task 6b, belongs to a higher level of understanding symmetry, one that cannot easily be completed with an analytic way. Similar results were found by Hoyles and Healy (1997), which strengthens the fact that it was grouped at the relational level. Finally, finding all lines of symmetry seems to be the most difficult task of the relational level. This is probably due to the fact that it requires viewing all aspects simultaneously and repeatedly, in order to decide the occurrence of symmetry and find the place of the line.

The structural model developed in this study may be useful for assessing students' development of knowledge in line symmetry. It can be used to develop an effective instructional program in understanding line symmetry and solving line symmetry tasks. Further research could evaluate the viability of using the model for teaching symmetry in regular classroom situations or with computer environments, to assess the ease with which classroom teachers are able to use the model to enhance students' learning. Such research would provide opportunities for improving the model and making it more efficient for generating instructional programs that build on students' prior knowledge, monitor and assess their understanding.

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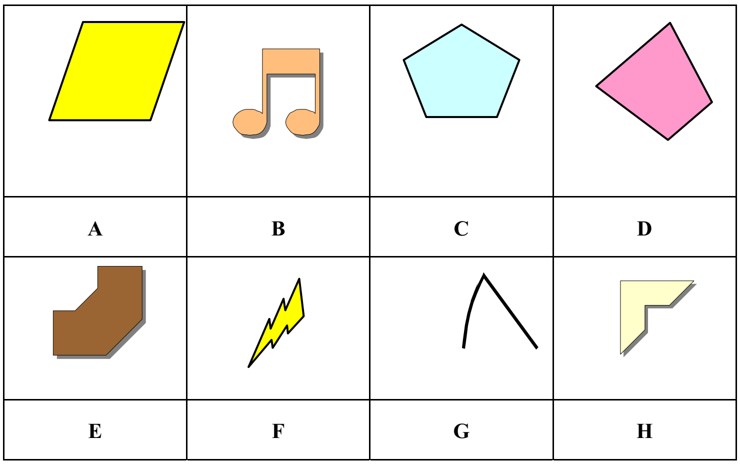
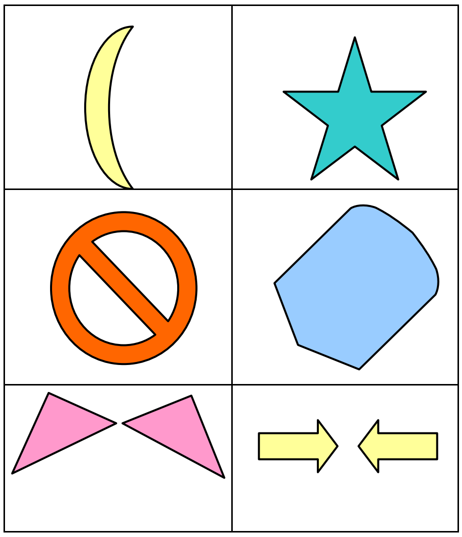
APPENDIX

The tasks

1. Circle the shapes that have line symmetry.

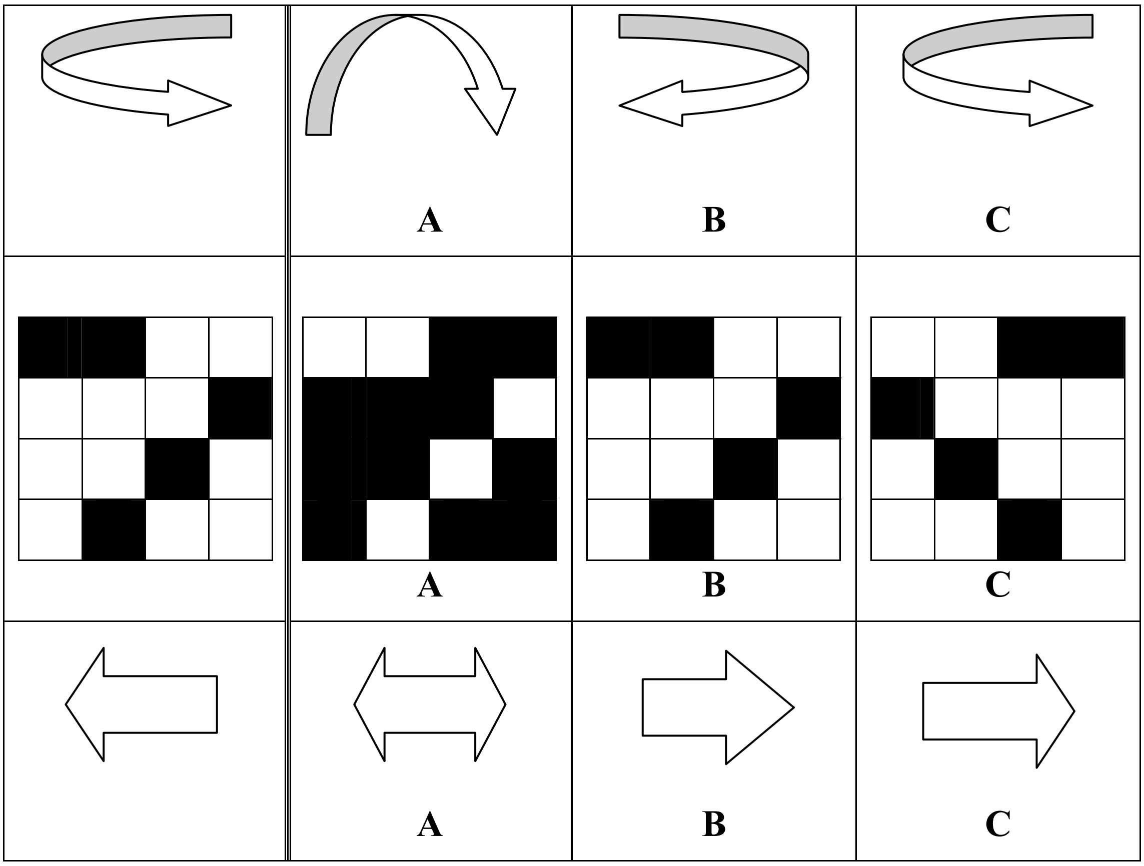


2. Draw **all** the lines of symmetry for the given shapes.

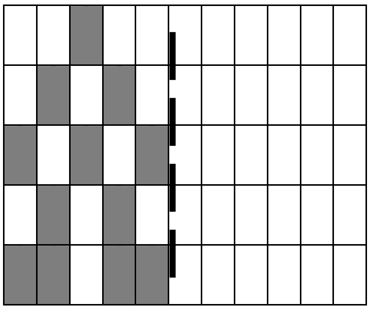


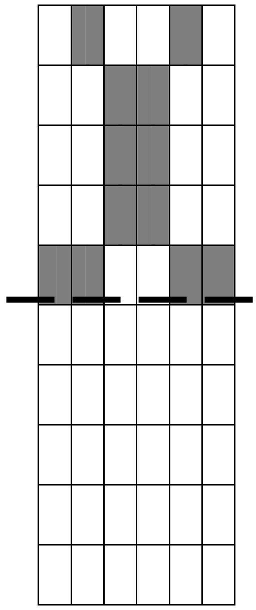
3. Identify which of the first eight letters of the Greek alphabet have exactly two lines of symmetry.

4. Which one of A, B, C is symmetrical to the first shape, on a vertical line?

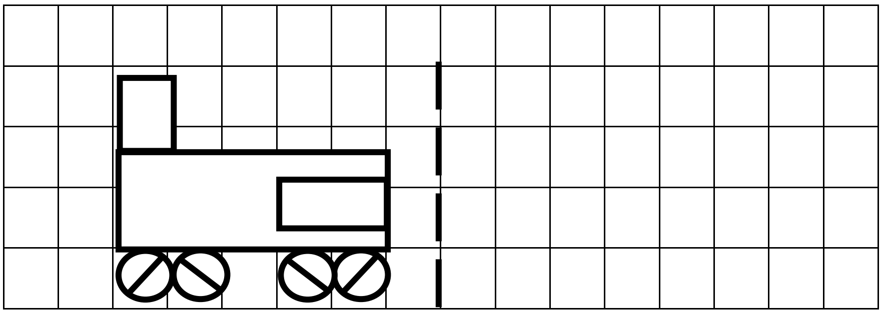


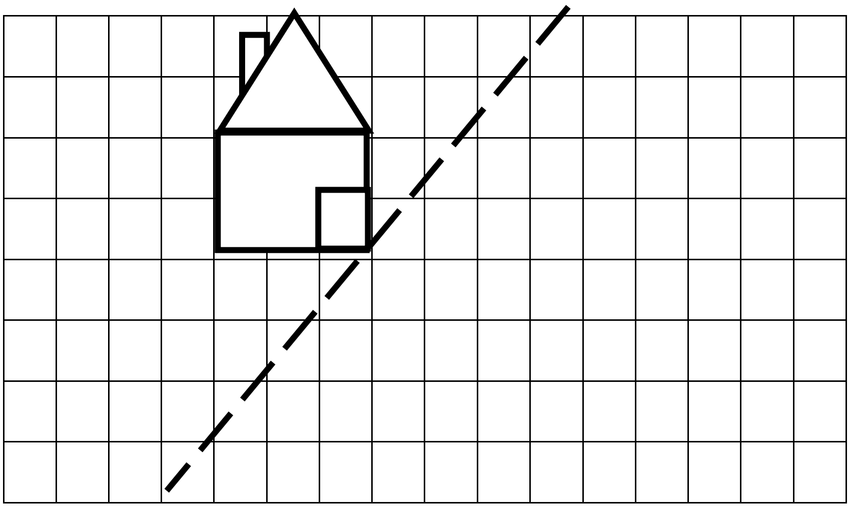
5. Fill in the boxes to draw the symmetrical of each shape for the given line of symmetry.





6. Draw the symmetrical of the shapes. The discontinuous line is the line of symmetry.





7. John discovered the rule: “I can say whether a quadrilateral has lines of symmetry. IF the triangles formed by the folding line are exactly the same, then the shape has a line of symmetry”. Explain whether you agree or not.